Chapter 1

Introduction

Stochastic modelling has been widely used to model the phenomena arising in many branches of science and industry such as biology, economics, mechanics, electronics and telecommunications. Theory of stochastic differential equations has attracted much attention since it is not only academically challenging but also of practical importance and has played an important role in many areas of study such as insurance, finance, population dynamics, control in [79, 36].

The inclusion of random effects in differential equations leads to two distinct classes of equations for which the solution processes have differentiable and non-differentiable sample paths. They require fundamentally different methods of analysis. The first and simpler class arises when an ordinary differential equation has random coefficients or a random initial value or is forced by a fairly regular stochastic process or when some combination of these holds. The equations are called random differential equations and are solved by sample path as ordinary differential equations. The sample paths of solution processes are then at least differential functions. The second class occurs when the forcing is an irregular stochastic process such as Gaussian white noise. The equations are then written symbolically as stochastic differentials, but are interpreted as integral equations with Itô or Stratonovich stochastic integrals. They are called stochastic differential equations and, in general, their solutions inherit the non-differentiability of sample paths from the Wiener processes in the stochastic integrals. In many applications such equations result from the incorporation of either internally or externally originating random fluctuations in the dynamical description of a system. An example of the former is the molecular bombardment of a speck of dust on a water surface which results in Brownian motion. In the second class, one kind
of the differential equations is Itô stochastic differential equations. The most of stochastic differential equations cannot be solved explicitly. So the construction of efficient numerical methods of these equations and study of the properties of these approximate schemes are very important.

1.1 Motivation

Stochastic differential equations of the Itô type which arise frequently in mathematical descriptions of physical phenomena depending on the effect of a Gaussian white noise random forces play a major role in the characterization of many physical phenomena in life sciences and engineering and recently in financial mathematics. Bearing in mind that a Gaussian white noise is an abstraction and not a physical process, mathematically described as a formal derivative of a Brownian motion process, all such phenomena are essentially based on stochastic differential equations of the Itô type. Moreover many phenomena analyzed in the natural sciences and engineering are described by solutions of nonlinear Itô stochastic differential equations [5, 69].

1.2.1 Population Models

One of the most used models is the system of Volterra integro-differential equation of the form

$$dx_i(t) = x_i(t) \left[ \alpha_i(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t) + \sum_{j=1}^{n} \int_0^t K_{ij}(t,s)x_j(s)ds \right] dt + \sin x_i(t)d\mathcal{W}(t)$$

which governs the population growth of interacting species $x_i(t), i = 1, 2, \cdots, n$. The above equation is a stochastic integro-differential system of the form

$$dx(t) = \left[ A(t)x(t) + f(t, x) + \int_0^t K(t, s)x(s)ds \right] dt + \sigma(t, x(t))d\mathcal{W}(t)$$

in which $A(t) = \text{diag}[a_1(t), a_2(t), \cdots, a_n(t)], a_i(t) \in C(\mathbb{R}, \mathbb{R})$. The integral term here specifies how much weight is to be attached to the population at various past times in order to arrive at their present effect on the resources availability [55].
1.2.2 Nonlinear Price Dynamics

A nonlinear stochastic continuous time model captures the main characteristics of price dynamics. Nonlinear phenomena in real data can be observed in different asset markets and are becoming more and more focus of attention in the mathematical modeling of financial price processes. It is a common practice in the financial modeling to assume that the price dynamics $X$ is modeled by the Itô stochastic differential equation [79]

$$dX = \mu(t, X(t), Z(t))dt + \sigma(t, X(t), Z(t))dW.$$ 

Here $Z(t)$ represents external, such as economic or political, effects and $W$ is a standard Wiener process with the property that $dW$ is distributed as $\mathcal{N}(0, dt)$ and $\mu$ and $\sigma$ satisfy Lipschitz and growth conditions sufficient for the existence of a continuous solution to the above equation. The interplay of nonlinearities in the dynamics and the stochastic influences in the system plays a very important role. These interactions may lead to effects which cannot be explained otherwise: for example, multi-modal distributions can be traced back to multiple states in the dynamical system; observed jumps and strong oscillations in the historical data can be explained by stochastic changes of attractors. Small random perturbations may push a balanced market from one equilibrium into another reflecting both regime switches and rare events.

1.2.3 Model in Commodity Markets

The dynamics of price, production and consumption of a particular commodity when we assume relative variations in the market price $P(t)$ is governed by the equation [9]

$$dP = P f(D(P_d), S(P_s))dt + \sigma(t)dW(t)$$

where $D(\cdot)$ and $S(\cdot)$, respectively, denote the demand and supply functions for the commodity in question. The arguments of the demand and supply schedules are given by $P_D$ (demand price) and $P_S$ (supply price) respectively and $\sigma(t)$ is the environmental noise like economic or political effects.

