Chapter 4

Stability of Milstein Method for Hybrid Stochastic Delay Integrodifferential Equations

4.1 Introduction

Delay integrodifferential equations occur in biology, medicine and many other fields. If we include random noise into delay integrodifferential equations, we obtain stochastic delay integrodifferential equations. In this chapter, we study numerical methods for hybrid systems of stochastic delay integrodifferential equations.

Koto [60] investigated the stability of the numerical methods for the integrodifferential equation

\[ \dot{x}(t) = \lambda x(t) + \mu x(t - \tau) + \kappa \int_{t-\tau}^{t} x(s) \, ds. \]

Shaikhet and Rorberts [101] studied the stability of the numerical methods for the linear stochastic functional differential equation of the form

\[ dx(t) = \left[ a x(t) + 3 \int_{0}^{t} x(s) \, ds \right] dt + \sigma x(t - \tau) \, dw(t). \]

Ding et al [32] studied the convergence and stability of the semi-implicit Euler method for the linear stochastic delay integrodifferential equation

\[ dx(t) = \left[ a x(t) + b \int_{t-\tau}^{t} x(s) \, ds \right] dt + c x(t - \tau) \, dw(t). \]
In this chapter, we consider a linear stochastic delay integrodifferential equation with Markovian switching of the form

\[
dx(t) = \left[ A(r(t))x(t) + B(r(t))x(t - \tau) + C(r(t)) \int_{t-\tau}^{t} x(s)ds \right] dt \\
+ [D(r(t))x(t) + E(r(t))x(t - \tau)] dW(t), \quad t \geq 0, 
\]

(4.1.1)

where the initial segment of solution is \( x(t) = \psi(t) \) for \( t \in [-\tau, 0] \), \( r(t) \) is a Markov chain taking values in \( S = \{1, 2, ..., N\} \) and \( A(\cdot), B(\cdot), C(\cdot), D(\cdot), E(\cdot) \in \mathbb{R} \), \( \{W(t)\} \) is a standard one-dimensional Brownian motion and \( \tau \) is a positive fixed delay.

In section 4.2, we discuss the exponential stability in mean square for (4.1.1). In section 4.3, the Milstein method is used to obtain the numerical solutions. Furthermore, our main result is proved in the section 4.4 and a numerical example is given in section 4.5.

### 4.2 Stability of Analytical Solutions

In this chapter, we keep the same assumptions as with single delay used in the previous chapter. So in the scalar test equation (4.1.1), we can match (4.1.1) to (3.1.1):

\[
f(x(t), x(t - \tau), t, r(t)) = A(r(t))x(t) + B(r(t))x(t - \tau) + C(r(t)) \int_{t-\tau}^{t} x(s)ds \\
g(x(t), x(t - \tau), t, r(t)) = D(r(t))x(t) + B(r(t))x(t - \tau).
\]

If we take \( \alpha_i = 2A_0(i) \), \( \rho_i = |B(i)| + \tau |C(i)| \), \( \sigma_i = |D(i)|(|D(i)| + |E(i)|) \) and \( \beta_i = |E(i)|(|D(i)| + |E(i)|) \), then the hypotheses (H1) and (H2) in the previous chapter with single delay case are satisfied naturally for (4.1.1). Then, for any given initial data \( \psi \in C_{\mathcal{F}_0}([-\tau, 0] : \mathbb{R}^n) \), (4.1.1) has a unique continuous solution denoted by \( x(t; \psi) \) on \( t \geq -\tau \). It is also easy to see that the equation admits the trivial solution \( x(t; 0) \equiv 0 \). By means of Lyapunov’s function method, we obtain the following lemma:

**Theorem 4.2.1.** Let hypotheses (H1) and (H2) hold. Define

\[
\mathcal{A} = \text{diag} \left( - (\alpha_1 + \rho_1 + \sigma_1), ..., - (\alpha_N + \rho_N + \sigma_N) \right) - \Gamma.
\]

(4.2.1)
Assume that $A$ is a non-singular M-matrix and
\[
\begin{pmatrix}
\frac{1}{\rho_1 + \beta_1} & \cdots & \frac{1}{\rho_N + \beta_N}
\end{pmatrix}^T \gg A^{-1} \Gamma,
\]
where $\Gamma = (1, 1, \ldots, 1)^T$. Then the trivial solution of (4.1.1) is exponentially stable in mean square.

