Chapter 5

Stability of Split-Step Forward Euler Method to Stochastic Delay Integrodifferential Equations

5.1 Introduction

In recent years, many researchers are interested to study the numerical solutions of stochastic differential equations by split-step methods [31, 54, 103, 110, 119]. Recently, Wang and Li [110] analysed split-step forward method for stochastic differential equations. In this chapter, we study the split-step forward Euler method for the linear stochastic delay integrodifferential equations and analyse its stability properties. Consider the following linear stochastic delay integrodifferential equation of the form

$$\begin{align*}
\frac{dx(t)}{dt} &= \left[ax(t) + bx(t - \tau) + c \int_{t-\tau}^{t} x(t) \, dt \right]dt \\
&\quad + \left[ax(t) + \beta x(t - \tau)\right]dW(t), \quad t \geq 0, \\
x(t) &= \psi(t), \quad t \in [-\tau, 0].
\end{align*}$$

(5.1.1)

where $a, b, c, \alpha, \beta \in \mathbb{R}$, $\tau$ is a positive fixed delay, $W(t)$ is an one-dimensional standard Wiener process and $\psi(t)$ is a $C([-\tau, 0]; \mathbb{R})$-valued initial segment.

In section 5.2, we discuss the exponential stability in mean square of the analytic solution for (5.1.1). In section 5.3, the split-step forward Euler methods are used to obtain the numerical solutions. Furthermore, the mean-square stability of split-step forward Euler methods are proved in the section 5.4 and a numerical example is given in section 5.5.
5.2 Stability of Analytical Solutions

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space with a filtration \((\mathcal{F}_t)_{t \geq 0}\), which satisfies the usual conditions (that is, it is increasing and right continuous and each \(\mathcal{F}_t, t \in [0, T]\) contains all \(P\)-null sets in \(\mathcal{F}\)). Let \(W(t), t \geq 0\), be \(\mathcal{F}_t\)-adapted and independent of \(\mathcal{F}_0\). Moreover, we assume \(\psi(t), t \in [-\tau, 0]\) to be \(\mathcal{F}_0\)-measurable and right continuous with norm \(\|\psi\| = \sup_{-\tau \leq t \leq 0} |\psi(t)|\) and \(E \|\psi\|^2 < \infty\), where \(E\) is the expectation with respect to \(P\). Under the above assumptions, (5.1.1) has a unique strong solution \(x(t) : [-\tau, +\infty) \rightarrow \mathbb{R}\) which is a measurable, sample-continuous and \(\mathcal{F}_t\)-adapted process.

Lemma 5.2.1. [69] If

\[
a + |b| + |c|\tau + \frac{1}{2}(|\alpha| + |\beta|)^2 < 0.
\]

the solution of (5.1.1) is asymptotically stable in the mean-square.

Lemma 5.2.2. [110] For any given \(0 < T < \infty\), there exist positive numbers \(C_1\) and \(C_2\) such that the solution of (5.1.1) satisfies

\[
E \left( \sup_{-\tau \leq s \leq t} |x(s)|^2 \right) \leq C_1 [1 + E \|\psi\|^2],
\]

for all \(t \in [-\tau, T]\),

\[
E |x(t) - x(s)|^2 \leq C_2 (t - s),
\]

for any \(0 \leq s < t \leq T, t - s < 1\).

5.3 The Split-Step Forward Euler Methods

Now we construct the split-step forward Euler schemes for solving a linear stochastic delay integrodifferential equation. We define a mesh with a uniform step size \(\Delta\) on the interval \([0, T]\) and \(\Delta = T / N, t_n = nD, n = 0, 1, 2, ...,\). We assume that there is an integer \(m\) such that delay can be expressed in terms of the step size \(\Delta\) as \(\tau = mn\). We construct the two fully explicit methods for solving (5.1.1) by \(\tilde{y}_n = \psi(n\Delta)\) when \(n = -m, -m + 1, ..., 0\) and when \(n \geq 0\). the drifting split-step forward Euler method (DRSFE) is
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\[ \tilde{y}_n^* = \tilde{y}_n + \left( a\tilde{y}_n + b\tilde{y}_{n-m} + \Delta c \sum_{j=1}^{k} \tilde{y}_{n-k} \right) \Delta, \]

\[ \tilde{y}_{n+1} = \tilde{y}_n^* + (\alpha \tilde{y}_n + \beta \tilde{y}_{n-m}) \Delta W_n \]  \hspace{1cm} (5.3.1)

and the diffused split-step forward Euler method (DISSFE) is

\[ \tilde{z}_n^* = \tilde{z}_n + (\alpha \tilde{z}_n + \beta \tilde{z}_{n-m}) \Delta W_n, \]

\[ \tilde{z}_{n+1} = \tilde{z}_n^* + \left( a\tilde{z}_n + b\tilde{z}_{n-m} + \Delta c \sum_{j=1}^{k} \tilde{z}_{n-k} \right) \Delta, \]  \hspace{1cm} (5.3.2)

where \( \tilde{y}_n \) and \( \tilde{z}_n \) are the numerical approximations of \( x(t_n) \) with \( t_n = n \Delta \). Moreover the increments \( \Delta W_n := W(t_{n+1}) - W(t_n) \) are independent Gaussian random variables with mean 0 and variance \( \Delta \). Further we assume that \( \tilde{y}_n \) and \( \tilde{z}_n \) are \( \mathcal{F}_{t_n} \)-measurable at the mesh point \( t_n \).

By (5.3.1), we have

\[ \tilde{y}_{n+1} = \tilde{y}_n + \left( a\tilde{y}_n + b\tilde{y}_{n-m} + \Delta c \sum_{j=1}^{m} \tilde{y}_{n-k} \right) \Delta \]

\[ + \left( \alpha(1 + \alpha \Delta)\tilde{y}_n + (b\alpha \Delta + \beta)\tilde{y}_{n-m} + \Delta^2 c a \sum_{j=1}^{m} \tilde{y}_{n-k} \right) \Delta W_n, \]

\[ = (1 + \alpha \Delta W_n) \left( 1 + a\Delta \right) \tilde{y}_n + b\tilde{y}_{n-m} + \Delta c \sum_{j=1}^{m} \tilde{y}_{n-k} \Delta \]

\[ + \beta \tilde{y}_{n-m} \Delta W_n. \]  \hspace{1cm} (5.3.3)

