CHAPTER V

TIME-OPTIMAL SYNTHESIS FOR A SPECIAL CLASS OF SECOND ORDER NONLINEAR SYSTEM

5.1 INTRODUCTION

Perhaps the most widely studied type of problem in the mathematical control theory, known as the time optimal control problem, is that of steering the initial state of a system to a desired final state in minimum time, with control subject to constraints. A comprehensive treatment of this problem is given in [4] and [43]. A survey paper has been published by Balachandran [8] on the nonlinear time optimal control problem.

Many of the peculiarities of nonlinear control problems appear in systems of only second order and there are many systems of technology and science whose dynamics are governed by second order ordinary differential equation of the form

\[ \ddot{x} = f(x, \dot{x}, u) \]  

(A)

Time optimal control problem of equation (A) has been studied by Boltyanskii [19,20]. When \( f \) is linear, the problem has been thoroughly investigated by Athans and Falb [4]. Lee and Markus [43] discussed the time optimal control problem for a class of second order nonlinear system of the form

\[ \ddot{x} + f(x, \dot{x}) = u, \quad |u| \leq 1. \]  

(B)
Similarly in [4] time optimal control for a special class of nonlinear system

\[ \ddot{x} + f(\dot{x}) = Ku, \quad K > 0, \quad |u| \leq 1 \quad (C) \]

is investigated and it is shown that for nonlinear systems (C), like linear ones, bang-bang control is indeed time optimal. Almuzara and Flugge-Lotz [2] considered the minimum time control of systems of the form

\[ \ddot{x} + f(x) = u, \quad |u| \leq 1 \quad (D) \]

where \( f \) is a periodic function. Davis [30] studied the problem for the Duffing equation. In [48] Markin and Makeev shows that the controllability region of the system

\[ \ddot{x} + \sin \dot{x} = u \quad (E) \]

is \( -\infty < x < +\infty, \quad -3\pi/2 < \dot{x} < +3\pi/2 \) when \( |u| \leq 1 \). If \( |u| \leq 1 + \delta \), where \( \delta \) is a small positive number, then the region of controllability coincides with the entire phase plane.

In [20] Boltyanskii gave an example as

\[ \ddot{x} = (1/2) \ u \exp(\dot{x}^2) \quad (F) \]

for which the region of controllability is not completely filled with optimal trajectories and in [21] he determined the exact region where the trajectories are optimal.
In this chapter, we shall study the problem of [20] for a special class of second order nonlinear systems. We prove that the region of controllability of certain second order nonlinear systems is not completely filled with the optimal trajectories. Further we determine the exact region where the trajectories are optimal.

5.2 ANALYSIS

We consider a control system described by the equation:

\[ \dot{x} = u f(\dot{x}), \quad |u| \leq 1 \]

or in phase coordinates \( x_1 = x; \quad x_2 = \dot{x} \)

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u f(x_2)
\end{align*} \quad (5.1) \]

where the controlling parameter \( u \) satisfies

\[ -1 \leq u \leq 1 \quad (5.2) \]

we shall assume that

(i) \( f \) is a function of \((x_2)^2\) only and \( f(0) \neq 0 \).

(ii) \( \int_{f(x_2)}^{x_2} \frac{dx_2}{g(x_2)} = g(x_2), \) where \( g(x_2) \) is finite for all \( x_2 \).

(iii) \( \lim_{x \to \infty} \int_{-x}^{0} \frac{dx_2}{f(x_2)} = \mu < \infty. \)
For this system, we consider time-optimal controls, that is, the determination of controls yielding the most rapid movement of the system to the point \( x_1 = x_2 = 0 \) from a given initial state.

