Chapter 6

Stability of $\theta$-Split-Step Methods for Hybrid Stochastic Delay Integrodifferential Equations

6.1 Motivation

Of late many authors are interested in studying split-step forward and backward Euler methods to stochastic differential and delay differential equations [54, 103, 110, 119]. Here we combine these two methods as a single method. So we use the $\theta$-split-step forward-backward Euler Methods for linear stochastic delay integro-differential equations with Markovian switching. Recently, Ding et. al [31] introduced the split-step $\theta$-methods for stochastic differential equations. In this chapter, we analyse the method proposed by Ding et. al [31] for the linear stochastic delay integro-differential equation with Markovian switching of the form

$$dx(t) = \left[ A(r(t))x(t) + B(r(t))x(t - \tau) + C(r(t)) \int_{t-\tau}^{t} x(s)ds \right] dt$$
$$+ [D(r(t))x(t) + E(r(t))x(t - \tau)] dW(t), \quad t \geq 0, (6.1.1)$$

with the initial data $x(t) = \xi(t) \in \mathcal{C}([-\tau, 0]; \mathbb{R}^d)$, where $r(t)$ is a Markov chain taking values in $\mathcal{S} = \{1, 2, \ldots, N\}$ and $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ and $E(\cdot) \in \mathbb{R}$. $\{W(t)\}$ is a standard one-dimensional Brownian motion and $\tau$ is a positive fixed delay.

In this chapter, we assume the same conditions and study stability of analytical solution of (6.1.1) use in the fourth chapter. The mean-square stability and general mean-square stability of the $\theta$-split-step methods are studied in this chapter.
6.2 The $\theta$-Split-Step Methods

Now we construct the $\theta$-split-step methods for solving a linear stochastic delay integro-differential equation with Markovian switching. We define a mesh with a uniform step size $\Delta$ on the interval $[0, T]$ and $\Delta = T/L$, $t_n = n\Delta$, $n = 0, 1, 2, \ldots, L$. We assume that there is an integer $m$ such that delay can be expressed in terms of the step size $\Delta$ as $\tau = mA$. Let $\bar{y}_n = \psi(n\Delta)$ when $n = -m, -m + 1, \ldots, 0$ and when $n \geq 0$, the $\theta$-split-step forward-backward Euler methods ($\theta$-SSFBE) of the form

$$\bar{y}_n = \bar{y}_n + \left( A \left( r^\Delta_n \right) (1 - \theta) \bar{y}_n + B \left( r^\Delta_n \right) \bar{y}_{n-m} \right) + \Delta C \left( r^\Delta_n \right) \sum_{j=1}^{m} \bar{y}_{n-j} \Delta W_n,$$

$$\bar{y}_{n+1} = \bar{y}_n + \left( D \left( r^\Delta_n \right) \bar{y}_n + E \left( r^\Delta_n \right) \bar{y}_{n-m} \right) \Delta W_n. \quad \text{(6.2.1)}$$

where $0 \leq \theta \leq 1$ and $\bar{y}_n$ is the numerical approximation of $x(t_n)$ with $t_n = n\Delta$. Moreover the increments $\Delta W_n := W(t_{n+1}) - W(t_n)$ are the independent Gaussian random variables with mean 0 and variance $\Delta$. Further we assume that $\bar{y}_n$ is $\mathcal{F}_{t_n}$-measurable at the mesh point $t_n$. To study $\theta$-split-step methods, in this paper, we always assume that $1 - \theta A(r^\Delta_n) \Delta \neq 0$.

Remark 6.2.1. In particular, if $\theta = 0$, then the method coincides with the split-step forward Euler method of Chapter 5 and it is called split-step backward Euler method for $\theta = 1$ of [54].

The convergence of this method is very similar to that of the method in the previous chapter, so we discuss the stability of this method.

6.3 Stability of $\theta$-Split-Step Methods

In this section, we study the mean-square stability and general mean-square stability of the $\theta$-split-step methods.

Theorem 6.3.1. Assume that the conditions of Theorem 4.2.1 are satisfied, for any $i \in S$.

$$A_i + |B_i| + \tau |C_i| + \frac{1}{2} (|D_i| + |E_i|)^2 < 0. \quad \text{(6.3.1)}$$
where $A_i = A(i)$, $B_i = B(i)$, $C_i = C(i)$, $D_i = D(i)$ and $E_i = E(i)$. Then the $\theta$-split-step methods are general mean-square stable if

GMS:(i) \[ L = 0, \text{ for any } i \in S \text{ and} \]
(a) $M \leq 0$ for $\theta = 1$, (b) $M \leq 0$ and $\min_{i \in S} \left\{ \frac{1}{(1-\theta)|A_i|}, \frac{-M + \sqrt{M^2 - 4LN}}{2L} \right\} \geq 1$ for $\theta \neq 1$.