Similarly, from (5.3.2), we have

\[ \tilde{z}_{n+1} = (1 + a\Delta)(\alpha \tilde{z}_n + \beta \tilde{z}_{n-m}) \Delta W_n \]

\[ + \left( \tilde{z}_n + a\Delta \tilde{z}_n + b\Delta \tilde{z}_{n-m} + \Delta^2 c \sum_{j=1}^{m} \tilde{z}_{n-k} \right). \]  \hspace{1cm} (5.3.4)

**Definition 5.3.1.** The local error of DRSSFE method (5.3.1) for the approximation of the solution \( x(t) \) of (5.1.1), for \( n = 0, 1, 2, \ldots, N - 1 \), is defined as

\[ \delta_{n+1} = x(t_{n+1}) - \left\{ x(t_n) + \left( a x(t_n) + b x(t_{n-m}) + \Delta c \sum_{j=1}^{m} x(t_{n-k}) \right) \Delta \right. \]

\[ + \left( \alpha(1 + a \Delta) x(t_n) + (b \alpha \Delta + \beta) x(t_{n-m}) \right) \Delta^2 c a \sum_{k=1}^{m} x(t_{n-k}) \Delta W_n \right\}, \]  \hspace{1cm} (5.3.5)
where \( x(t_n) \) denotes the value of the exact solution of (5.1.1) at the mesh-point \( t_n \).

**Definition 5.3.2.** The global error of DRSSFE method (5.3.1) for the approximation of the solution \( x(t) \) of (5.1.1), for \( n = 1, 2, \ldots, N \) is defined as

\[
\epsilon_n = x(t_n) - \bar{y}_n. \quad (5.3.6)
\]

Note that \( \epsilon_n \) is \( \mathcal{F}_{t_n} \)-measurable since \( x(t_n), \bar{y}_n \) and \( \bar{z}_n \) are \( \mathcal{F}_{t_n} \)-measurable random variables.

**Definition 5.3.3.** The local error of DISSFE method (5.3.2) for the approximation of the solution \( x(t) \) of (5.1.1), for \( n = 0, 1, 2, \ldots, N-1 \), is defined as

\[
\tilde{\epsilon}_{n+1} = x(t_{n+1}) - \{(1 + a\Delta)(ax(t_n) + \beta x(t_{n-m})\Delta W_n \\
+ (x(t_n) + a\Delta x(t_n) + b\Delta x(t_{n-m}) \\
+ \Delta^2 c \sum_{k=1}^m x(t_{n-k})) \}, \quad (5.3.7)
\]

where \( x(t_n) \) denotes the value of the exact solution of (5.1.1) at the mesh-point \( t_n \).

**Definition 5.3.4.** The global error of DISSFE method (5.3.2) for the approximation of the solution \( x(t) \) of (5.1.1), for \( n = 1, 2, \ldots, N \) is defined as

\[
\tilde{\epsilon}_n = x(t_n) - \bar{z}_n. \quad (5.3.8)
\]

**Theorem 5.3.1.** The numerical solution produced by the DRSSFE method (5.3.1) to approximate the solution of (5.1.1) satisfies

\[
\max_{0 \leq n \leq N-1} |E(\delta_{n+1})| \leq C_3 \Delta^{1/2}, \quad \text{as} \quad \Delta \to 0. \quad (5.3.9)
\]

and

\[
\max_{0 \leq n \leq N-1} \left( E(\delta_{n+1})^2 \right)^{1/2} \leq C_4 \Delta, \quad \text{as} \quad \Delta \to 0. \quad (5.3.10)
\]

where \( C_3 \) and \( C_4 \) are positive constants that are independent of \( \Delta \), but may depend on \( T \) and the initial segment \( \psi \) of (5.1.1).

**Proof.** When \( 0 \leq s \leq t \leq T \), we have

\[
x(t) - x(s) = \int_s^t \left( ax(u) + br(u - \tau) + c \int_{\nu - \tau}^u x(s) ds \right) du \\
+ \int_s^t \left( ax(u) + 3x(u - \tau) \right) dW(u). \quad (5.3.11)
\]
First, we prove the inequality (5.3.9). By (5.1.1), (5.3.3) and (5.3.11), we can derive that

\[
\delta_{n+1} = x(t_{n+1}) - \left\{ x(t_n) + \left( ax(t_n) + bx(t_{n-m}) + \Delta c \sum_{k=1}^{m} x(t_{n-k}) \right) \Delta \\
+ \left( \alpha (1 + a\Delta) x(t_n) + (b\alpha \Delta + \beta) x(t_{n-m}) \right) \right\} \\
+ \Delta^2 c\alpha \sum_{k=1}^{m} x(t_{n-k}) \Delta W_n,
\]

\[
= x(t_n) + \int_{t_n}^{t_{n+1}} \left( ax(u) + bx(u - \tau) + c \int_{u-\tau}^{u} x(s) ds \right) du \\
+ \int_{t_n}^{t_{n+1}} (\alpha x(u) + \beta x(u - \tau)) dW(u) \\
- \left\{ x(t_n) + \left( ax(t_n) + bx(t_{n-m}) + \Delta c \sum_{k=1}^{m} x(t_{n-k}) \right) \Delta \\
+ \left( \alpha (1 + a\Delta) x(t_n) + (b\alpha \Delta + \beta) x(t_{n-m}) \right) \right\} \\
+ \Delta^2 c\alpha \sum_{k=1}^{m} x(t_{n-k}) \Delta W_n,
\]

\[
= \int_{t_n}^{t_{n+1}} \left[ a(x(u) - x(t_n)) + b(x(u - \tau) - x(t_{n-m})) \\
+ c \left( \int_{u-\tau}^{u} x(s) ds - \Delta \sum_{k=1}^{m} x(t_{n-k}) \right) \right] du \\
+ \int_{t_n}^{t_{n+1}} [\alpha (x(u) - x(t_n)) - a\alpha \Delta x(t_n)] dW(u) \\
+ \int_{t_n}^{t_{n+1}} [\beta (x(u - \tau) - x(t_{n-m})) - b\alpha \Delta x(t_{n-m})] dW(u) \\
- \Delta^2 c\alpha \sum_{k=1}^{m} x(t_{n-k}) \Delta W_n. \tag{5.3.12}
\]