In specifying how consumer behavior affects commodity demand, we assume that this behavior is governed by an integration of information regarding past prices. Thus demand for a commodity is a weighted function $(P_D)$ of past prices. In calculating this weighted function, we assume that, at time $t$, the consumer
attaches a weight \( K_P(t - s) \) to a past market price \( P(s) \), where \( 0 \leq s \leq t \), and that the weighted average of all of these past prices is just the demand price

\[
P_D(t) = \int_0^t K_P(t - s)P(s)ds.
\]

To complete the formulation of the model, the relation between the current market price \( P \) and the supply price \( P_S \) must be specified. We assume that producers, like consumers, take past market prices into account when making decisions to initiate alterations in production using an analogous normalized weight function, namely, the supply price kernel \( K_S(q) \). Then the supply price \( P_S \) is given by

\[
P_S(t) = \int_0^t K_S(t - T_{\text{min}} - s)P(s)ds
\]
in complete analogy with the demand price.

### 1.2 Basic Notions of Probability Theory

Probability theory is concerned with the mathematical analysis of the intuitive notion of chance or randomness which is born of experience. The quantitative idea of randomness first took form at the gaming tables, and probability theory was begun by Pascal and Fermat (1654), as a theory of games of chance. Since then, the notion of chance has found its way into almost all branches of knowledge. The probability theory deals with mathematical models of trials whose outcomes depend on chance. All possible outcomes (the elementary events) are grouped together to form a set \( \Omega \) with typical element \( \omega \in \Omega \). Not every subset of \( \Omega \) is in general an observable or interesting event. So we only group these observable or interesting events together as a family \( \mathcal{F} \) of subsets of \( \Omega \).

**Definition 1.2.1.** If \( \Omega \) is a given set, then a \( \sigma \)-algebra \( \mathcal{F} \) on \( \Omega \) is a family \( \mathcal{F} \) of subsets of \( \Omega \) with the following properties:

1. \( \emptyset \in \mathcal{F} \)
2. \( F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F} \), where \( F^c = \Omega \setminus F \) is the complement of \( F \) in \( \Omega \).
3. \( A_1, A_2, \ldots \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F} \).

The pair \( (\Omega, \mathcal{F}) \) is called a measurable space. A probability measure \( \mathbb{P} \) on a measurable space \( (\Omega, \mathcal{F}) \) is a function \( \mathbb{P} : \mathcal{F} \rightarrow [0, 1] \) such that
(a) $P(\emptyset) = 0$, $P(\Omega) = 1$,

(b) if $A_1, A_2, \ldots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint (that is, $A_i \cap A_j = \emptyset$ if $i \neq j$), then

$$
P\left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} P(A_i).
$$

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space. It is called a complete probability space if $\mathcal{F}$ contains all subsets $G$ of $\Omega$ with $P$-outer measure zero, that is,

$$
P^*(G) := \inf \{P(F) : F \in \mathcal{F}, G \subseteq F \} = 0.
$$

**Definition 1.2.2.** If $(\Omega, \mathcal{F}, P)$ is a given probability space, then a function $Y : \Omega \rightarrow \mathbb{R}^n$ is called $\mathcal{F}$-measurable if

$$
Y^{-1}(U) := \{ \omega \in \Omega ; Y(\omega) \in U \} \in \mathcal{F}
$$

for all open sets $U \in \mathbb{R}^n$ (or, equivalently, for all Borel sets $U \subseteq \mathbb{R}^n$).

The mathematical model for a random quantity is a random variable. A random variable is simply a characterization of the outcome of a probabilistic experiment which assigns a numerical value to that outcome.

**Definition 1.2.3.** A random variable $X$ is an $\mathcal{F}$-measurable function $X : \Omega \rightarrow \mathbb{R}^n$. Every random variable induces a probability measure $\mu_X$ on $\mathbb{R}^n$ defined by

$$
\mu_X(B) = P(X^{-1}(B)),
$$

where $\mu_X$ is called the distribution of $X$.

**Definition 1.2.4.** If $\int_{\Omega} |X(\omega)| dP(\omega) < \infty$, then the number

$$
E[X] := \int_{\Omega} X(\omega) dP(\omega) = \int_{\mathbb{R}^n} xd\mu_X(x)
$$

is called the expectation of $X$ (with respect to $\mathbb{P}$).

More generally, if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is Borel measurable and $\int_{\Omega} |f(X(\omega))| dP(\omega) < \infty$, then we have

$$
E[f(X)] := \int_{\Omega} f(X(\omega)) dP(\omega) = \int_{\mathbb{R}^n} f(x) d\mu_X(x).
$$

**Definition 1.2.5.** Two subsets $A, B \in \mathcal{F}$ are called independent if $P(A \cap B) = P(A) \cdot P(B)$.

**Remark 1.2.1.** If two random variables $X, Y : \Omega \rightarrow \mathbb{R}$ are independent, then

$$
E[XY] = E[X]E[Y]
$$

provided $E[|X|] < \infty$, $E[|Y|] < \infty$. 
1.3 Stochastic Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration is a family of $\{\mathcal{F}_t\}_{t \geq 0}$ of increasing sub-$\sigma$-algebras of $\mathcal{F}$ (that is, $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for all $0 \leq t < s < \infty$). The filtration is said to be right-continuous if $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \geq 0$. When the probability space is complete, the filtration is said to satisfy the usual conditions if it is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. From now on, unless otherwise specified, we shall always work on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions.