### 4.3 Milstein Method

The general nonlinear multi-delay systems without Markovian switching is
\[
dx(t) = b^0(x(t), x(t-\tau), t)dt + \sum_{j=1}^d b^{(j)}(x(t), x(t-\tau), t)dW^{(j)}(t), \quad t \geq 0, (4.3.1)
\]
where the initial segment of solution is $x(t) = \psi(t)$ for $t \in [-\tau, 0]$. Küchler and Planten [61] presented the so-called order 1 strong Taylor approximation formula for (4.3.1). Applying the formula to the linear one-delay integrodifferential system with Markovian switching, we use the Milstein method to (4.1.1) leading to a numerical process of the following type:
\[
\begin{align*}
\bar{y}_{n+1} &= \bar{y}_n + \left( A (r_n^\Delta) \bar{y}_n + B (r_n^\Delta) \bar{y}_{n-m} + C (r_n^\Delta) \Delta \sum_{k=1}^m \bar{y}_{n-k} \right) \Delta \\
&\quad + \left( D (r_n^\Delta) \bar{y}_n + E (r_n^\Delta) \bar{y}_{n-m} \right) \Delta W_n \\
&\quad + D (r_n^\Delta) \left( D (r_n^\Delta) \bar{y}_n + E (r_n^\Delta) \bar{y}_{n-m} \right) I_1 \\
&\quad + E (r_n^\Delta) \left( D (r_n^\Delta) \bar{y}_{n-m} + E (r_n^\Delta) \bar{y}_{n-2m} \right) I_2,
\end{align*}
\]
where $\Delta > 0$ is a stepsize which satisfies $\tau = m\Delta$ for a positive integer $m$, $t_n = n\Delta$, and $r_n^\Delta = r(t_n) \in S$. $\bar{y}_n$ is an approximation to $x(t_n) = x_n$, if $t_n \leq 0$ and we have $\bar{y}_n = \psi(t_n)$. Moreover the increments $\Delta W_n := W(t_{n+1}) - W(t_n)$ are independently $\mathcal{N}(0, \Delta)$-distributed Gaussian random variables. We assume that $\bar{y}_n$ is $\mathcal{F}_{t_n}$-measurable at the mesh-points $t_n$. Let $I_1$ and $I_2$ denote the two double integrals defined, respectively, by
\[
I_1 = \int_{t_{n+1}}^{t_n} \int_{t_n}^t dW(t) dW(s) = \frac{1}{2} (\Delta W_n^2 - \Delta) \quad \text{and} \quad I_2 = \int_{t_{n+1}}^{t_n} \int_{t_n}^t dW(t-\tau) dW(s).
\]

We refer to the numerical scheme (4.3.2) as Milstein method since the scheme is just Milstein method when applied to a system without delay and Markovian
swichting. The convergence of the method (4.3.2) without Markovian switching is discussed in [61]. Küchler and Planten [61] proved that the order 1 strong Taylor approximation formula covers strongly with order 1 whenever the coefficients \( b^{(j)} (j=1,2,\ldots,d) \) of system (4.3.1) without Markovian switching are homogeneous and satisfy both the generalized Lipschitz condition and the generalized growth condition. A simple check shows that the Milstein method (4.3.2) satisfies these conditions and hence it is strongly convergent of order 1. The following lemma [113] will be key to the proof.

**Lemma 4.3.1.** The double integrals \( I_1 \) and \( I_2 \) have the same expectation and variation and satisfy \( E[I_1] = E[I_2] = 0, \quad E[I_1^2] = E[I_2^2] = \frac{\Delta^2}{2}, \quad E[I_1 I_2] = 0. \)

### 4.4 Mean-square Stability of Numerical Scheme

We investigate the mean-square stability of the Milstein method in this section.