Taking mathematical expectation and using properties of the Itô integral, we ob-
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\[ |\mathbb{E}(\delta_{n+1})| \leq \int_{t_n}^{t_{n+1}} \left[ |a|\mathbb{E}|x(t) - x(t_n)|dt + |b|\mathbb{E}|x(t) - x(t_{n-m})| \right. \\
+ \left. |c| \left( \sum_{i=n-m}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}|x(s) - x(t_i)|ds \right) \right] dt. \]

It follows from (5.2.3) and \( m\Delta = \tau \) that

\[ |\mathbb{E}(\delta_{n+1})| \leq |a|C_2 \int_{t_n}^{t_{n+1}} (t - t_n)^{\frac{3}{2}} dt + |b|C_2 \int_{t_n}^{t_{n+1}} (t - t_n)^{\frac{3}{2}} dt \]
\[ + |c|C_2 \int_{t_n}^{t_{n+1}} \left( \sum_{i=n-m}^{n-1} \int_{t_i}^{t_{i+1}} (s - t_i)^{\frac{1}{2}} ds \right) dt \]
\[ = \frac{2}{3} |a|C_2 \Delta^{\frac{3}{2}} + \frac{2}{3} |b|C_2 \Delta^{\frac{3}{2}} + \frac{2}{3} |c| \tau C_2 \Delta^{\frac{3}{2}} = C_3 \Delta^{\frac{3}{2}}. \]

Now we have to prove (5.3.10), when \( t_k \leq t \leq t_{k+1} \). Let

\[ \xi(t) = a(x(t) - x(t_n)) + b(x(t) - t) - x(t_{n-m})) \\
+ c \left( \int_{t-\tau}^{t} x(s) ds - \Delta \sum_{k=1}^{m} x(t_{n-k}) \right) \]

and

\[ \eta(t) = a(x(t) - x(t_n)) + b(x(t) - \tau) - x(t_{n-m})) - \Delta a \alpha x(t_n) \\
- \Delta b \alpha x(t_{n-m}) - \Delta^2 \sigma \sum_{k=1}^{m} x(t_{n-k}). \]

It follows that \( \delta_{n+1} = \int_{t_n}^{t_{n+1}} \xi(t) dt + \int_{t_n}^{t_{n+1}} \eta(t) dW(t) \) and applying the inequality \(|a + b|^2 \leq 2(|a|^2 + |b|^2)\), we have

\[ \mathbb{E}|\delta_{n+1}|^2 = 2 \left[ \mathbb{E} \left| \int_{t_n}^{t_{n+1}} \xi(t) dt \right|^2 + \mathbb{E} \left| \int_{t_n}^{t_{n+1}} \eta(t) dW(t) \right|^2 \right]. \]

Now, using the Cauchy-Schwartz inequality and Itô isometry, we get

\[ \mathbb{E}|\delta_{n+1}|^2 \leq 2(t_{n+1} - t_n) \mathbb{E} \int_{t_n}^{t_{n+1}} |\xi(t)|^2 dt + 2 \int_{t_n}^{t_{n+1}} \mathbb{E}|\eta(t)|^2 dt \]
\[ = 2 \Delta \int_{t_n}^{t_{n+1}} \mathbb{E}|\xi(t)|^2 dt + 2 \int_{t_n}^{t_{n+1}} \mathbb{E}|\eta(t)|^2 dt. \quad (5.3.13) \]
Making use of the elementary inequality \((\sum_{i=1}^n x_i)^2 \leq n \sum_{i=1}^n x_i^2, x_i \in \mathbb{R}\), we obtain

\[
|\xi(t)|^2 \leq 3a^2 |x(t) - x(t_n)|^2 + 3b^2 |x(t - \tau) - x(t_n-m)|^2 + 3c^2 m^2 \Delta^2 \left( \sum_{i=n-m}^{n-1} \sum_{\tau=0}^{t_i} |x(s) - x(t_i)|^2 ds \right)
\]

and

\[
|\eta(t)|^2 \leq 5a^2 |x(t) - x(t_n)|^2 + 5b^2 |x(t - \tau) - x(t_n-m)|^2 + 5a^2 \alpha^2 \Delta^2 |x(t_n)|^2 + 5b^2 \alpha^2 \Delta^2 |x(t_n-m)|^2 + 5c^2 \alpha^2 \Delta^2 m \sum_{k=1}^{m} |x(t_n-k)|^2.
\]

It follows from (5.2.2) and (5.2.3) that

\[
\mathbb{E} |\xi(t)|^2 \leq 3a^2 C_2(t - t_n) + 3b^2 C_2(t - t_n) + \frac{3}{2} c^2 \tau^2 C_2 \Delta
\]

and

\[
\mathbb{E} |\eta(t)|^2 \leq 5a^2 C_2(t - t_n) + 5b^2 C_2(t - t_n) + 5a^2 \alpha^2 \Delta^2 C_1(1 + \mathbb{E} \|\psi\|^2) + 5b^2 \alpha^2 \Delta^2 C_1(1 + \mathbb{E} \|\psi\|^2).
\]

Setting \(D = \max \{a^2, b^2, c^2 \tau^2, \alpha^2, \beta^2\}\), we estimate

\[
\mathbb{E} |\xi(t)|^2 \leq \frac{15}{2} D C_2 \Delta.
\]  \hspace{1cm} (5.3.14)

and

\[
\mathbb{E} |\eta(t)|^2 \leq \left[10 D C_2 + 15 \Delta D^2 C_1(1 + \mathbb{E} \|\psi\|^2)\right] \Delta.
\]  \hspace{1cm} (5.3.15)

Substituting (5.3.14) and (5.3.15) into (5.3.13), we have

\[
\mathbb{E} |\delta_{n+1}|^2 \leq 15 D C_2 \Delta^2 + 2 \left[10 D C_2 + 15 \Delta D^2 C_1(1 + \mathbb{E} \|\psi\|^2)\right] \Delta^2
\]

\[
= 2 \left[15 \frac{D C_2 \Delta + 10 D C_2 + 15 \Delta D^2 C_1(1 + \mathbb{E} \|\psi\|^2)}{2}\right] \Delta^2
\]

\[
= 2 \left[15 \frac{D C_2 \Delta + 10 D C_2 + 15 \Delta D^2 C_1(1 + \mathbb{E} \|\psi\|^2)}{2}\right] \Delta^2
\]

\[
\leq \left[35 D C_2 + 30 D^2 C_1(1 + \mathbb{E} \|\psi\|^2)\right] \Delta^2.
\]

Then the inequality follows by letting

\[
C_1' = \left[35 D C_2 + 30 D^2 C_1(1 + \mathbb{E} \|\psi\|^2)\right]^{\frac{1}{2}}
\]

The proof is completed. \(\square\)
Similarly we can prove the following theorem for the DISSFE method.