**Definition 1.3.1.** A stochastic process is a parametrized collection of random variables $\{X_t\}_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ assuming values in $\mathbb{R}^n$.

1.3.1 Brownian Motion

Brownian motion is the heart of most stochastic models in practice. Its name comes from the Scottish botanist Robert Brown who, around 1827, reported experimental observations involving the erratic behaviour of a pollen grain when bombarded by (relatively small and effectively invisible) water molecules. A mathematical theory for Brownian motion has been developed by Albert Einstein and Norbert Wiener making significant contributions.

**Definition 1.3.2.** A Brownian motion is a stochastic process $\{B_t\}_{t \in \mathbb{R}^+}$ with the following properties.

(i). The increments on disjoint intervals are independent: if $0 < t_1 < t_2 < \ldots < t_n$, the random variables $B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$, are independent.

(ii). If $s < t$, the increment $B_t - B_s$ of the process on the interval $[s, t]$ is normally distributed with mean 0 and variance $t - s$.

(iii). The process starts almost surely at 0: $B_0 = 0$ with probability one.

(iv). The paths of the process $B_t$ are all continuous.

**Definition 1.3.3.** A stochastic process $\{x(t), t \geq 0\}$ on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be continuous-time Markov chain if, for all $s, t \geq 0$ and non-negative integers $i, j$, $x(u), 0 \leq u \leq s$.

$$\mathbb{P}\{x(t+s) = j \mid x(s) = i, x(u) = \pi(u), 0 \leq u \leq s\} = \mathbb{P}\{x(t+s) = j \mid x(s) = i\}.$$
1.4 Stochastic Differential Equations

In this section, we discuss the construction of stochastic differential equations, Itô formula, use of Itô formula in finding analytical solutions of stochastic differential equations and their stability properties. During the last decade, the theory of stochastic integrals and stochastic differential equations founded by K. Ito [51, 52] has developed rapidly.

Following Einstein’s explanation of observed Brownian motion, attempts were made by Langevin and others to formulate the dynamics of such motions in terms of differential equations. The resulting equations were written in the form

\[ dx(t) = f(t, x(t))dt + g(t, x(t))\xi(t)dt \]  

with a deterministic or averaged drift term in the ordinary differential equation perturbed by a noisy, diffusive term \( g(t, x(t))\xi(t) \), where the \( \xi(t) \) were standard Gaussian random variables for each \( t \) and \( g(t, x) \) is a space-time dependent intensity factor. This symbolic differential was interpreted as an integral equation

\[ x(\omega)(t) = x(\omega)(t_0) + \int_{t_0}^{t} f(s, x(\omega)(s))ds + \int_{t_0}^{t} g(s, x(\omega)(s))\xi(\omega)(s)ds, \]  

for each sample path \( \omega \). When extrapolated to a limit, the observations of Brownian motion seemed to suggest that the covariance \( c(t) = E(\xi(s)\xi(s + t)) \) of the process \( \xi(t) \) had a constant spectral density, that is with all time frequencies equally weighted in any Fourier transform of \( c(t) \). Such a process became known as Gaussian white noise, particularly in the engineering literature. For the special case of (1.4.2) with \( f = 0 \) and \( g = 1 \), we see that \( \xi(t) \) should be the derivative of pure Brownian motion, that is, the derivative of a Wiener process \( W(t) \), thus suggesting that we could write (1.4.2) alternatively as

\[ x(\omega)(t) = x(\omega)(t_0) + \int_{t_0}^{t} f(s, x(\omega)(s))ds + \int_{t_0}^{t} g(s, x(\omega)(s))dW(\omega)(s). \]  

The problem with this is that a Wiener process \( W(t) \) is nowhere differentiable, so strictly speaking the white noise process \( \xi(t) \) does not exist as a conventional function of \( t \); indeed, a flat spectral density implies that its covariance function \( c(t) \) is a constant multiple of the Dirac delta function \( \delta(t) \). Thus the second integral in (1.4.3) cannot be an ordinary Riemann or Lebesgue integral. Worse still, the continuous sample paths of a Wiener process are not of bounded variation on any
bounded time interval, so the second integral in (1.4.3) cannot even be interpreted as a Riemann-Stieltjes integral for each sample path.

For constant $g(t, x) = c$ we would expect the second integral in (1.4.3), to equal $c(W_\omega(t) - W_\omega(t_0))$. This is the starting point for Itô’s definition of a stochastic integral. To fix ideas, we shall consider such an integral of a random function $g$ over the time interval $0 \leq S < T$, denoting it by $I(g)$, where

$$I(g)(\omega) = \int_S^T g(s, \omega)dW_\omega(s). \quad (1.4.4)$$

**Definition 1.4.1.** Let $\{\mathcal{M}_t\}_{t \geq 0}$ be an increasing family of $\sigma$-algebras of subsets of $\Omega$. A process $g(t, \omega) : [0, \infty) \times \Omega \mapsto \mathbb{R}$ is called $\mathcal{M}_t$-adapted if, for each $t \geq 0$, the function $\omega \mapsto g(t, \omega)$ is $\mathcal{M}_t$-measurable.