**Definition 4.4.1.** Under the conditions (4.2.2) and for any \( i \in \mathbb{S} \),

\[
A_i + |B_i| + \tau |C_i| + \frac{1}{2} (|D_i| + |E_i|)^2 < 0. \quad (4.4.1)
\]

where \( A_i = A(i) \), \( B_i = B(i) \), \( C_i = C(i) \), \( D_i = D(i) \) and \( E_i = E(i) \), a numerical method is said to be mean square stable, if there exists a \( h_0(A_i, B_i, C_i, D_i, E_i) > 0 \) such that any application of the method to problem (4.1.1) generates numerical approximation \( \{\tilde{y}_n\} \), which satisfies

\[
\lim_{n \to \infty} E[\tilde{y}_n^2] = 0 \quad (4.4.2)
\]

for all \( \Delta \in (0, h_0(A_i, B_i, C_i, D_i, E_i)) \) with \( \Delta = \tau / m \) for an integer \( m \).

Now we prove the main theorem of this chapter.

**Theorem 4.4.1.** Assume that the conditions (4.2.2) and (4.4.1) are satisfied. Then the Milstein method applied to (4.1.1) is mean-square stable.

**Proof.** By re-arranging the right-hand side of (4.3.2), we have

\[
\tilde{y}_{n+1} = \tilde{y}_n + \left( A \left( r_n^{\Delta} \right) \Delta + D \left( r_n^{\Delta} \right) \Delta W_n \right) \tilde{y}_n + C \left( r_n^{\Delta} \right) \Delta^2 \sum_{k=1}^{m} \tilde{y}_{n-k} \\
+ \left( B \left( r_n^{\Delta} \right) \Delta + E \left( r_n^{\Delta} \right) \Delta W_n \right) \tilde{y}_{n-m} + D \left( r_n^{\Delta} \right) \left( D \left( r_n^{\Delta} \right) \tilde{y}_n + E \left( r_n^{\Delta} \right) \tilde{y}_{n-m} \right) I_1 \\
+ E \left( r_n^{\Delta} \right) \left( D \left( r_n^{\Delta} \right) \tilde{y}_{n-m} + E \left( r_n^{\Delta} \right) \tilde{y}_{n-2m} \right) I_2.
\]
By Lemma 3.2.1, the generation of $r_n^\Delta$ occurs before computing $\tilde{y}_{n+1}$; then $r_n^\Delta$ is known. Since $r_n^\Delta \in \mathbb{S}$, the above expression can be rewritten as

$$
\tilde{y}_{n+1} = (1 + A_i h + D_i \Delta W_n) \tilde{y}_n + C_i \Delta^2 \sum_{k=1}^{m} \tilde{y}_{n-k} + (B_i h + E_i \Delta W_n) \tilde{y}_{n-m} + D_i (D_i \tilde{y}_n + E_i \tilde{y}_{n-m}) I_1 + E_i (D_i \tilde{y}_{n-m} + E_i \tilde{y}_{n-2m}) I_2,
$$

where $A_i = A(i)$, $B_i = B(i)$, $C_i = C(i)$, $D_i = D(i)$ and $E_i = E(i)$ for any $i \in \mathbb{S}$.