**Theorem 5.3.2.** The numerical solution produced by the split-step forward Euler method (5.3.1) to approximate the solution of (5.1.1) satisfies

\[
\max_{0 \leq n \leq N-1} |E(\tilde{\delta}_{n+1})| \leq \bar{C}_3 \Delta^{3/2}, \quad \text{as} \quad \Delta \to 0, \quad (5.3.16)
\]

and

\[
\max_{0 \leq n \leq N-1} \left( E(\tilde{\delta}_{n+1})^2 \right)^{1/2} \leq \bar{C}_4 \Delta, \quad \text{as} \quad \Delta \to 0, \quad (5.3.17)
\]

where \( \bar{C}_3 \) and \( \bar{C}_4 \) are positive constants that are independent of \( \Delta \), but may depend on \( T \) and the initial segment \( \psi \) of (5.1.1).

The following theorem shows the strong order of convergence of DRSSFE method.

**Theorem 5.3.3.** The numerical solution produced by the DRSSFE method (5.3.1) converges to the exact solution of (5.1.1) on the mesh-points in the mean-square sense with strong order \( \gamma = \frac{1}{2} \), that is, there exists a positive constant \( C_0 \) such that

\[
\max_{0 \leq n \leq N} \left( E(\epsilon_n)^2 \right)^{1/2} \leq C_0 \Delta^{3/2}, \quad \text{as} \quad \Delta \to 0.
\]

**Proof.** Since we have the exact initial values, we set \( \epsilon_n = 0 \) for \( n = -m, -(m - 1), \ldots, 0 \). Now, with \( \epsilon_n = x(t_n) - \tilde{y}_n \), \( \epsilon_n \) is \( \mathcal{F}_{t_n} \)-measurable since both \( x(t_n) \) and \( \tilde{y}_n \) are \( \mathcal{F}_{t_n} \)-measurable random variables. From (5.3.3), (5.3.5) and (5.3.6), we have

\[
\epsilon_{n+1} = x(t_{n+1}) - \tilde{y}_{n+1}
\]

\[
= x(t_{n+1}) - \left\{ y_n + \left( a\tilde{y}_n + b\tilde{y}_{n-m} + \Delta c \sum_{k=1}^{m} \tilde{y}_{n-k} \right) \Delta + \left( \alpha(1 + a\Delta)\tilde{y}_n + (b\alpha \Delta + \beta)\tilde{y}_{n-m} + \Delta^2 c\alpha \sum_{k=1}^{m} \tilde{y}_{n-k} \right) \Delta W_n \right\}
\]

\[
= \epsilon_n + \delta_{n+1} + u_n, \quad (5.3.18)
\]

where

\[
u_n = \left( a(x(t_n) - \tilde{y}_n) + b(x(t_{n-m}) - \tilde{y}_{n-m}) + \Delta c \sum_{k=1}^{m} (x(t_{n-k}) - \tilde{y}_{n-k}) \right) \Delta + \left( \alpha(1 + a\Delta)(x(t_n) - \tilde{y}_n) + (b\alpha \Delta + \beta)(x(t_{n-m}) - \tilde{y}_{n-m}) \right)
\]

\[
+ \Delta^2 c\alpha \sum_{k=1}^{m} (x(t_{n-k}) - \tilde{y}_{n-k}) \Delta W_n. \quad (5.3.19)
\]
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Squaring both sides and applying the conditional mean with respect to the \( \sigma \) algebra \( \mathcal{F}_0 \) and taking absolute values, we obtain

\[
\mathbb{E}(\epsilon_n^2 | \mathcal{F}_0) \leq \mathbb{E}(\epsilon_n^2 | \mathcal{F}_0) + \mathbb{E}(\sigma_n^2 | \mathcal{F}_0) + \mathbb{E}(u_n^2 | \mathcal{F}_0) + 2|\mathbb{E}(u_n\delta_{n+1} | \mathcal{F}_0)| + 2|\mathbb{E}(u_n u_{n+1} | \mathcal{F}_0)|. \tag{5.3.20}
\]

Clearly the Brownian increments satisfy \( \mathbb{E}(\Delta W_n) = 0 \), \( \mathbb{E}(\Delta W_n)^2 = \Delta \); hence taking conditional expectation with respect to the \( \sigma \) algebra \( \mathcal{F}_0 \) and using the independence of the increment, we find that

\[
|\mathbb{E}(u_n | \mathcal{F}_n)| \leq \left[ |a|\mathbb{E}(|\epsilon_n| | \mathcal{F}_n) + |b|\mathbb{E}(|\epsilon_{n-m}| | \mathcal{F}_n) + |c|\Delta \sum_{n=k}^m \mathbb{E}(|\epsilon_{n-k}| | \mathcal{F}_n) \right] \Delta
\
\leq K' \Delta \left[ \mathbb{E}(|\epsilon_n| | \mathcal{F}_n) + \mathbb{E}(|\epsilon_{n-m}| | \mathcal{F}_n) + \max_{1 \leq k \leq m} \mathbb{E}(|\epsilon_{n-k}| | \mathcal{F}_n) \right] \leq K' \left( \mathbb{E}(|\epsilon_n|) + \mathbb{E}(|\epsilon_{n-m}|) + \max_{1 \leq k \leq m} \mathbb{E}(|\epsilon_{n-k}|) \right) \Delta, \tag{5.3.21}
\]

where \( K' = \max \{|a|, |b|, |c|\} \). Squaring both sides of (5.3.19) and applying the inequality \( \left( \sum_{i=1}^n x_i \right)^2 \leq n \sum_{i=1}^n x_i^2 \), where \( x_i \in \mathbb{R} \), we have