GMS:(ii) \[ L > 0, \text{ for any } i \in S \text{ and} \]
\[ \min_{i \in S} \left\{ \min \left\{ \frac{1}{(1-\theta)|A_i|}, \frac{-M + \sqrt{M^2 - 4LN}}{2L} \right\} \right\} \geq 1. \]

GMS:(iii) \[ L < 0, \text{ for any } i \in S \text{ and} \]
\[ M^2 - 4LN < 0 \text{ for any } i \in S. \]

The $\theta$-split-step method is mean-square stable if

MS:(i) \[ L = 0, \text{ for any } i \in S \text{ and} \]
\[ M > 0 \text{ and } \Delta \in (0, h), \text{ where} \]
\[ h = \min_{i \in S} \left\{ \frac{1}{(1-\theta)|A_i|}, \frac{-M}{M} \right\} < 1. \]

MS:(ii) \[ L > 0, \text{ for any } i \in S \]
\[ \min_{i \in S} \left\{ \min \left\{ \frac{1}{(1-\theta)|A_i|}, \frac{-M + \sqrt{M^2 - 4LN}}{2L} \right\} \right\} < 1 \text{ and } \Delta \in (0, h), \text{ where} \]
\[ h = \min_{i \in S} \left\{ \min \left\{ \frac{1}{(1-\theta)|A_i|}, \frac{-M + \sqrt{M^2 - 4LN}}{2L} \right\} \right\}. \]

MS:(iii) \[ L < 0, \text{ for any } i \in S \]
\[ M^2 - 4LN \geq 0 \text{ for any } i \in S \text{ and } \Delta \in (0, h), \text{ where} \]
\[ h = \min_{i \in S} \left\{ \min \left\{ \frac{1}{(1-\theta)|A_i|}, \frac{-M + \sqrt{M^2 - 4LN}}{2L} \right\} \right\}. \]
Proof. Note that $1 - \theta A_i \Delta \neq 0$ and (6.3.1) implies $A_i < 0$. From (6.2.1), we have

\[
\bar{y}_{n+1} = \frac{1 + D_i \Delta W_n}{1 - \theta A_i (r_n^\Delta) \Delta} \left[ (1 + (1 - \theta)A_i \Delta) \bar{y}_n + B_i \Delta \bar{y}_{n-m} + \Delta^2 C_i \sum_{k=1}^{m} \bar{y}_{n-k} \right] + \Delta^2 C_i (r_n^\Delta) \sum_{j=1}^{k} \bar{y}_{n-k} + E_i (r_n^\Delta) \Delta W_n.
\]  

(6.3.2)

By Lemma 3.2.1, the generation of $r_n^\Delta$ occurs before computing $\bar{y}_{n+1}$; then $r_n^\Delta$ is known. Since $r_n^\Delta \in S$, for any $i \in S$, $A_i = A(i), B_i = B(i), C_i = C(i), D_i = D(i), E_i = E(i)$ and squaring both sides of the above equality, we have

\[
\bar{y}_{n+1}^2 = \frac{(1 + D_i \Delta W_n)^2}{(1 - \theta A_i \Delta)^2} \left[ (1 + (1 - \theta)A_i \Delta) \bar{y}_n + B_i \Delta \bar{y}_{n-m} + \Delta^2 C_i \sum_{k=1}^{m} \bar{y}_{n-k} \right]^2
\]

\[
+ 2(F_i \bar{y}_{n-m} \Delta W_n) + 2(F_i \bar{y}_{n-m} \Delta W_n) \left( \frac{1 + D_i \Delta W_n}{1 - \theta A_i \Delta} \right)
\]

\[
\times \left( (1 + (1 - \theta)A_i \Delta) \bar{y}_n + B_i \Delta \bar{y}_{n-m} + \Delta^2 C_i \sum_{k=1}^{m} \bar{y}_{n-k} \right). \quad (6.3.3)
\]