**Definition 1.4.2.** Let $\mathcal{Y}(S, T)$ be the class of functions $f(t, \omega) : [0, \infty) \times \Omega \mapsto \mathbb{R}$ such that

(i) $f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel $\sigma$-algebra on $[0, \infty)$.

(ii) $f(t, \omega)$ is $\mathcal{F}_t$-adapted.

(iii) $\mathbb{E} \left[ \int_S^T f(s, \omega)^2 dt \right] < \infty$.

Here we also consider the following properties of the Itô Integral (1.4.4).

**Definition 1.4.3.** Let $f, g \in \mathcal{Y}(0, T)$ and let $0 \leq S < U < T$. Then

(i) $\mathbb{E} \left[ \left( \int_S^T f(t, \omega)dW(t) \right)^2 \right] = \mathbb{E} \left[ \int_S^T f^2(t, \omega)dt \right]$ for all $f \in \mathcal{Y}(S, T)$.

(ii) $\int_S^T f dW(t) = \int_S^U f dW(t) + \int_U^T f dW(t)$ for almost all $\omega$.

(iii) $\int_S^T (cf + g)dW(t) = c \int_S^T f dW(t) + \int_S^T g dW(t)$ (c constant) for almost all $\omega$.

(iv) $\mathbb{E} \left[ \int_S^T f dW(t) \right] = 0$.

(v) $\int_S^T f dW(t)$ is $\mathcal{F}_T$-measurable.
1.4.1 Itô Stochastic Differential Equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. We let $W(t) = (W_1(t), \ldots, W_m(t))$, $t \geq 0$ be an $m$-dimensional Wiener process defined on the space. Let $0 \leq t_0 < T < \infty$. Let $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$, that is, an $\mathcal{F}_{t_0}$-measurable $\mathbb{R}^n$-valued random variable such that $\mathbb{E}|x_0|^2 < \infty$. Let $f : [t_0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ and $g : [t_0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$ be both Borel measurable. Consider the $n$-dimensional stochastic differential equation of Itô type

$$dx(t) = f(t, x(t))dt + g(t, x(t))dW(t), \quad t_0 \leq t \leq T \quad (1.4.5)$$

with initial value $x(t_0) = x_0$. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds + \int_{t_0}^{t} g(s, x(s))dW(s), \quad \forall s \in [t_0, T]. \quad (1.4.6)$$

Let us first give the definition of the solution.

**Definition 1.4.4.** An $\mathbb{R}^n$-valued stochastic process $\{x(t)\}_{t_0 \leq t \leq T}$ is called a solution of (1.4.5) if it has the following properties:

(i). $\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted;

(ii). $\{f(t, x(t))\} \in \mathcal{L}^1([t_0, T]; \mathbb{R}^n)$ and $\{g(t, x(t))\} \in \mathcal{L}^2([t_0, T]; \mathbb{R}^{n \times m})$;

(iii). (1.4.6) holds with probability 1.

**Definition 1.4.5.** A solution $\{x(t)\}$ is said to be unique if any other solution $\tilde{x}(t)$ is indistinguishable from $\{x(t)\}$, that is,

$$\mathbb{P}\{x(t) = \tilde{x}(t) \text{ for all } t_0 \leq t \leq T\} = 1.$$

Let us now turn to finding the conditions that guarantee the existence and uniqueness of the solution to (1.4.5). The following theorem and its proof are in [74].

**Theorem 1.4.1.** Assume that there exist two positive constants $\hat{K}$ and $K$ such that

(Lipschitz condition) For all $x, y \in \mathbb{R}^n$ and $t \in [t_0, T]$

$$|f(t, x) - f(t, y)|^2 \vee |g(t, x) - g(t, y)|^2 \leq \hat{K}|x - y|^2;$$
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(Linear growth condition) For all \((t, x) \in [t_0, T] \times \mathbb{R}^n\)

\[ |f(t, x)|^2 + |g(t, x)|^2 \leq K(1 + |x|^2). \]

Then there exists a unique solution \(x(t)\) to (1.4.5) and the solution \(x(t)\) is adapted to the filtration \(\mathcal{F}_t\) and \(\mathbb{E} \left[ \int_{t_0}^{T} |x(t)|^2 dt \right] < \infty.\)

1.4.2 Itô Formula

However we have no differentiation theory, only integration theory. Nevertheless it turns out that it is possible to establish an Itô integral version of the chain rule, called the Itô formula. The Itô formula is very useful for evaluating Itô integrals. First we consider the one-dimensional Itô formula. One can see the proof of this formula in [79].