We apply Pontryagin’s Maximum Principle [19,20]. For that, consider the function

\[
H = \phi_1 x_2 + \phi_2 u f(x_2)
\]

where the unknown functions \( \phi_1, \phi_2 \) satisfy

\[
\begin{align*}
\dot{\phi}_1 &= -\frac{\partial H}{\partial x_1} = 0 \\
\dot{\phi}_2 &= -\frac{\partial H}{\partial x_2} = -\phi_1 - \phi_2 u \frac{\partial f}{\partial x_2}.
\end{align*}
\] (5.3)

Clearly, for given \( x_1, x_2, \phi_1, \phi_2 \) the function \( H \) attains its maximum as a function of \( u \) simultaneously with the function \( \phi_2 u \); hence the maximum condition for \( H \) is equivalent to

\[
u = \text{sign}\phi_2 \] (5.4)

We shall prove that each optimal control takes only the values \( u = \pm 1 \), and has not more than one reversal from one value to the other. In fact let \( u(t), t_0 \leq t \leq t_1 \) be an optimal control, and let \( x(t) = (x_1(t), x_2(t)) \) be the corresponding trajectory (starting at a point \( x_0 \) and terminating at the origin). The maximum principle implies that there is a nontrivial solution \( \phi(t) = (\phi_1(t), \phi_2(t)) \) of (5.3) such that, for all \( t \) in the interval \( t_0 \leq t \leq t_1 \), the functions \( u(t) \) and \( \phi_2(t) \) satisfy the maximum condition (5.4). It is thus sufficient to establish that \( \phi_2(t) \) has not more than one zero for \( t_0 \leq t \leq t_1 \). Assume the
contrary. That is, that \( \phi_2(t) \) has more than one zero on this interval, and let \( \alpha \) and \( \beta \) be two adjacent zeros. (We cannot have \( \phi_2(t) = 0 \) on an interval, since then (5.3) would imply that \( \phi_1(t) = 0 \).) Then on the interval \( \alpha < t < \beta \), the function \( \phi_2(t) \) has constant sign. Let \( \phi_2(t) < 0 \) and let \( \alpha < t < \beta \) (if \( \phi_2 \) is positive on this interval the reasoning is similar); in this case \( \phi_2(\alpha) \leq 0 \) and \( \phi_2(\beta) \geq 0 \). Since \( \phi_2(\alpha) = \phi_2(\beta) = 0 \), the second equation in (5.3) implies \( \phi_1(\alpha) = -\phi_2(\alpha) \geq 0, \phi_1(\beta) = -\phi_2(\beta) \leq 0 \). Since \( \phi_1(t) = \text{constant} \) (by virtue of the first of the equations (5.3)), we have \( \phi_1(t) = 0 \). In particular \( \phi_1(\alpha) = 0 \) and \( \phi_2(\alpha) = 0 \), which contradicts the fact that \( \phi_2(t) \) is a nontrivial solution of (5.3). This contradiction proves our assertion.

For the time interval on which \( u = +1 \), it follows from (5.1) that

\[
\frac{dx_1}{dx_2} = \frac{x_2}{f(x_2)}
\]

hence

\[
x_1 = \int \frac{x_2 \, dx_2}{f(x_2)} = g(x_2) + c. \tag{5.5}
\]

Thus the part of the phase trajectory for which \( u = +1 \) is an arc of the curve (5.5). Similarly, for the time interval on which \( u = -1 \), we have

\[
x_1 = -\int \frac{x_2 \, dx_2}{f(x_2)} = -g(x_2) - c. \tag{5.6}
\]
The phase point moves upwards on the curves (5.5) and downwards on the curves (5.6).

As shown above, each optimal control is a piecewise-constant function taking values ±1 and changing sign not more than once. If $u(t)$ is equal to +1 for a certain time and then equal to -1, the phase trajectory consists of two parts of the curves (5.5) and (5.6) (the continuous curve in Fig. 1) abutting one another, and the second of the parts lies on the curve (5.6) passing through the origin. (since the required trajectory must end at the origin.) That is, on the curve $x_1 = B_1 - g(x_2)$, where $B_1 = g(0)$. Conversely if $u = -1$ first and then $u = +1$ we have the phase trajectory shown by the dotted curve in Fig. 1. Figure 2 shows a family of trajectories obtained in this way. Here AO is part of the curve $x_1 = - B_1 + g(x_2)$ in the lower half-plane, and BO is part of the curve $x_1 = B_1 - g(x_2)$ in the upper half-plane.

The maximum principle implies that only the trajectories shown in Fig. 2 can be optimal. The region $G$ filled by these trajectories is determined by the inequality

$$|x_1| < B_1 + g(x_2);$$

the boundary of $G$ is shown by the dashed curves in Fig. 2.