We apply the following inequalities $2uvxy \leq |uv| (x^2 + y^2)$, where $u, v \in \mathbb{R}, m\Delta = \tau$ and $\sum_{k=1}^{m} y_{n-k} \leq m \max_{1 \leq k \leq m} y_{n-k}$ into (6.3.3) we get

\[
\bar{y}_{n+1}^2 \leq \frac{(1 + D_i^2 (\Delta W_n)^2)}{(1 - \theta A_i \Delta)^2} \left[ (1 + (1 - \theta)A_i \Delta) \bar{y}_n + B_i^2 \bar{y}_{n-m} \Delta^2 + \Delta^2 C_i^2 \max_{1 \leq k \leq m} \bar{y}_{n-k}^2 + \Delta (\bar{y}_n^2 + \bar{y}_{n-m}^2) \right]
\]

\[
+ |1 + (1 - \theta)A_i \Delta| C_i |\Delta \tau \left( \max_{1 \leq k \leq m} \bar{y}_{n-k}^2 + \bar{y}_n^2 \right) + |C_i| B_i |\Delta^2 \tau \left( \max_{1 \leq k \leq m} \bar{y}_{n-k}^2 + \bar{y}_n^2 \right)
\]

\[
+ \left[ \frac{(2D_i \Delta W_n)}{(1 - \theta A_i \Delta)^2} + \frac{2(F_i \bar{y}_{n-m} \Delta W_n)}{(1 - \theta A_i \Delta)} \right]
\]

\[
\times \left( (1 + (1 - \theta)A_i \Delta) \bar{y}_n + B_i \Delta \bar{y}_{n-m} + \Delta^2 C_i \sum_{k=1}^{m} \bar{y}_{n-k} \right)^2
\]

\[
+ \left( \frac{(\Delta W_n)^2}{(1 - \theta A_i \Delta)} \right) \left[ |D_i| |F_i| (1 + (1 - \theta)A_i \Delta) (\bar{y}_n^2 + \bar{y}_{n-m}^2) + 2B_i D_i F_i \Delta \bar{y}_{n-m} \Delta \tau C_i D_i F_i \left( \max_{1 \leq k \leq m} \bar{y}_{n-k}^2 + \bar{y}_n^2 \right) \right]
\]

\[
+ F_i^2 \bar{y}_{n-m} (\Delta W_n)^2. \quad (6.3.4)
\]
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Note that $\mathbb{E}(\Delta W_n) = 0$, $\mathbb{E}[(\Delta W_n)^2] = \Delta$ and $\tilde{y}_n$, $\tilde{y}_{n-k}$ and $\tilde{y}_{n-m}$ are $\mathcal{F}$-measurable; hence

$$
\mathbb{E}(\Delta W_n X_i X_j) = \mathbb{E}[X_i X_j \mathbb{E}(\Delta W_n / \mathcal{F})] = 0,
$$

$$
\mathbb{E}[(\Delta W_n)^2 X_i^2] = \mathbb{E}[X_i^2 \mathbb{E}((\Delta W_n)^2 / \mathcal{F})] = \Delta \mathbb{E}(X_i^2),
$$

$$
i, j \in \{n - m, n - m + 1, ..., n\}. \quad (6.3.5)
$$

Let $Y_n = \mathbb{E}|X_n|^2$ and without loss of generality, we assume that $1 + (1 - \theta) A_i \Delta \geq 0$. Then we have from (6.3.5)

$$
Y_{n+1} \leq P(A_i, ..., \Delta) Y_n + Q(A_i, ..., \Delta) Y_{n-m} + R(A_i, ..., \Delta) \max_{0 \leq k \leq m} Y_{n-k}, \quad (6.3.6)
$$

where

$$
P(A_i, ..., \Delta) = \frac{(1 + D_i^2 (\Delta W_n)^2)}{(1 - \theta A_i \Delta)^2} \left[ (1 + (1 - \theta) A_i \Delta)^2 + |1 + (1 - \theta) A_i \Delta||B_i|\Delta + |1 + (1 - \theta) A_i \Delta||C_i|\Delta\tau \right]
$$

$$
+ \Delta |D_i E_i (1 + (1 - \theta) A_i \Delta)| \frac{(1 - \theta A_i \Delta)}{(1 - \theta A_i \Delta)},
$$

$$
Q(A_i, ..., \Delta) = \frac{(1 + D_i^2 (\Delta W_n)^2)}{(1 - \theta A_i \Delta)^2} \left[ (B_i \Delta)^2 + |1 + (1 - \theta) A_i \Delta||B_i|\Delta + |B_i C_i|\Delta^2 \tau \right]
$$