**Theorem 1.4.2.** Let \(x(t)\) be an Itô process given by

\[ dx(t) = f(t, x(t))dt + g(t, x(t))dW(t). \]

Let \(h(t, x)\) be twice continuously differentiable on \([0, \infty) \times \mathbb{R}\). Then \(y(t) = h(t, x(t))\) is again an Itô process and

\[ dy(t) = \frac{\partial h}{\partial t} dt + \frac{\partial h}{\partial x} dx + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} (dx)^2. \] (1.4.7)

where \((dx)^2 = dx.dx\) is computed according to the rules

\[ dt.dt = dt.dW(t) = dW(t).dt = 0, \quad dW(t).dW(t) = dt. \] (1.4.8)

**Remark 1.4.1.** (1.4.7) takes the another form

\[ y(t) - y(s) = \int_s^t \left\{ \frac{\partial h}{\partial t}(u, x(u)) + f_u \frac{\partial h}{\partial x}(u, x(u)) + \frac{1}{2} g_u^2 \frac{\partial^2 h}{\partial x^2}(u, x(u)) \right\} du \]

\[ + \int_s^t g_u \frac{\partial h}{\partial x}(u, x(u))du, \]

where \(f_u = f(u, x(u))\) and \(g_u = g(u, x(u)).\)

**Example 1.4.1.** Let us turn to the Itô Integral

\[ I = \int_{s}^{t} W(s) dW(s). \] (1.4.9)
To evaluate this integral, we choose \( x(t) = W(t) \) and \( h(t,x) = \frac{1}{2}x^2 \). Then
\[
y(t) = h(t,W(t)) = \frac{1}{2}(W(t))^2.
\]
Then, by above Itô formula, we get
\[
dy(t) = \frac{\partial h}{\partial t} dt + \frac{\partial h}{\partial x} dW(t) + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} (dW(t))^2
= W(t)dW(t) + \frac{1}{2}(dW(t))^2 = W(t)dW(t) + \frac{1}{2}dt.
\]
Hence
\[
d\left(\frac{1}{2}(W(t))^2\right) = W(t)dW(t) + \frac{1}{2}dt.
\]
In other words,
\[
I = \int_0^t W(s)dW(s) = \frac{1}{2}(W(t))^2 - \frac{1}{2}t.
\]
Next we consider the multi-dimensional Itô formula in the following theorem.

**Theorem 1.4.3.** Let
\[
dx(t) = f(t,x(t))dt + g(t,x(t))dW(t)
\]
be an \( n \)-dimensional Itô process. Let \( h(t,x) = (h_1(t,x),...,h_p(t,x)) \) be a twice continuously differentiable on \( [0,\infty) \times \mathbb{R}^n \) into \( \mathbb{R}^p \). Then the process \( y(t,\omega) = h(t,x(t)) \) is again an Itô process, whose component corresponding to number \( k \) namely, \( y_k \), is given by
\[
dy_k(t) = \frac{\partial h_k}{\partial t} dt + \sum_i \frac{\partial h_k}{\partial x_i} dx_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 h_k}{\partial x_i \partial x_j} (dx_i)(dx_j).
\]
where \( dW_i dW_j = \delta_{ij}dt \), \( dW_idt = dt dW_i = 0 \). \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \).

### 1.4.3 Analytical Solutions

The general form of a scalar linear stochastic differential equation is
\[
dx(t) = (a_1(t)x(t) + a_2(t)) dt + (b_1(t)x(t) + b_2(t)) dW(t).
\]
In [56], the explicit solution of (1.4.11) derived by Itô formula is given as
\[
x(t) = \Phi_{t_0,t_0}\left(x(t_0) + \int_{t_0}^{t} (a_2(s) - b_1(s)b_2(s))\Phi^{-1}_{s,t_0} ds + \int_{t_0}^{t} b_2(s)\Phi^{-1}_{s,t_0} dW(s)\right),
\]
with \( \Phi_{t_0,t_0} = exp\left(\int_{t_0}^{t} (a_1(s) - \frac{1}{2}b_1^2(s)) ds + \int_{t_0}^{t} b_1(s) dW(s)\right)\).
Example 1.4.2. Consider a simple homogeneous linear stochastic differential equation with additive noise and constant coefficients such as
\[ dx(t) = -\alpha x(t) dt + \sigma dW(t). \] (1.4.13)

Using the formula (1.4.12), we can get
\[ x(t) = \exp(-\alpha t) \left( x(0) + \sigma \int_0^t \exp(\alpha s) dW(s) \right). \]

Example 1.4.3. Consider a inhomogeneous linear stochastic differential equation with additive noise and constant coefficients such as
\[ dx(t) = (ax(t) + b) dt + c dW(t). \] (1.4.14)

Using the formula (1.4.12), we can get
\[ x(t) = \exp(at) \left( x(0) + b \left( 1 - \exp(-at) \right) + c \int_0^t \exp(-as) dW(s) \right). \]

Example 1.4.4. Consider a simple homogeneous linear stochastic differential equation with multiplicative noise and constant coefficients such as
\[ dx(t) = ax(t) dt + bx(t) dW(t). \] (1.4.15)

Using the formula (1.4.12), we have
\[ x(t) = x(0) \exp \left( \left( a - \frac{1}{2} b^2 \right) t + b W(t) \right). \]

Example 1.4.5. An inhomogeneous linear stochastic differential equation with additive noise and variable coefficients such as
\[ dx(t) = \left( \frac{2}{1 + t} - x(t) + b(1 + t)^2 \right) dt + b(1 + t)^2 dW(t). \] (1.4.16)

has the fundamental solution \( \Phi_{t,t_0} = \left( \frac{1 + t}{1 + t_0} \right)^2 \) and general solution
\[ x(t) = \left( \frac{1 + t}{1 + t_0} \right)^2 x(t_0) + b(1 + t_0)^2 (W(t) - W(t_0) + t - t_0). \]
1.4.4 Stability of Analytical Solutions