Squaring both sides of the above equality, we have

$$
\tilde{y}_{n+1}^2 = (1 + A_i \Delta + D_i \Delta W_n)^2 \tilde{y}_n^2 + \left[ C_i \Delta^2 \sum_{k=1}^{m} \tilde{y}_{n-k} \right]^2 + (B_i \Delta + E_i \Delta W_n)^2 \tilde{y}_{n-m}^2 + D_i^2 (D_i \tilde{y}_n + E_i \tilde{y}_{n-m})^2 I_1^2 + E_i^2 (D_i \tilde{y}_{n-m} + E_i \tilde{y}_{n-2m})^2 I_2^2 + 2 (1 + A_i \Delta + D_i \Delta W_n) C_i \Delta^2 \sum_{k=1}^{m} \tilde{y}_{n-k} \tilde{y}_n + 2 (1 + A_i \Delta + D_i \Delta W_n) (B_i \Delta + E_i \Delta W_n) \tilde{y}_{n-m} \tilde{y}_n + 2 (1 + A_i \Delta + D_i \Delta W_n) D_i (D_i \tilde{y}_n + E_i \tilde{y}_{n-m}) I_1 \tilde{y}_n + 2 (1 + A_i \Delta + D_i \Delta W_n) E_i (D_i \tilde{y}_{n-m} + E_i \tilde{y}_{n-2m}) I_2 \tilde{y}_n + 2 C_i \Delta^2 (B_i \Delta + E_i \Delta W_n) \sum_{k=1}^{m} \tilde{y}_{n-k} \tilde{y}_{n-m} + 2 C_i \Delta^2 D_i (D_i \tilde{y}_n + E_i \tilde{y}_{n-m}) I_1 \sum_{k=1}^{m} \tilde{y}_{n-k} + 2 C_i \Delta^2 E_i (D_i \tilde{y}_{n-m} + E_i \tilde{y}_{n-2m}) I_2 \sum_{k=1}^{m} \tilde{y}_{n-k} + 2 (B_i \Delta + E_i \Delta W_n) \tilde{y}_{n-m} D_i (D_i \tilde{y}_n + E_i \tilde{y}_{n-m}) I_1 + 2 (B_i \Delta + E_i \Delta W_n) \tilde{y}_{n-m} E_i (D_i \tilde{y}_{n-m} + E_i \tilde{y}_{n-2m}) I_2 + 2 D_i (D_i \tilde{y}_n + E_i \tilde{y}_{n-m}) E_i (D_i \tilde{y}_{n-m} + E_i \tilde{y}_{n-2m}) I_1 I_2.
$$

It follows from $2xy \leq x^2 + y^2 \ (\forall x, y \in \mathbb{R})$ that $\sum_{k=1}^{m} \tilde{y}_{n-k} \leq m \max_{1 \leq k \leq m} \tilde{y}_{n-k}$.
and \( mh = \tau \). We get
\[
\hat{y}_{n+1}^2 = \left(1 + A_i \Delta + D_i \Delta W_n\right)^2 \hat{y}_n^2 + C_i^2 \tau^2 \Delta^2 \max_{(1 \leq k \leq m)} \hat{y}_{n-k}^2
+ (B_i \Delta + E_i \Delta W_n)^2 \hat{y}_{n-m}^2
+ D_i^2 \left[(D_i^2 + |E_i D_i|)^2 \hat{y}_n^2 + (E_i^2 + |E_i D_i|)^2 \hat{y}_{n-m}^2\right] I_1^2
+ E_i^2 \left[(D_i^2 + |E_i D_i|)^2 \hat{y}_{n-m}^2 + (E_i^2 + |E_i D_i|)^2 \hat{y}_{n-2m}^2\right] I_2^2
+ \left|1 + A_i \Delta\right| |B_i| \Delta \left(\hat{y}_n^2 + \hat{y}_{n-m}^2\right) + |D_i E_i| \left(\Delta W_n^2\right)^2 \left(\hat{y}_n^2 + \hat{y}_{n-m}^2\right)
+ 2 \left|1 + A_i \Delta\right| E_i + B_i D_i \Delta \Delta W_n \hat{y}_{n-m} \hat{y}_n
+ \left|1 + A_i \Delta\right| |C_i| \tau \Delta \max_{(1 \leq k \leq m)} \left(\hat{y}_{n-k}^2 + \hat{y}_n^2\right)
+ |C_i B_i| \tau \Delta^2 \max_{(1 \leq k \leq m)} \left(\hat{y}_{n-k}^2 + \hat{y}_n^2\right)
+ 2C_i E_i \tau \Delta W_n \max_{(1 \leq k \leq m)} \hat{y}_{n-k} \hat{y}_{n-m} + 2D_i C_i \tau \Delta W_n \max_{(1 \leq k \leq m)} \hat{y}_{n-k} \hat{y}_n
+ 2 \left(1 + A_i \Delta + D_i \Delta W_n\right) D_i \left(D_i \hat{y}_n + E_i \hat{y}_{n-m}\right) I_1 \hat{y}_n
+ 2 \left(1 + A_i \Delta + D_i \Delta W_n\right) E_i \left(D_i \hat{y}_n + E_i \hat{y}_{n-m}\right) I_2 \hat{y}_n
+ 2C_i \tau \Delta D_i \left(D_i \hat{y}_n + E_i \hat{y}_{n-m}\right) I_1 \max_{(1 \leq k \leq m)} \hat{y}_{n-k}
+ 2C_i \tau \Delta E_i \left(D_i \hat{y}_{n-m} + E_i \hat{y}_{n-2m}\right) I_2 \max_{(1 \leq k \leq m)} \hat{y}_{n-k}
+ 2 \left(B_i \Delta + E_i \Delta W_n\right) \hat{y}_{n-m} \left(D_i \hat{y}_{n-m} + E_i \hat{y}_{n-m}\right) I_1
+ 2 \left(B_i \Delta + E_i \Delta W_n\right) \hat{y}_{n-m} \left(D_i \hat{y}_{n-m} + E_i \hat{y}_{n-m}\right) I_2
+ 2D_i \left(D_i \hat{y}_n + E_i \hat{y}_{n-m}\right) \hat{y}_{n-m} \left(D_i \hat{y}_{n-m} + E_i \hat{y}_{n-m}\right) I_1 I_2. \tag{4.4.3}
\]