If $x_0$ is not in $G$, there is no optimal trajectory starting at $x_0$ and ending at the origin. However, for any $x_0$ in $G$, there is a trajectory (not optimal) corresponding to an admissible control and ending at the origin. That is, the region
Fig. 1
of controllability is the whole plane. For a point to the left of G, the curve MNPO in Fig. 3 is such a trajectory. Here MN is the arc of a curve (5.5) corresponding to the control \( u = +1 \) and ending above the horizontal axis, and NP is a straight-line segment parallel to this axis and corresponding to \( u = 0 \) (so that, by virtue of (5.1), \( \dot{x}_2 = 0 \) and \( \dot{x}_1 = x_2 = \text{constant} > 0 \) on this section). For any positive number \( \epsilon \), the time for motion along the straight line segment NP can be made smaller than \( \epsilon \) by having NP sufficiently high. The velocity \( \dot{x}_1 = x_2 \) on this section can be arbitrarily large if NP is sufficiently high.

If \( x_0 \) is to the right of G, the origin can be reached along \( M'N'P'O \) (Fig. 3), and the time of motion along \( N'P' \) tends to zero when this segment moves downwards.

5.3 TIME ESTIMATE AND BELLMAN'S EQUATION

We now estimate the time required for the phase point to move from an arbitrary initial point \( x_0 \) to the origin. We start by assuming that \( x_0 \) is inside G. Then there is a unique trajectory from \( x_0 \) to the origin corresponding to \( u = \pm 1 \) and having not more than one switching (Fig. 2). The time of motion from \( x_0 \) to the origin along this trajectory will be denoted by \( T(x_0) \). If \( x_0 = (x_{10}, x_{20}) \) lies on \( AP' \) (Fig. 3)

\[
T(x_0) = \int_{x_{20}}^{0} \frac{dx_2}{f(x_2)}
\]

58
(since (5.1) implies that $\frac{dt}{dx} = \frac{dx}{f(x)}$). Hence if $x_0$ moves along AO towards infinity (That is, $x_{20} \rightarrow -\infty$).

$$\lim_{x_0 \to 0} T(x_0) = \int_{-\infty}^{0} \frac{dx_2}{f(x_2)} = \mu$$

Now, let $x_0$ be any point to the right of G (or on its boundary), That is, $x_{10} > \beta_1 + g(x_{20})$. Let a curve (5.6) pass through this point. The time required for the motion along this curve from $x_0$ downwards (to infinity) is shown by (5.1) to be

$$\int_{x_{20}}^{-\infty} \frac{-dx_2}{f(x_2)} = \int_{-\infty}^{x_{20}} \frac{dx_2}{f(x_2)} = \int_{-\infty}^{0} \frac{dx_2}{f(x_2)} + \int_{0}^{x_{20}} \frac{dx_2}{f(x_2)}$$

$$= \mu + 2\mu \Phi(x_{20}),$$

where

$$\Phi(x) = \frac{1}{2\mu} \int_{0}^{x} \frac{dx_2}{f(x_2)}$$

It is clear that (for any $\epsilon > 0$) motion from $x_0$ to the origin is completed during a time shorter than $2\mu(1 + \Phi(x_{20})) + \epsilon$. In fact at a time shorter than $\mu + 2\mu \Phi(x_{20})$, the phase point can travel downwards an arbitrarily large distance; then after a time less than $\epsilon$ the phase point can move along $x_2 = \text{constant}$ to the left and reach some position on AO; finally, after a time
less than $\mu$ the point can move along AO from this position to the origin.

An analogous calculation shows that, if $x_0 = (x_{10}, x_{20})$ is to the left of $G$ (or on its left boundary) and $\epsilon > 0$ is arbitrary, the phase point can move from $x_0$ to the origin in a time shorter than $2\mu(1 + \Phi(-x_{20})) + \epsilon$.