To discuss the stability, we consider a nonlinear n-dimensional stochastic differential equation of the form

\[ dx(t) = f(x(t))dt + g(x(t))dW(t), \quad t > 0 \]  

(1.4.17)

with \( x(0) \in \mathbb{R}^n \) and \( W(t) \) is an m-dimensional Wiener process and \( f : \mathbb{R}^n \to \mathbb{R}^n \), \( g : \mathbb{R}^n \to \mathbb{R}^{n \times m} \) are locally Lipschitz continuous functions with \( f(0) = 0 \) and \( g(0) = 0 \). (1.4.17) has an equilibrium solution \( x \equiv 0 \) and Protter \cite{[84]} showed that unique solutions exist on the the interval \( [0, \tau_e) \), where \( \tau_e \) is a (possibly infinite) random explosion time that depends on the initial value \( x(0) \). We give standard definitions of p-moment stability, global asymptotic stability and exponentially p-stability of the equilibrium solutions of (1.4.17).

Definition 1.4.6.  
(i). The equilibrium solution of (1.4.17) is pth-moment stable if and only if, for each \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that

\[ \mathbb{E}|x(t)|^p < \epsilon, \quad t \geq 0, \]

whenever \( \mathbb{E}|x(0)|^p < \delta \).

(ii). The equilibrium is globally pth-moment asymptotically stable if and only if it is pth-moment stable and, for all \( x(0) \in \mathbb{R} \),

\[ \lim_{t \to \infty} \mathbb{E}|x(t)|^p = 0. \]

(iii). If \( p = 2 \), the equilibrium is said to be mean-square asymptotically stable.

Definition 1.4.7.  
(i). The equilibrium solution of (1.4.17) is almost surely stable if and only if, for each \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that

\[ |x(t)| < \epsilon, \quad t > 0, \text{ almost surely}, \]

whenever \( |x(0)| < \delta \) almost surely.

(ii). The equilibrium is globally almost surely asymptotically stable if and only if, for all \( x(0) \in \mathbb{R} \),

\[ \lim_{t \to \infty} x(t) = 0 \quad \text{almost surely}. \]
Definition 1.4.8. (i). The equilibrium solution of (1.4.17) is said to be almost surely exponentially stable if

\[ \limsup_{t \to \infty} \frac{1}{t} \log |x(t)| < 0 \quad \text{almost surely,} \tag{1.4.18} \]

for all \( x(0) \in \mathbb{R}^n \). The quantity in the left hand side of (1.4.18) is called the sample Lyapunov exponent.

(ii). Assume that \( p > 0 \). The trivial solution of (1.4.17) is said to be \( p \)th moment exponentially stable if there is a pair of constants \( \lambda > 0 \) and \( C > 0 \) such that

\[ \mathbb{E} [ |x(t)|^p ] \leq C |x(0)|^p \exp(-\lambda t) \quad \text{for all } t \geq 0, \tag{1.4.19} \]

for all \( x(0) \in \mathbb{R}^n \). In this case, we call the quantity \( \limsup_{t \to \infty} \frac{1}{t} \log (\mathbb{E} [ |x(t)|^p ] ) \) as the \( p \)th moment Lyapunov exponent.

The following theorem in [46] gives the exponential stability result for (1.4.17).

Theorem 1.4.4. Let

\[ |f(x)| \vee |g(x)| \leq K |x|, \quad \forall x \in \mathbb{R}^n \tag{1.4.20} \]

hold. If

\[ -\lambda := \sup_{x \in \mathbb{R}^n, x \neq 0} \left( \frac{\langle x, f(x) \rangle + \frac{1}{2} |g(x)|^2}{|x|^2} - \frac{\langle x, g(x) \rangle^2}{|x|^4} \right) < 0, \tag{1.4.21} \]

then the solution of (1.4.17) obeys

\[ \limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \leq -\lambda \quad \text{almost surely} \tag{1.4.22} \]

and, given any \( \epsilon \in (0, \lambda) \), there exists a \( p^* \in (0, 1) \) such that, for all \( 0 < p < p^* \).

\[ \limsup_{t \to \infty} \frac{1}{t} \log (\mathbb{E} [ |x(t)|^p ] ) \leq -p(\lambda - \epsilon) \quad \text{almost surely}, \tag{1.4.23} \]

Appleby et al. in [6] developed conditions on \( f \) and \( g \) ensuring almost surely asymptotic stability of (1.4.17) by the following theorem.