Note that \( E(\Delta W_n) = 0, E[(\Delta W_n)^2] = \Delta \) and any two of \( \Delta W_n, I_1 \) and \( I_2 \) are independent. Furthermore \( \hat{y}_n, \hat{y}_{n-1}, \ldots, \hat{y}_{n-m} \) and \( \hat{y}_{n-2m} \) are all \( F_{t_n} \)-measurable. Hence
\[
E(\Delta W_n \hat{y}_n \hat{y}_{n-m}) = E(\hat{y}_n \hat{y}_{n-m} E(\Delta W_n | F_{t_n})) = 0.
\]
\[
E \left[(\Delta W_n)^2 \hat{y}_n^2\right] = E \left(\hat{y}_n^2 E((\Delta W_n)^2 | F_{t_n})\right) = \Delta E(\hat{y}_n^2), \; j \in \{n, n - m\}.
\]

Similarly it can be derived that
\[
E \left[I_1^2 \hat{y}_j^2\right] = \frac{\Delta^2}{2} E \left[\hat{y}_j^2\right] \cdot j \in \{n, n - m\}.
\]
\[
E \left[I_2^2 \hat{y}_k^2\right] = \frac{\Delta^2}{2} E \left[\hat{y}_k^2\right] \cdot k \in \{n - m, n - 2m\}
\]
and
\[
E \left[\hat{y}^i_n \hat{y}^j_{n-m} \Delta W_n^j I_1\right] = 0, \; i_1, i_2 \in \{0, 1, 2\}, \; j_1 \in \{0, 1\}.
\]
With a recursive calculation and Lemma 4.3.1, we obtain
\[ E\left[ \tilde{g}_{n}^{i_{1}} \tilde{g}_{n-m}^{i_{2}} \Delta W_{n}^{j_{1}} I_{1}I_{2} \right] = 0, \ i_1, i_2 \in \{0, 1, 2\}, \ i_3, j_1, j_2 \in \{0, 1\}, \]
and \( j_1 + j_2 \leq 1. \) Let \( Y_n = \mathbb{E}|y_n|^2 \) and take expectation on both sides of inequality (4.4.3). Then it holds that
\[ Y_{n+1} \leq P(A_i,..) Y_n + Q(A_i,..) Y_{n-m} + R(A_i,..) Y_{n-2m} + U(A_i,..) \max_{1 \leq k \leq m} Y_{n-k}, \]
where
\[
P(A_i,..) = (1 + A_i \Delta)^2 + D_i^2 \Delta + |1 + A_i \Delta| |B_i| \Delta + |D_i E_i| \Delta + \frac{\Delta^2}{2} D_i^2 (D_i^2 + |D_i E_i|) + |1 + A_i \Delta| |C_i| \tau \Delta, \\
Q(A_i,..) = B_i^2 h^2 + E_i^2 \Delta + |1 + A_i \Delta| |B_i| \Delta + |D_i E_i| \Delta + \frac{\Delta^2}{2} E_i^2 (D_i^2 + |D_i E_i|) + |B_i C_i| \tau \Delta^2, \\
R(A_i,..) = \frac{\Delta^2}{2} E_i^2 (E_i^2 + |D_i E_i|), \\
U(A_i,..) = C_i^2 \tau^2 \Delta^2 + |1 + A_i \Delta| |C_i| \tau \Delta + |B_i C_i| \tau \Delta^2.
\]
This implies that
\[ Y_{n+1} \leq \max_{0 \leq k \leq m} \left\{ Y_{n-k}, Y_{n-2m} \right\}. \]
By the above inequality, we conclude that \( Y_n \to 0 (n \to \infty) \) if
\[ P(A_i,..) + Q(A_i,..) + R(A_i,..) + U(A_i,..) \leq 1, \]
that is,
\[
\left[ (1 + A_i \Delta)^2 + |B_i| + \tau |C_i|^2 \Delta^2 + (|D_i| + |E_i|)^2 \Delta + 2|1 + A_i \Delta| |B_i| + \tau |C_i| \Delta + \frac{\Delta^2}{2} (D_i^2 + E_i^2) (|D_i| + |E_i|)^2 \right] < 1. \quad (4.4.4)
\]
Write
\[
h_1(A_i,..) = \min_i \left\{ \frac{\alpha(i)}{\beta(i)} \right\}, \\
h_2(A_i,..) = \min_i \left\{ \min \left\{ \frac{1}{|A_i|} \right\}, \min \left\{ \frac{\alpha(i)}{\kappa(i)} \right\} \right\}.
\]
\( \alpha(i) = -\left[ 2A_i + 2|B_i| + 2|C_i| \tau + (|D_i| + |E_i|)^2 \right] \),
\( \beta(i) = (|A_i| + |B_i| + |C_i| \tau)^2 + \frac{1}{2} (D_i^2 + E_i^2) (|D_i| + |E_i|)^2 \)
and
\( \kappa(i) = (A_i + |B_i| + |C_i| \tau)^2 + \frac{1}{2} (D_i^2 + E_i^2) (|D_i| + |E_i|)^2 \).