We now prove that

$$w(x) = \begin{cases} 
-T(x), & \text{if } x \text{ is in } G; \\
-2\mu (1 + \Phi(x_2)), & \text{if } x = (x_1, x_2) \text{ is to the right of } G \\
& \text{or on its right boundary;} \\
-2\mu (1 + \Phi(-x_2)), & \text{if } x = (x_1, x_2) \text{ is to the left of } G \\
& \text{or on its left boundary;}
\end{cases}$$

It is clear from the foregoing discussion that, for any $\epsilon > 0$, the phase point can move from any point $x = (x_1, x_2)$ to the origin in a time smaller than $-w(x) + \epsilon$ (and for $x \in G$ in exactly the time $-w(x)$). It is easily seen that $w(x)$ is continuous in the phase plane.

The Bellman equation for the process (5.1) is

$$\max \left( x_2 \frac{\partial w}{\partial x_1} + u f(x_2) \frac{\partial w}{\partial x_2} \right) = 1. \quad (5.7)$$

We shall prove that the function $w(x)$ constructed above has continuous derivatives $\frac{\partial w(x)}{\partial x_1}$ and $\frac{\partial w(x)}{\partial x_2}$ and a continuous Bellman function (5.7), except on the switching curve AOB and on the boundary of $G$. 

60
Let $x$ be to the right of $G$ (not on its boundary). Then $w(x) = -2\mu(1 + \phi(x_2))$; hence $\partial w/\partial x_1 = 0$. Moreover

$$\partial w/\partial x_2 = -2\mu \left\{ \partial \phi(x_2)/\partial x_2 \right\}$$

$$= -2\mu \left\{ (1/2\mu) \left( 1/f(x_2) \right) \right\} = \left\{ -1/f(x_2) \right\}.$$ 

Hence the derivatives $\partial w(x)/\partial x_1$ and $\partial w(x)/\partial x_2$ exist and are continuous to the right of $G$. Direct substitution of the values of the derivatives in (5.7) shows that $w(x)$ satisfies Bellman's equation (5.7) to the right of $G$. It can be proved in the same way that $w(x)$ satisfies Bellman's equation to the left of $G$. Finally, everywhere inside $G$ except at points on AOB, $w(x)$ also satisfies (5.7); this follows directly from Theorem 3.19 in [20] (pp. 266-274), since the trajectories in Fig. 2 satisfy maximum principle.

Hence $w(x)$ is continuous in the phase plane, and has continuous derivatives and satisfies Bellman's equation everywhere except at points of the piecewise-smooth set $M$ consisting of the curve AOB and the boundary of $G$. It follows from the Fundamental Lemma of [20] that the movement of the phase point from any point $x_0$ of the phase plane to the origin takes at least as long as $-w(x_0)$. If $x_0$ is in $G$, movement from $x_0$ to the origin takes exactly the time $-w(x_0) = T(x_0)$. That is, all trajectories shown in Fig. 2 are optimal. If $x_0$ is not in $G$, then for any $\epsilon > 0$, the phase point can move from $x_0$ to the origin in a
time smaller than $-w(x_0) + \varepsilon$. In this case, however, it is impossible to attain exactly the time $-w(x)$ (in fact if this were possible there would be an optimal trajectory from $x_0$ to the origin, but since $x_0 \not\in G$ there is no optimal trajectory).

Hence the region of controllability of the object (5.1), (5.2) coincides with the whole phase plane, and the region $G$, consisting of points from which there are optimal trajectories to the origin, does not coincide with the region of controllability. For any point $x_0$ of the phase plane, the number $-w(x)$ is the greatest lower bound of times for movement from $x_0$ to the origin, but outside $G$ this lower bound is not attained.

5.4 EXAMPLES

The assumptions (i), (ii) and (iii) are valid for the following examples:

5.4(a). Systems.

Ex: (1). $\dot{x}_1 = x_2; \dot{x}_2 = \frac{1}{2} u \exp((x_2)^2)$, $|u| \leq 1$.

Ex: (2). $\dot{x}_1 = x_2; \dot{x}_2 = u \cosh((x_2)^2)$, $|u| \leq 1$.

Ex: (3). $\dot{x}_1 = x_2; \dot{x}_2 = \frac{1}{2} u \sec((x_2)^2)$, $|u| \leq 1$.

Ex: (4). $\dot{x}_1 = x_2; \dot{x}_2 = \frac{1}{2} u \cosec((x_2)^2)$, $|u| \leq 1$. 
Example (1) was discussed in detail by Boltyanskii [21].