Theorem 1.4.5. Let \( x \) be a solution of (1.4.17). If there exists \( \phi \in (0, 1) \) such that, for \( x \in \mathbb{R}^n \),

\[ |x|^2 (2 \langle x, f(x) \rangle + |g(x)|^2) - (2 - \phi)|x^T g(x)|^2 \leq 0, \tag{1.4.24} \]

and for every \( L > 0 \), \( \min_{|x|=L} |x^T g(x)| > 0 \). then \( \lim_{t \to \infty} x(t) = 0 \).
1.5 Numerical Methods

Many nonlinear stochastic differential equations have no closed-form solutions (analytical solutions). We have thus to be content with the numerical demonstrations. Several authors proposed various numerical methods for stochastic differential equations. Using MATLAB, we achieve the numerical simulation of all stochastic differential equations. In this section, we consider very basic numerical method, namely, Euler-Maruyama method to the one-dimensional stochastic differential equation of the form

\[ dx = f(x(t))dt + g(x(t))dW(t), \quad t > 0, \]  

with \( x(0) = x_0 \), where \( W \) is an one-dimensional Wiener process.

1.5.1 Euler–Maruyama Approximation

To apply a numerical method to (1.5.1) over \([0, T]\), we first discretize the interval. Let \( \Delta = T/L \) for some positive integer \( L \) and \( t_n = n\Delta \). Our numerical approximation to \( x(t_n) \) will be denoted \( y_n \) and \( x(0) = y_0 \). The Euler–Maruyama method takes the form

\[ y_{n+1} = y_n + f(y_n)\Delta + g(y_n)\Delta W_n, \]  

where \( n = 1, 2, \ldots, L \) and \( \Delta W_n = W(t_n) - W(t_{n-1}) \).

Now we apply this Euler–Maruyama method to the linear stochastic differential equation of the form

\[ dx(t) = \lambda x(t)dt + \mu x(t)dW(t) \]  

in which has the analytical solution

\[ x(t) = x(0) \left( \left( \lambda - \frac{1}{2} \mu^2 \right) t + \mu W(t) \right). \]  

To simulate exact solution, we consider

\[ x(t_n) = x(0) \left( \left( \lambda - \frac{1}{2} \mu^2 \right) t_n + \mu \sum_{i=1}^{n} \Delta W_{i-1} \right). \]

The Euler–Maruyama method is

\[ y_{n+1} = y_n + \lambda y_n \Delta + \mu y_n \Delta W_n, \]  

where \( y_0 = x(0) \), \( n = 1, 2, \ldots, L \) and \( \Delta W_n = W(t_n) - W(t_{n-1}) \).
1.5.2 Stability of Numerical Solutions

Stability analysis of numerical methods for stochastic differential equations has recently received the attention of many researchers. Consider a multi-dimensional non-linear stochastic differential equation of the form

\[ dx = f(x(t))dt + g(x(t))dW(t), \ t > 0, \]  

with \( x(0) = x_0 \), where \( W \) is a standard Wiener process. We give standard definitions of \( p \)-moment stability, global asymptotic stability and exponentially \( p \)-stability of the equilibrium solutions of (1.5.6).

Definition 1.5.1. (i). For a given step size \( \Delta t > 0 \), a numerical method \( y_n \) is said to be \( p \)-th-moment stable to the equilibrium solution of (1.5.6) if and only if, for each \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that

\[ \mathbb{E}|y_n|^p < \epsilon, \quad n \geq 0, \]

whenever \( \mathbb{E}|y_0|^p < \delta \).

(ii). For a given step size \( \Delta t > 0 \), a numerical method \( y_n \) is said to be globally \( p \)-th-moment asymptotically stable to the equilibrium solution of (1.5.6) if and only if it is \( p \)-th-moment stable and, for all \( y_0 \in \mathbb{R} \),

\[ \lim_{n \to \infty} \mathbb{E}|y_n|^p = 0. \]

(iii). If \( p = 2 \), the equilibrium is said to be mean-square asymptotically stable.

Definition 1.5.2. (i). For a given step size \( \Delta t > 0 \), a numerical method \( y_n \) is said to be almost surely stable to the equilibrium solution of (1.5.6) if and only if, for each \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that

\[ |y_n| < \epsilon, \quad n \geq 0, \text{ almost surely} \]

whenever \( |y_0| < \delta \) almost surely.

(ii). For a given step size \( \Delta t > 0 \), a numerical method \( y_n \) is said to be globally almost surely asymptotically stable to the equilibrium solution of (1.5.6) if and only if, for all \( y_0 \in \mathbb{R} \),

\[ \lim_{n \to \infty} y_n = 0, \text{ almost surely}. \]
Definition 1.5.3. (i). For a given step size $\Delta t > 0$, a numerical method $y_n$ is said to be almost surely exponentially stable to the equilibrium solution of (1.5.6) if

$$\limsup_{n \to \infty} \frac{1}{n} \log |y_n| < 0, \text{ almost surely},$$

for all $y_0 \in \mathbb{R}^n$.

(ii). Assume that $p > 0$. For a given step size $\Delta t > 0$, a numerical method $y_n$ is said to be $p$th moment exponentially stable to the trivial solution of (1.5.6) if there is a pair of constants $\lambda > 0$ and $N > 0$ such that

$$E[|y_n|^p] \leq N E[|y_0|^p] \exp(-\lambda(n\Delta)),$$

for all $n \geq 0$.