It follows from that condition (4.4.1) that \( h_1(A_i, \ldots) > 0 \) and \( h_2(A_i, \ldots) > 0 \). If \( \Delta \in (0, h_1(A_i, \ldots)) \), then we have

\[
\begin{aligned}
&\left[ (|A_i| + |B_i| + |C_i| \tau)^2 + \frac{1}{2} (D_i^2 + E_i^2) (|D_i| + |E_i|)^2 \right] \Delta^2 + \\
&\left( 2A_i + 2|B_i| + 2|C_i| \tau + \frac{1}{2} (|D_i| + |E_i|)^2 \right) \Delta < 0,
\end{aligned}
\]

which implies that (4.4.4) holds. If \( \Delta \in (0, h_2(A_i, \ldots)) \), then we have \( 1 + A_i \Delta > 0 \) and

\[
\begin{aligned}
&\left[ (A_i + |B_i| + |C_i| \tau)^2 + \frac{1}{2} (D_i^2 + E_i^2) (|D_i| + |E_i|)^2 \right] \Delta^2 + \\
&\left( 2A_i + 2|B_i| + 2|C_i| \tau + \frac{1}{2} (|D_i| + |E_i|)^2 \right) \Delta < 0,
\end{aligned}
\]

which implies that (4.4.4) still holds. Let

\[ h_0(A_i, B_i, C_i, D_i, E_i) = \max \{ h_1(A_i, \ldots), h_2(A_i, \ldots) \}. \tag{4.4.5} \]

Then we can conclude that (4.4.4) holds whenever

\[ \Delta \in (0, h_0(A_i, B_i, C_i, D_i, E_i)) . \]

The proof is completed .

### 4.5 Numerical Example

We discuss an example to illustrate our theory. Let \( W(t) \) be a scalar Brownian motion. Let \( r(t) \) be a right continuous Markov chain taking values in \( S = \{1, 2\} \) with the generator

\[
\Gamma = (\gamma_{ij})_{2 \times 2} = \begin{pmatrix} -1 & 1 \\ \gamma & -\gamma \end{pmatrix},
\]
Of course $W(t)$ and $r(t)$ are assumed to be independent. Consider an one-dimensional linear stochastic delay integrodifferential equation with Markovian switching

$$dx(t) = \left[ A(r(t)) x(t) + B(r(t)) x(t - 1) + C(r(t)) \int_{t-1}^{t} x(s) ds \right] dt + E(r(t)) x(t - 1) dW(t), \quad (4.5.1)$$

on $t \geq 0$ with initial segment $\psi(t) = t + 1$ for $t \in [-1,0]$ as a function, $r(0) = 1$.