The equations (5.5) and (5.6) take the following form for the Examples (2) to (4) respectively,

\[ x_1 = \arctg(\exp((x_2)^2)) + c \]  \hspace{1cm} (5.8)

\[ x_1 = -\arctg(\exp((x_2)^2)) - c \]

\[ x_1 = \sin((x_2)^2) + c \]  \hspace{1cm} (5.9)

\[ x_1 = -\sin((x_2)^2) - c \]

and

\[ x_1 = -\cos((x_2)^2) + c \]  \hspace{1cm} (5.10)

\[ x_1 = \cos((x_2)^2) - c \]

5.4(b). Controllable region

The region \( G \) filled by the optimal trajectories in each of the above examples is determined by the inequality

for \( \text{EX}:(2), \) \[ |x_1| < \pi/4 + \arctg(\exp((x_2)^2)); \]

for \( \text{EX}:(3), \) \[ |x_1| < \sin((x_2)^2); \]

for \( \text{EX}:(4), \) \[ |x_1| < 1 + \cos((x_2)^2); \]

the boundary of \( G \) is shown by the dashed curves in Figs.2a,2b,2c.
5.4(c). Time estimate

We now estimate the time required for the phase point to move from an arbitrary initial point \( x_0 \) to the origin. We start by assuming that \( x_0 \) is inside \( G \). The time of motion from \( x_0 \) to the origin is denoted by \( T(x_0) \) and for Ex.(2),

\[
T(x_0) = \int_{x_{20}}^{0} \frac{dx_2}{\cosh((x_2)^2)}
\]

and

\[
\lim_{x_2 \to 0} T(x_0) = \int_{-\infty}^{0} \frac{dx_2}{\cosh((x_2)^2)} = \frac{m}{\pi},
\]

where \( m = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} < 1 \).

For Ex.(3),

\[
T(x_0) = \int_{x_{20}}^{0} 2 \cos((x_2)^2) \, dx_2
\]

and

\[
\lim_{x_2 \to 0} T(x_0) = \int_{-\infty}^{0} 2 \cos((x_2)^2) \, dx_2 = \sqrt{\pi/2}.
\]

For Ex(4),

\[
T(x_0) = \int_{x_{20}}^{0} 2 \sin((x_2)^2) \, dx_2
\]
and \[ \lim_{x \to \infty} T(x_0) = \int_{-\infty}^{0} 2 \sin((x_2)^2) \, dx_2 = \sqrt{(\pi/2)}. \]

Now, let \( x_0 \) be any point to the right of \( G \) (or on its boundary). The motion time along this curve from \( x_0 \) downwards (to infinity) is given by, for Ex.(2),

\[
\int_{x_0}^{-\infty} -\frac{dx_2}{\cosh((x_2)^2)} = m \sqrt{\pi} + 2/\pi \, \Phi_1(x_20, \sqrt{2}),
\]

where \( \Phi_1(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{x} \frac{dt}{\cosh(t^2/2)} \).

For Ex(3),

\[
\int_{x_0}^{-\infty} -2\cos((x_2)^2) \, dx_2 = \sqrt{(\pi/2)} + 2/(\pi/2) \, C(x_20),
\]

where \( C(x) = \frac{1}{\sqrt{(\pi/2)}} \int_{0}^{x} \cos(t^2) \, dt \), is the Fresnel's cosine integral.

For Ex(4),

\[
\int_{x_20}^{-\infty} -2\sin((x_2)^2) \, dx_2 = \sqrt{(\pi/2)} + 2/(\pi/2) \, S(x_20),
\]
where

\[ S(x) = \frac{1}{\sqrt{\pi/2}} \int_{0}^{x} \sin(t^2) \, dt, \]
is the Fresnel's sine integral.