\[
u_n^2 \leq 2 \left( a(x(t_n) - \bar{y}_n) + b(x(t_{n-m}) - \bar{y}_{n-m}) + \Delta c \sum_{k=1}^m (x(t_{n-k}) - \bar{y}_{n-k}) \right)^2 \Delta^2
\]
\[
+2 \left( a(1 + a\Delta)(x(t_n) - \bar{y}_n) + (ba\Delta + \beta)(x(t_{n-m}) - \bar{y}_{n-m}) + \Delta^2 c \sum_{k=1}^m (x(t_{n-k}) - \bar{y}_{n-k}) \right)^2 \langle \Delta W_n \rangle^2
\]
\[
\leq 6 \left( a^2(x(t_n) - \bar{y}_n)^2 + b^2(x(t_{n-m}) - \bar{y}_{n-m})^2 + \Delta^2 m \sum_{k=1}^m (x(t_{n-k}) - \bar{y}_{n-k})^2 \right) \Delta^2
\]
\[
+6 \left( a^2(1 + a\Delta)^2(x(t_n) - \bar{y}_n)^2 + (ba\Delta + \beta)^2(x(t_{n-m}) - \bar{y}_{n-m})^2 + \Delta^4 a^2 m \sum_{k=1}^m (x(t_{n-k}) - \bar{y}_{n-k})^2 \right) \langle \Delta W_n \rangle^2
\]
\[
\leq 6 \left[ a^2 \Delta_n^2 + b^2 \Delta_{n-m}^2 + c^2 \tau^2 \max_{1 \leq k \leq m} \epsilon_{n-k}^2 \right] \Delta^2 \\
+6 \left[ \alpha^2 (1 + a\Delta)^2 \epsilon_n^2 + (b\alpha + \beta)^2 \Delta^2 \epsilon_{n-m}^2 + c^2 \alpha^2 \tau^2 \Delta^2 \max_{1 \leq k \leq m} \epsilon_{n-k}^2 \right] \Delta.
\]

So, taking conditional expectation with respect to the \( \sigma \)-algebra \( \mathcal{F}_0 \), we have

\[
\mathbb{E}(u_n^2 | \mathcal{F}_0) \\
\leq 6 \left[ a^2 \mathbb{E}(\epsilon_n^2 | \mathcal{F}_0) + b^2 \mathbb{E}(\epsilon_{n-m}^2 | \mathcal{F}_0) + c^2 \tau^2 \max_{1 \leq k \leq m} \mathbb{E}(\epsilon_{n-k}^2 | \mathcal{F}_0) \right] \Delta^2 \\
+6 \left[ \alpha^2 (1 + a\Delta)^2 \mathbb{E}(\epsilon_n^2 | \mathcal{F}_0) + (b\alpha + \beta)^2 \Delta^2 \mathbb{E}(\epsilon_{n-m}^2 | \mathcal{F}_0) \\
+ c^2 \alpha^2 \tau^2 \Delta^2 \max_{1 \leq k \leq m} \mathbb{E}(\epsilon_{n-k}^2 | \mathcal{F}_0) \right] \Delta.
\]

(5.3.22)

By Theorem 5.3.1, we have

\[
\mathbb{E}(\delta_{n+1}^2 | \mathcal{F}_0) \leq \mathbb{E} \left[ \mathbb{E}(\delta_{n+1}^2 | \mathcal{F}_n) | \mathcal{F}_0 \right] \leq (C_4 \Delta)^2.
\]

(5.3.23)

Using the Holder inequality, the properties of conditional expectation and inequalities (5.3.16), (5.3.21)-(5.3.23), we have

\[
2 | \mathbb{E}[u_n \delta_{n+1} | \mathcal{F}_0] | \leq 2 \mathbb{E}[\delta_{n+1}^2 | \mathcal{F}_0]^{1/2} \mathbb{E}[u_n^2 | \mathcal{F}_0]^{1/2} \\
\leq \mathbb{E}[\delta_{n+1}^2 | \mathcal{F}_0] + \mathbb{E}[u_n^2 | \mathcal{F}_0] \\
\leq (C_4 \Delta)^2 + \mathbb{E}[u_n^2 | \mathcal{F}_0].
\]

(5.3.24)

\[
2 | \mathbb{E}[\epsilon_n \delta_{n+1} | \mathcal{F}_0] | \leq 2 \mathbb{E}[\mathbb{E}[\delta_{n+1} | \mathcal{F}_n] | \epsilon_n] | \mathcal{F}_0] \\
\leq 2 \mathbb{E}[\mathbb{E}[\delta_{n+1}^2 | \mathcal{F}_n]^2 | \mathcal{F}_0]^{1/2} (\mathbb{E}[\epsilon_n^2 | \mathcal{F}_n])^{1/2} \\
\leq 2 \mathbb{E}[C_4^2 \Delta^2]^{1/2} (\mathbb{E}[\epsilon_n^2 | \mathcal{F}_n])^{1/2} \\
= 2 \mathbb{E}[C_4^2 \Delta^2]^{1/2} \Delta \mathbb{E}[\epsilon_n^2 | \mathcal{F}_0]]^{1/2} \\
\leq C_3^2 \Delta^2 + \Delta \mathbb{E}[\epsilon_n^2 | \mathcal{F}_0].
\]

(5.3.25)

\[
2 | \mathbb{E}[\epsilon_n u_n | \mathcal{F}_0] | \\
\leq 2 \mathbb{E}[|\mathbb{E}[u_n | \mathcal{F}_n] | \epsilon_n] | \mathcal{F}_0] \\
\leq 2 \mathbb{C}(\mathbb{E}[\epsilon_n^2 | \mathcal{F}_n] + \mathbb{E}[\epsilon_{n-m}] + \max_{1 \leq k \leq m} \mathbb{E}[\epsilon_{n-k}] | \mathcal{F}_n] | \mathcal{F}_0] | \Delta \\
\leq 2 \mathbb{C} \mathbb{E}[\epsilon_n^2 | \mathcal{F}_n] + \mathbb{E}[\epsilon_{n-m}] | \mathcal{F}_n] + \max_{1 \leq k \leq m} \mathbb{E}[\epsilon_{n-k} | \mathcal{F}_n] | \Delta \\
\leq 4 C \Delta \mathbb{E}[\epsilon_n^2 | \mathcal{F}_n] + C \Delta \mathbb{E}[\epsilon_{n-m}^2 | \mathcal{F}_n] + \max_{1 \leq k \leq m} \mathbb{E}[\epsilon_{n-k}^2 | \mathcal{F}_n]].
\]