**Case 1.** Let $A(1) = -2$, $B(1) = 1$, $C(1) = \frac{1}{2}$, $E(1) = \frac{2}{5}$ and $A(2) = -9$, $B(2) = 4$, $C(2) = 3$, $E(2) = \frac{4}{5}$. It is interesting to regard (4.5.1) as the result of the following two equations:

$$dx(t) = \left[ -2x(t) + x(t - 1) + \frac{1}{2} \int_{t-1}^{t} x(s) ds \right] dt + \frac{2}{5} x(t - 1) dW(t) \quad (4.5.2)$$

and

$$dx(t) = \left[ -9x(t) + 4x(t - 1) + 3 \int_{t-1}^{t} x(s) ds \right] dt + \frac{4}{5} x(t - 1) dW(t) \quad (4.5.3)$$

switching from one to the other according to the movement of the Markov chain $r(t)$. It is known that (4.5.2) and (4.5.3) are almost surely exponentially stable: however, as the result of Markovian switching, the overall behaviour, that is, (4.5.1), will be exponentially stable if $0 < \gamma < 2.354$. So we take $\gamma = 1$.

In the following tests, we show the influence of stepsize $\Delta$ on the mean-square stability of the Milstein method. The data used in all the figures are obtained by the mean square of data by 100 trajectories, that is, $\omega_r : 1 \leq r \leq 100$, $y_0 = \frac{1}{100} \sum_{r=1}^{100} | \bar{y}_n(\omega_r) |^2$. In all figures $\omega_n$ denotes the mesh-point. In this case $h_0(.) = \frac{1}{5}$. Thus, when a method of form (4.3.2) with stepsize $\Delta : 0 < \Delta < \frac{1}{5}$ is used to solve the above system, the corresponding numerical solution is mean-square stable by Theorem 4.4.1.

With the numerical tests, we can intuitively see the stepsize’s influence on the stability of the method. In the Milstein method, by taking stepsizes $\Delta = \frac{1}{2}$, $\Delta = \frac{1}{21}$, and $\Delta = \frac{1}{21}$, respectively, we obtain four groups of numerical solutions of (4.5.1) on the interval $[0, 0.15]$, which are displayed in Fig. 4.1 and Fig.4.2.

One may be interested to know about the stability of the numerical methods when stepsize are outside the stability range $(0, \frac{1}{5})$. Here, we make an insight
into the problem. Taking stepsizes $\Delta = \frac{1}{2^6}$, $\Delta = \frac{1}{2^5}$, respectively, we obtain two groups of numerical solutions of (4.5.1) on the interval [0, 15], which are displayed in Fig.4.3.

**Case 2.** $A(1) = -2$, $B(1) = 1$, $C(1) = 0.5$, $E(1) = 0.4$ and $A(2) = 0.5$, $B(2) = 0.2$, $C(2) = 0.1$, $E(2) = 0.8$. For these values, neither conditions of Theorem 4.2.1 nor conditions of Theorem 4.4.1 are satisfied. To carry out the numerical simulation we choose the step size $\Delta = \frac{1}{1024}$. The computer simulation result is shown in Fig.4.4. Clearly the Milstein method reveals the unstable property of the solution.

![Figure 4.1: Numerical simulation of case.1 with $\Delta = \frac{1}{2^6}$ and $\Delta = \frac{1}{2^5}$.](image1)

![Figure 4.2: Numerical simulation of case.1 with $\Delta = \frac{1}{2^6}$ and $\Delta = \frac{1}{2^5}$.](image2)
Figure 4.3: Numerical simulation of case.1 with $\Delta = \frac{1}{2^4}$ and $\Delta = \frac{1}{2}$.

Figure 4.4: Numerical simulation of case.2 with $\Delta = \frac{1}{2^{10}}$. 