(iii). If $p = 2$ in (ii) of Definition 1.4.3, the equilibrium is said to be exponentially stable in mean-square sense.

Stability analysis of numerical methods for ordinary differential equations is motivated by the question “for what choices of the step-size does the numerical method reproduce the characteristics of the test equation?”. It was in this spirit that Mitsui and his coworkers [58, 97, 98] studied the asymptotic stability of numerical methods with respect to linear test of stochastic differential equation

$$dx(t) = \mu x(t)dt + \sigma x(t) dW(t).$$

To explain their results more precisely, let us recall the Euler-Maruyama method applied to the linear stochastic differential equation (1.5.9): given a stepsize $\Delta > 0$, the discrete Euler-Maruyama approximation $y_k \approx x(k\Delta)$ is formed by setting $y_0 = x(0)$ and generally

$$y_{k+1} = y_k(1 + \mu \Delta + \sigma \Delta W_k), \quad k = 0, 1, 2, \ldots,$$

where $\Delta W_k = W((k + 1)\Delta) - W(k\Delta)$. One of the main results of Mitsui et al. is that the Euler-Maruyama approximate solution is exponentially stable in mean square for a sufficiently small stepsize if the true solution of the linear stochastic differential equation (1.5.9) is exponentially stable in mean square (namely, $2\mu + \sigma^2 < 0$); that is, when we choose a step size $\Delta \in (0, h)$, where $h = \min \left\{ \frac{1}{|\mu|}, \frac{2\mu + \sigma^2}{\sigma^2} \right\}$, then the Euler-Maruyama method (1.5.10) is mean-square stable. The same linear test equation is also considered by many authors [33, 39, 49, 47, 99]. This result was
generalized by Higham et al. in [45] to a multi-dimensional non-linear stochastic
differential equation (1.5.6) where they find that under the global Lipschitz condi­
tion on the coefficients f and g, the Euler-Maruyama approximate solution to the
stochastic differential equation (1.5.6) is exponentially stable in mean square for
a sufficiently small stepsize if and only if the true solution of the stochastic differen­
tial equation (1.5.6) is exponentially stable in mean square. Motivated by the
work of [58, 97, 98], many researchers got interested in working in the area of mean­­
square stability of numerical solutions of the linear stochastic delay differential and
integro-differential equations [32, 67, 93, 104, 113].

Tocino and Ardanuy proposed a new weak truncated Itô-Taylor expansion of
functionals of Itô processes in [105, 107]. Using this new expansion, the same au­
thors studied Runge-Kutta methods for numerical solution of stochastic differential
equations in [106]. Tocino and Vigo-Aguiar [108] introduced weak second order
conditions for stochastic Runge-Kutta methods for multi-dimensional stochastic
differential equations. Also Tocino studied mean-square stability of second-order
Runge-Kutta methods for a linear test equation of stochastic differential equations.
At the same time, Robler [86] carried out rooted tree analysis for order condi­
tions of stochastic Runge-Kutta methods for the weak approximation of stochastic
differential equations. Using rooted tree analysis, he introduced second order
Runge-Kutta methods for Ito stochastic differential equations in [87]. Debrabant
and Robler discussed a families of efficient second order Runge-Kutta methods for
the weak approximation of Itô stochastic differential equations in [29] and also an­
alyzed diagonally drift-implicit Runge-Kutta methods of weak order one and two
for Itô stochastic differential equations and stability. Many types of Runge-Kutta
methods for stochastic differential equations have been studied by several authors
in [26, 28, 27, 68].

Komori and Mitsui [59, 58] analyzed numerical mean-square stability for a two-
dimensional stochastic differential equation in a special case, that is. simultane­
ously diagonalizable case. Saito and Mitsui [98] analyzed numerical mean-square
stability for Euler-Maruyama scheme applied to a general two-dimensional stochas­
tic differential system with one multiplicative noise. Abukhaled [1, 2] analyzed the
numerical mean-square stability of second-order Runge-Kutta schemes proposed
in [3] for a two-dimensional stochastic differential system with one multiplicative
noise under some particular cases. Recently Buckwar and Kelly carried out lin­
ear stability analysis of numerical methods for system of stochastic differential
square stability of the split-step backward Euler scheme for stochastic delay integrodifferential equations with Markovian switching is studied by Jiang et al. in [54]. Very recently Ding et al. introduced the split-step $\theta$-method for stochastic differential equations and proved the convergence and obtained stability results in [31]. Numerical solutions by multi-step methods are studied by many authors [15, 16, 50, 102]. Recently, many researchers started constructing numerical methods to various stochastic differential equations and proving their convergence to analytical solutions in [10, 48, 53, 66, 75, 73, 85, 94, 90, 114, 117, 116, 120, 118] and proving stability results of some numerical methods in [4, 17, 80, 88]. Currently, many stochastic models have been studied through numerical methods by many authors in [81, 91, 92, 111].

1.6 Author’s Contributions

To the best of our knowledge, the stability analysis of numerical methods of stochastic differential equations is only of recent origin. In the light of the description of the problem and methodology explained in the preceding sections, the author has obtained some significant results on the following topics:


The rest of the thesis presents the various results established by the author on the above topics.