(5.3.26)
Taking the above equations (5.3.22)-(5.3.26) into (5.3.20), we have

\[
\mathbb{E}[\epsilon_{n+1}^2 | \mathcal{F}_t] \\
\leq \mathbb{E}[\epsilon_{n}^2 | \mathcal{F}_t] + 2(C_4 \Delta)^2 + (C_3 \Delta)^2 + \Delta \mathbb{E}[\epsilon_{n}^2 | \mathcal{F}_t] \\
+ 12 \left[ a^2 \mathbb{E}(\epsilon_{n}^2 | \mathcal{F}_t) + b^2 \mathbb{E}(\epsilon_{n-m}^2 | \mathcal{F}_t) + c^2 \tau^2 \max_{1 \leq k \leq m} \mathbb{E}(\epsilon_{n-k}^2 | \mathcal{F}_t) \right] \Delta^2 \\
+ 12 \left[ a^2 (1 + a \Delta)^2 \mathbb{E}(\epsilon_{n}^2 | \mathcal{F}_t) + (b \Delta + \beta)^2 \Delta^2 \mathbb{E}(\epsilon_{n-m}^2 | \mathcal{F}_t) \right] \\
+ c^2 \alpha^2 \tau^2 \Delta^2 \max_{1 \leq k \leq m} \mathbb{E}(\epsilon_{n-m}^2 | \mathcal{F}_t) \Delta + 4c \Delta \mathbb{E}[\epsilon_{n}^2 | \mathcal{F}_t] \\
+ c \Delta \left[ \mathbb{E}[\epsilon_{n-m}^2 | \mathcal{F}_t] + \max_{1 \leq k \leq m} \mathbb{E}[\epsilon_{n-k}^2 | \mathcal{F}_t] \right] \\
= (1 + 4C \Delta + \Delta + 12a^2 (1 + a \Delta)^2 \Delta + 12a^2 \Delta^2) \mathbb{E}[\epsilon_{n}^2 | \mathcal{F}_t] \\
+(C \Delta + 12b^2 \Delta^2 + 12(b \Delta + \beta)^2 \Delta^3 \mathbb{E}[\epsilon_{n-m}^2 | \mathcal{F}_t] + (2C_4^2 + C_3^2) \Delta^2 \\
+(C \Delta + 12c^2 \tau^2 \Delta^2 + 12c^2 \alpha^2 \tau^2 \Delta^3) \max_{1 \leq k \leq m} \mathbb{E}[\epsilon_{n-k}^2 | \mathcal{F}_t].
\]

Let \( \mathcal{E}_n = \max_{0 \leq i \leq n} \mathbb{E}[\epsilon_i^2 | \mathcal{F}_t] \), since \( \epsilon_n = 0 \) for \( n = -m, -(m-1), \ldots, 0 \), where \( m = \max_{1 \leq j \leq k} \{ m_j, j \} \); then we can get

\[
\mathbb{E}(\epsilon_{n-m}^2 | \mathcal{F}_t) \leq \mathcal{E}_n, \quad \max_{1 \leq j \leq l} \mathbb{E}[\epsilon_{n-j}^2 | \mathcal{F}_t] \leq \mathcal{E}_n.
\]

\[
\mathcal{E}_{n+1} \leq (1 + 4C \Delta + \Delta + 12a^2 (1 + a \Delta)^2 \Delta + 12a^2 \Delta^2) \mathcal{E}_n \\
+(C \Delta + 12b^2 \Delta^2 + 12(b \Delta + \beta)^2 \Delta^3 \mathcal{E}_n \\
+(C \Delta + 12c^2 \tau^2 \Delta^2 + 12c^2 \alpha^2 \tau^2 \Delta^3) \mathcal{E}_n + (2C_4^2 + C_3^2) \Delta^2.
\]

Since \( \Delta \in (0, 1) \), we have

\[
\mathcal{E}_{n+1} \leq \left[ 1 + 6C \Delta + \Delta + 12 \left( a^2 + b^2 + c^2 \tau^2 + a^2 (1 + a \Delta)^2 \right. \right. \\
\left. \left. + (b \Delta + \beta)^2 + c^2 \alpha^2 \tau^2 \right) \right] \mathcal{E}_n + (2C_4^2 + C_3^2) \Delta^2. \quad (5.3.27)
\]

Let

\[
C_5 = 1 + 6C \Delta + 12 \left( a^2 + b^2 + c^2 \tau^2 + a^2 (1 + a \Delta)^2 + (b \Delta + \beta)^2 + c^2 \alpha^2 \tau^2. \right. \\
C_6 = 2C_4^2 + C_3^2.
\]
Then inequality (5.3.22) becomes

\[ \varepsilon_{n+1} \leq (1 + C_5 \Delta) \varepsilon_n + C_6 \Delta^2 \]

\[ \leq (1 + C_5 \Delta) \left[ (1 + C_5 \Delta) \varepsilon_{n-1} + C_6 \Delta^2 \right] + C_6 \Delta^2 \]

\[ \leq C_6 \Delta^2 \sum_{i=0}^{n} (1 + C_5 \Delta)^i = C_0 \Delta^2 \left( \frac{(1 + C_5 \Delta)^{n+1} - 1}{1 + C_5 \Delta - 1} \right) \leq C_7 \Delta, \]

where \( C_7 = \frac{C_6(e^{C_5 \tau} - 1)}{C_5} \), which implies

\[ (\varepsilon_{n+1})^{1/2} \leq C_0 \Delta^{1/2}, \]

that is

\[ \max_{0 \leq n \leq N} (\mathbb{E}(\varepsilon_n^2))^{1/2} \leq C_0 \Delta^{1/2}. \]

where \( C_0 = \sqrt{\frac{C_6(e^{C_5 \tau} - 1)}{C_5}} \)

The proof is completed.

By the same way, we can easily show the strong order of convergence of DISSFE method.

**Theorem 5.3.4.** The numerical solution produced by the DISSFE method (5.3.1) converges to the exact solution of (5.1.1) on the mesh-points in the mean-square sense with strong order \( \gamma = \frac{1}{2} \). that is, there exists a positive constant \( C_0 \) such that

\[ \max_{0 \leq n \leq N} (\mathbb{E}(\tau_n^2))^{1/2} \leq C_0 \Delta^{1/2} \]

as \( \Delta \to 0 \).

### 5.4 Mean-square Stability of Split-Step Forward Euler Methods

In this section, we study the mean-square stability of the split-step forward Euler method.