$$
+ \Delta |D_i E_i (1 + (1 - \theta) A_i \Delta)| + 2B_i D_i E_i \Delta
$$

$$
+ \Delta \tau |C_i D_i E_i| + E_i^2 \Delta,
$$

$$
R(A_i, ..., \Delta) = \frac{(1 + D_i^2 (\Delta W_n)^2)}{(1 - \theta A_i \Delta)^2} \left[ (C_i \tau \Delta)^2 + |1 + (1 - \theta) A_i \Delta||C_i|\tau \Delta + |B_i C_i|\Delta^2 \tau \right]
$$

$$
+ \Delta^2 \tau |C_i D_i E_i| \frac{(1 - \theta A_i \Delta)}{(1 - \theta A_i \Delta)}.
$$

So we have

$$
Y_{n+1} \leq (P(A_i, ..., \Delta) + Q(A_i, ..., \Delta) + R(A_i, ..., \Delta)) \max_{0 \leq k \leq m} \{Y_{n-k}\}.
$$

By recursive calculation, we conclude that $Y_n \to 0$ ($n \to \infty$) if

$$(P(A_i, ..., \Delta) + Q(A_i, ..., \Delta) + R(A_i, ..., \Delta)) < 1.$$
which is equivalent to
\[ \mathcal{L} \Delta^2 + \mathcal{M} \Delta + \mathcal{N} < 0, \]
where
\[
\mathcal{L} = D_t^2((1 - \theta)A_i + |B_i| + |C_i|) + \theta A_i,  \\
\mathcal{M} = ((1 - \theta)A_i + |B_i| + |C_i|) + 2D_t^2((1 - \theta)A_i + |B_i| + |C_i|) \\
+ 2D_t^2((1 - 2\theta)A_i + |B_i| + |C_i|) + 2B_iD_i |B_i| + 2E_i^2 |A_i| - \theta^2 A_i^2, \\
\mathcal{N} = 2(A_i + |B_i| + |C_i|) + (|B_i| + |C_i|)^2.
\]

**Case 1.** If \( \mathcal{L} = 0 \) for any \( i \in \mathcal{S} \), the condition reduces to \( \mathcal{M} \Delta + \mathcal{N} < 0 \).

(i) When \( \mathcal{M} \leq 0 \), we obtain \( \mathcal{M} \Delta + \mathcal{N} < 0 \) because \( \mathcal{N} < 0 \). So the \( \theta \)-split-step method is general mean-square stable if \( \theta = 1 \) and is general mean-square stable for \( \theta \neq 1 \) if \( \min_{i \in \mathcal{S}} \left\{ \frac{1}{(1-\theta)|A_i|} \right\} \geq 1 \). The methods are mean-square stable for \( \theta \neq 1 \) if \( \min_{i \in \mathcal{S}} \left\{ \frac{1}{(1-\theta)|A_i|} \right\} < 1 \) and \( \Delta \in (0, h) \) where \( h = \min_{i \in \mathcal{S}} \left\{ \frac{1}{(1-\theta)|A_i|} \right\} \).

(ii) When \( \mathcal{M} > 0 \), the condition of case 1 is true if \( \Delta < \frac{1}{\mathcal{M}} \) holds for any \( i \in \mathcal{S} \). Therefore the \( \theta \)-split-step methods are mean-square stable if \( \Delta \in (0, h) \) where
\[ h = \min_{i \in \mathcal{S}} \left\{ \min \left\{ \frac{1}{(1-\theta)|A_i|}, \frac{-\mathcal{N}}{\mathcal{M}} \right\} \right\} . \]

**Case 2.** If \( \mathcal{L} > 0 \) for any \( i \in \mathcal{S} \) and since \( \mathcal{N} < 0 \), we have \( \mathcal{M}^2 - 4\mathcal{L}\mathcal{N} > 0 \).

(i) When \( \min_{i \in \mathcal{S}} \left\{ \frac{1}{(1-\theta)|A_i|}, \frac{-\mathcal{M} + \sqrt{\mathcal{M}^2 - 4\mathcal{L}\mathcal{N}}}{2\mathcal{L}} \right\} \geq 1 \), the \( \theta \)-split-step methods are general mean-square stable.