An analogous calculation shows that, if \( x_0 = (x_1, x_2) \) is to the left of \( G \) (or on its left boundary) and \( \epsilon > 0 \) is arbitrary, the phase point can move from \( x_0 \) to the origin in a time shorter than

\[
2/\pi (m + \Phi_1(-x_2/2)) + \epsilon, \quad \text{for Ex.}(2)
\]
\[
2/(\pi/2) (1 + C(-x_2)) + \epsilon, \quad \text{for Ex.}(3)
\]
\[
2/(\pi/2) (1 + S(-x_2)) + \epsilon, \quad \text{for Ex.}(4).
\]

We now prove that, for Ex. (2),

\[
w(x) = \begin{cases} 
-T(x), & \text{if } x \text{ is in } G; \\
-2/\pi (m + \Phi_1(x_2/2)), & \text{if } x = (x_1, x_2) \text{ is to the right of } G \\
& \text{or on its right boundary;}
-2/\pi (m + \Phi_1(-x_2/2)), & \text{if } x = (x_1, x_2) \text{ is to the left of } G \\
& \text{or on its left boundary;}
\end{cases}
\]
For Ex(3)

\[
\begin{align*}
    w(x) = & \begin{cases} 
        -T(x), & \text{if } x \text{ is in } G; \\
        -2/(\pi/2) \left( 1 + C(x_{20}) \right), & \text{if } x = (x_1, x_2) \text{ is to the right of } G \text{ or on its right boundary}; \\
        -2/(\pi/2) \left( 1 + C(-x_{20}) \right), & \text{if } x = (x_1, x_2) \text{ is to the left of } G \text{ or on its left boundary};
    \end{cases}
\end{align*}
\]

For Ex(4)

\[
\begin{align*}
    w(x) = & \begin{cases} 
        -T(x), & \text{if } x \text{ is in } G; \\
        -2/(\pi/2) \left( 1 + S(x_{20}) \right), & \text{if } x = (x_1, x_2) \text{ is to the right of } G \text{ or on its right boundary}; \\
        -2/(\pi/2) \left( 1 + S(-x_{20}) \right), & \text{if } x = (x_1, x_2) \text{ is to the left of } G \text{ or on its left boundary};
    \end{cases}
\end{align*}
\]

5.4(d). Bellman's equation

We shall prove that the functions \( w(x) \) constructed above have continuous derivatives \( \partial w(x)/\partial x_1 \) and \( \partial w(x)/\partial x_2 \) and continuous Bellman functions (5.7) [16], except on the switching curve AOB and on the boundary of G.
Let \( x \) be to the right of \( G \) (not on its boundary). Then, for Ex(2),
\[
\begin{align*}
w(x) &= -2/\pi(m + \Phi_1(x_{20}/2)) \text{; hence } \partial w/\partial x_1 = 0 \text{ and } \partial w/\partial x_2 = -2/\pi \left( \partial \Phi_1(x_{20}/2)/\partial x_2 \right) \\
&= -\text{sech}(x_2).
\end{align*}
\]
For Ex(3),
\[
\begin{align*}
w(x) &= -2/(\pi/2) \left( 1 + C(x_{20}) \right) \text{; hence } \partial w/\partial x_1 = 0 \text{ and } \partial w/\partial x_2 = -2/(\pi/2) \left( \partial C(x_2)/\partial x_2 \right) \\
&= -2 \cos((x_2)^2).
\end{align*}
\]
For Ex(4),
\[
\begin{align*}
w(x) &= -2/(\pi/2) \left( 1 + S(x_{20}) \right) \text{; hence } \partial w/\partial x_1 = 0 \text{ and } \partial w/\partial x_2 = -2/(\pi/2) \left( \partial S(x_2)/\partial x_2 \right) \\
&= -2 \sin((x_2)^2).
\end{align*}
\]
Hence, in all the above examples, the derivatives \( \partial w(x)/\partial x_1 \) and \( \partial w(x)/\partial x_2 \) exist and are continuous to the right of \( G \). Direct substitution of the values of the derivatives in (5.7) shows that \( w(x) \) satisfies Bellman’s equation (5.7) to the right of \( G \). It can be proved in the same way that \( w(x) \) satisfies Bellman’s equation to the left of \( G \). Finally, everywhere inside \( G \) except at points on AOB, \( w(x) \) also satisfies (5.7).
Thus we have verified the conclusions of the theory that the region of controllability of each object coincides with the whole phase plane; the region $G$, consisting of points from which there are optimal trajectories to the origin, does not coincide with the region of controllability; and for any point $x_0$ of the phase plane, the number $-w(x)$ is the greatest lower bound of times for movement from $x_0$ to the origin but outside $G$ this lower bound is not attained.

******