**Definition 5.4.1.** Under the condition (5.2.1), a numerical method is said to be mean square stable. if there exists a \( h > 0 \) such that any application of the method to (5.1.1) generates numerical approximations \( \{\tilde{y}_n\} \), which satisfy

\[ \lim_{n \to \infty} \mathbb{E} |\tilde{y}_n|^2 = 0. \quad (5.4.1) \]

for all \( \Delta \in (0, h) \) with \( \Delta = \tau/m \) for an integer \( m \).
Theorem 5.4.1. Assume that (5.2.1) is satisfied and let the step size satisfy $\Delta \in (0, h)$, where

$$h = \begin{cases} \min \left\{ \frac{1}{|a|}, \frac{-\mathcal{A} + \sqrt{\mathcal{A}^2 - 4\mathcal{B}\mathcal{C}}}{2\mathcal{B}} \right\} & \text{if } d \neq 0, \\ \min \left\{ \frac{1}{|a|}, \frac{-\mathcal{A}}{2\mathcal{B}} \right\} & \text{if } d = 0. \end{cases} \tag{5.4.2}$$

$$\mathcal{A} = \alpha^2(a + |b| + |c|\tau)^2,$$
$$\mathcal{B} = (a + |b| + |c|\tau)^2 + 2(\alpha^2 + |\alpha|\beta)(a + |b| + |c|\tau),$$
$$\mathcal{C} = 2(a + |b| + |c|\tau) + (|\alpha| + |\beta|)^2,$$

then the DRSSFE method of (5.1.1) is mean-square stable.

Proof. From 5.3.3, we have

$$\bar{y}_{n+1} = (1 + \alpha\Delta W_n) \left( (1 + a\Delta)\bar{y}_n + b\bar{y}_{n-m}\Delta + \Delta^2 c \sum_{k=1}^{m} \bar{y}_{n-k} \right) + \beta\bar{y}_{n-m}\Delta W_n.$$  

Squaring both sides of the above equality, we have

$$\begin{align*}
\bar{y}_{n+1}^2 &= (1 + \alpha\Delta W_n)^2 \left( (1 + a\Delta)\bar{y}_n + b\bar{y}_{n-m}\Delta + \Delta^2 c \sum_{k=1}^{m} \bar{y}_{n-k} \right)^2 \\
&\quad + (\beta\bar{y}_{n-m}\Delta W_n)^2 \\
&\quad + 2\beta(1 + \alpha\Delta W_n) \left( (1 + a\Delta)\bar{y}_n + b\bar{y}_{n-m}\Delta \\
&\quad + \Delta^2 c \sum_{k=1}^{m} \bar{y}_{n-k} \right) \bar{y}_{n-m}\Delta W_n. \tag{5.4.3}
\end{align*}$$

It follows from $2uvxy \leq |uv| (x^2 + y^2)$, where $u, v \in \mathbb{R}$, and (5.4.3), we have

$$\begin{align*}
\bar{y}_{n+1}^2 &\leq (1 + \alpha^2(\Delta W_n)^2 + 2\alpha \Delta W_n) \left[ (1 + a\Delta)\bar{y}_n^2 + b^2\bar{y}_{n-m}^2\Delta^2 + \Delta^4 \sum_{k=1}^{m} \bar{y}_{n-k}^2 \\
&\quad + |1 + a\Delta| |b|\Delta(\bar{y}_n^2 + \bar{y}_{n-m}^2) + |1 + a\Delta| |c|\Delta^2 \left( \sum_{k=1}^{m} \bar{y}_{n-k}^2 + m\bar{y}_n^2 \right) \\
&\quad + |c| |b|\Delta^2 \left( \sum_{k=1}^{m} \bar{y}_{n-k}^2 + m\bar{y}_n^2 \right) \right].
\end{align*}$$
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\[ + \beta^2 \tilde{y}_{n-m}^2 (\Delta W_n)^2 + \|a\| \|\beta\| \left[ 1 + a \Delta \left( \tilde{y}_n^2 + \tilde{y}_{n-m}^2 \right) + 2|b| \Delta \tilde{y}_{n-m}^2 \right. \\
\left. + \Delta^2 |c| \sum_{k=1}^{m} (\tilde{y}_{n-k} + m \tilde{y}_{n-m}^2) \right] (\Delta W_n)^2. \] 

(5.4.4)

Note that \( E(\Delta W_n) = 0, E[(\Delta W_n)^2] = \Delta \) and \( \tilde{y}_n, \tilde{y}_{n-k} \) and \( \tilde{y}_{n-m} \) are \( \mathcal{F} \)-measurable; hence

\[ E(\Delta W_n, X_i, X_j) = E(X_i X_j E(\Delta W_n) / \mathcal{F}) = 0, \]

\[ E[(\Delta W_n)^2 X_i^2] = E[X_i^2 E((\Delta W_n)^2) / \mathcal{F}] = \Delta E(X_i^2), \]

\[ i, j \in \{n - m, n - m + 1, \ldots, n\}. \] \hspace{1cm} (5.4.5)

Let \( Y_n = E[X_n^2] \) and, without loss of generality, we assume that \( 1 + a \Delta \geq 0 \); we have from (5.4.4)

\[ Y_{n+1} \leq P(a, b, c, a, \beta, \Delta) Y_n + Q(a, b, c, a, \beta, \Delta) Y_{n-m} + R(a, b, c, a, \beta, \Delta) \sum_{k=1}^{m} Y_{n-k}, \] \hspace{1cm} (5.4.6)

where

\[ P(a, b, c, a, \beta, \Delta) = (1 + \alpha^2 \Delta)(1 + a \Delta)^2 + (1 + a \Delta)|b| \Delta + \left( 1 + a \Delta \right)|c| m \Delta^2 \]

\[ + |a| |\beta| (1 + a \Delta) \Delta \]

\[ Q(a, b, c, a, \beta, \Delta) = (1 + \alpha^2 \Delta)|b|^2 \Delta^2 + (1 + a \Delta)|b| \Delta + \left( 1 + a \Delta \right)|c| m \Delta^3 + \beta^2 \Delta \]

\[ + |a| |\beta| (1 + a \Delta) \Delta + 2|b| |a| |\beta| \Delta^2 + |c| |a| |\beta| m \Delta^3. \]

\[ R(a, b, c, a, \beta, \Delta) = (1 + \alpha^2 \Delta)|c|^2 \Delta^4 m + (1 + a \Delta)|c| \Delta^2 + |b| |c| \Delta^3 + |c| |a| |\beta| \Delta^3. \]