(ii) When \( \min_{i \in \mathcal{S}} \left\{ \frac{1}{(1-\theta)|A_i|}, \frac{-\mathcal{M} + \sqrt{\mathcal{M}^2 - 4\mathcal{L}\mathcal{N}}}{2\mathcal{L}} \right\} < 1 \), the \( \theta \)-split-step methods are mean-square stable if \( \Delta \in (0, h) \) where
\[ h = \min_{i \in \mathcal{S}} \left\{ \min \left\{ \frac{1}{(1-\theta)|A_i|}, \frac{-\mathcal{M} + \sqrt{\mathcal{M}^2 - 4\mathcal{L}\mathcal{N}}}{2\mathcal{L}} \right\} \right\} . \]

**Case 3.** If \( \mathcal{L} < 0 \) for any \( i \in \mathcal{S} \) and since \( \mathcal{N} < 0 \), we have

(i) when \( \mathcal{M}^2 - 4\mathcal{L}\mathcal{N} < 0 \) for any \( i \in \mathcal{S} \), we can easily see that the \( \theta \)-split-step methods are general mean-square stable.

(ii) when \( \mathcal{M}^2 - 4\mathcal{L}\mathcal{N} \geq 0 \) for any \( i \in \mathcal{S} \), we can easily see that the \( \theta \)-split-step methods are mean-square stable in \((0,h)\) where
\[ h = \min_{i \in \mathcal{S}} \left\{ \min \left\{ \frac{1}{(1-\theta)|A_i|}, \frac{-\mathcal{M} + \sqrt{\mathcal{M}^2 - 4\mathcal{L}\mathcal{N}}}{2\mathcal{L}} \right\} \right\} . \]

The proof is completed.
6.4 Numerical Experiments

We consider the same test equation discussed in the chapter 4. In that equation, we consider the following three different cases in which we investigate the mean-square stability of $\theta$-split-step method with $\theta$ values zero and one.

Case 1. Let $A(1) = -2, B(1) = 1, C(1) = \frac{1}{2}, D(1) = 0, E(1) = \frac{2}{5}$ and $A(2) = -9, B(2) = 4, C(2) = 3, D(2) = 0, E(2) = \frac{3}{5}$. By Theorem 6.4.1, we know that the $\theta$-split-step methods are general mean-square stable if $\theta = 1$. It is mean-square stable for $\theta = 0$ with the stepsizes $0 < \Delta < 1/9$. For $\theta = 0.5$, the method is mean-square stable with the stepsizes $0 < \Delta < 2/9$. Fig. 6.1 illustrates general mean-square stability of the $\theta$-split-step methods for $\theta = 1$ and mean-square stability of the $\theta$-split-step methods for $\theta = 0$ and $\theta = 0.5$.

Case 2. Let $A(1) = -2, B(1) = 1, C(1) = 0, D(1) = 0, E(1) = \frac{1}{2}$ and $A(2) = -9, B(2) = 7, C(2) = 0, D(2) = 0, E(2) = 1$. By Theorem 6.4.1, $\theta$-split-step methods for $\theta = 1$ is mean-square stable when stepsize $\Delta \in (0, 0.297)$ and is mean-square stable for $\theta = 0$ when stepsize $\Delta \in (0, 0.111)$. Fig. 6.2 illustrates the mean-square stability of $\theta$-split-step methods for $\theta = 1$ and $\theta = 0$.

Case 3. $A(1) = -2, B(1) = 1, C(1) = 0.5, E(1) = 0.4$ and $A(2) = 0.5, B(2) = 0.2, C(2) = 0.1, E(2) = 0.8$. For these values, neither condition of Theorem 3.2.1 nor condition of Theorem 6.4.1 is satisfied. To carry out the numerical simulation we choose the step size $\Delta = \frac{1}{100}$. The computer simulation result is shown in Fig.6.3. Clearly the $\theta$-split-step methods reveal the unstable property of the solution.
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Theta-SSFBE method with $\theta=0$ and stepsize $=1/10$

Theta-SSFBE method with $\theta=1$ and stepsize $=1/4$

Theta-SSFBE method with $\theta=0.5$ and stepsize $=1/6$

Figure 6.1: Numerical simulation of case.1.
Theta-SSFBE method with theta=0 and stepsize =1/10
Theta-SSFBE method with theta=1 and stepsize =1/10

Figure 6.2: Numerical simulation of case.2.
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Theta-SSFBE method with $\theta=0$ and stepsize $=1/100$
Theta-SSFBE method with $\theta=1$ and stepsize $=1/100$

Figure 6.3: Numerical simulation of case 3.