Note that \( \sum_{k=1}^{m} Y_{n-k} \leq m \max_{1 \leq k \leq m} Y_{n-k} \) and \( m \Delta = \tau \). Then we have

\[ Y_{n+1} \leq \left( P(a, b, c, a, \beta, \Delta) + Q(a, b, c, a, \beta, \Delta) + m R(a, b, c, a, \beta, \Delta) \right) \max_{1 \leq k \leq m} \{ Y_{n-k} \}. \]

By recursive calculation, we conclude that \( Y_n \to 0 \) \((n \to \infty)\) if

\[ (P(a, b, c, a, \beta, \Delta) + Q(a, b, c, a, \beta, \Delta) + m R(a, b, c, a, \beta, \Delta)) < 1, \]

which is equivalent to

\[ (1 + \alpha^2 \Delta)(1 + a \Delta + |b| \Delta + |c| \tau \Delta)^2 + 2 |a| |\beta| (1 + a \Delta + |b| \Delta + |c| \tau \Delta) \Delta + \beta^2 \Delta < 1. \]
that is,
\[
\alpha^2(a + |b| + |c|^2)\Delta^2 + [(a + |b| + |c|^2)^2 + 2(\alpha^2 + |\alpha||\beta|)(a + |b| + |c|^2)]\Delta
+ 2(a + |b| + |c|^2) + (|\alpha| + |\beta|)^2 < 0.
\]
that is,
\[
\mathcal{A} h^2 + \mathcal{B} h + \mathcal{C} < 0
\]
where
\[
\mathcal{A} = \alpha^2(a + |b| + |c|^2)^2
\]
\[
\mathcal{B} = (a + |b| + |c|^2)^2 + 2(\alpha^2 + |\alpha||\beta|)(a + |b| + |c|^2)
\]
\[
\mathcal{C} = 2(a + |b| + |c|^2) + (|\alpha| + |\beta|)^2.
\]
Since \( \mathcal{C} < 0 \) and \( \mathcal{A} \geq 0 \), we have \( \mathcal{B}^2 - 4\mathcal{A}\mathcal{C} > 0 \).

So we can easily obtain that \( \mathcal{A} h^2 + \mathcal{B} h + \mathcal{C} < 0 \) holds when \( \Delta \in (0, h) \), where \( h \) is defined by
\[
h = \begin{cases} 
\min \left\{ \frac{1}{|a|}, \frac{-\mathcal{A} + \sqrt{\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}}}{2\mathcal{A}} \right\} & \text{if } \alpha \neq 0 \\
\min \left\{ \frac{1}{|a|}, \frac{-\mathcal{A}}{|a|} \right\} & \text{if } \alpha = 0.
\end{cases}
\tag{5.4.7}
\]
Since \( \mathcal{C} < 0 \) and \( \mathcal{A} \geq 0 \), we know that \( \mathcal{B} \leq \sqrt{\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}} \). Consequently, the DRSSFE method of (5.1.1) is mean-square stable. The proof is completed. \( \square \)

The following theorem for mean-square stability of DISSFE method of (5.1.1) and its proof is very similar to that of Theorem 5.4.1; so it is omitted.

**Theorem 5.4.2.** Assume that (5.2.1) is satisfied and let the step size satisfy \( \Delta \in (0, h) \), where
\[
h = \begin{cases} 
\min \left\{ \frac{1}{|a|}, \frac{-\mathcal{A} + \sqrt{\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}}}{2\mathcal{A}} \right\} & \text{if } \alpha \neq 0 \text{ or } \beta \neq 0 \\
\min \left\{ \frac{1}{|a|}, \frac{-\mathcal{A}}{|a|} \right\} & \text{if } \alpha = 0 \text{ and } \beta = 0.
\end{cases}
\tag{5.4.8}
\]
\[
\mathcal{P} = a^2(|\alpha| + |\beta|)^2.
\]
\[
\mathcal{Q} = (a + |b| + |c|^2)^2 + 2a(|\alpha| + |\beta|)^2.
\]
\[
\mathcal{R} = 2(a + |b| + |c|^2) + (|\alpha| + |\beta|)^2.
\]
then the DISSFE method of (5.1.1) is mean-square stable.


5.5 Numerical Results

We discuss an example to illustrate the stability.

Consider an one-dimensional linear stochastic delay integrodifferential equation of the form,

\[ dx(t) = (ax(t) + bx(t - 1) + c \int_{t-1}^{t} x(s)ds)dt \]

\[ + (\alpha x(t) + \beta x(t - 1))dW(t), \quad t \geq 0, \quad (5.5.1) \]

with initial segment \( \xi(t) = t + 1 \) for \( t \in [-1, 0] \). Here \( W(t) \) be a scalar Brownian motion. In the following tests, we find the stepsize interval \((0, h)\) on the mean-square stability of the DRSSFE and DISSFE methods. The data used in all figures are obtained by the mean square of data by 100 trajectories, that is, \( \omega_r : 1 \leq r \leq 100, Y_n = \mathbb{E}g_n^2 = \frac{1}{100} \sum_{r=1}^{100} |g_n(\omega_r)|^2 \). In all the figures, \( t_n \) denotes the mesh-point.

Case 1. We choose the coefficients of test equation (5.5.1) as \( a = -9, b = 4, c = 3, \alpha = 0, \) and \( \beta = \frac{4}{3} \). By Theorem 5.4.2 and Theorem 5.4.3, we know that the DRSSFE and DISSFE methods are mean-square stable if \( \Delta \in (0, \frac{1}{3}) \). Fig. 5.1 shows that the numerical solution produced by the DRSSFE and DISSFE schemes are mean-square stable when \( \Delta = \frac{1}{10} \). Fig. 5.2 shows that the numerical solution produced by the DRSSFE and DISSFE schemes are unstable when \( \Delta = \frac{1}{6} \).

Case 2. We consider the test equation (5.5.1) with coefficients \( a = -0.4, b = 0.2, c = 0.6, \alpha = 0.1, \) and \( \beta = 1. \) By Lemma 5.2.1, we get the result that the analytic solution of the (5.5.1) is unstable and from the Theorem 5.4.2 and Theorem 5.4.3, we know that the DRSSFE and DISSFE methods are also unstable. Fig. 5.3 shows that the numerical solution produced by the DRSSFE and DISSFE schemes are unstable when \( \Delta = \frac{1}{100} \).
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Figure 5.1: Split-step forward Euler method for case 1 (Mean-square stable)
Figure 5.2: Split-step forward Euler method for case 1 (unstable)
Figure 5.3: Split-step forward Euler method for case 2 (unstable)