Gregarious Path Factorizations of Product Graphs
Chapter 4

Gregarious Path Factorizations of Product Graphs

4.1 Introduction

Path decompositions of graphs, complete multipartite graphs has been studied by many authors, see [15, 37, 60, 69, 70, 73]. Ushio proved the existence of $P_3$-factorization of bipartite and tripartite symmetric digraphs in [69, 70]. They left open the problem of existence of $P_k$-factorization of complete $k$-partite symmetric digraphs when $k \geq 4$.

Necessary condition for the existence of $P_m$-factorizations in $K_m \circ K_n$ and $K_m \times K_n$ are $n(m - 1) | m(m - 1)n^2/2$ and $n(m - 1) | m(m - 1)n(n - 1)/2$, respectively. In this chapter it has been proved that the above necessary conditions are also sufficient for the existence of $P_m$-factorizations in $K_m \circ K_n$ and $K_m \times K_n$. Further, it is shown that

$$
P_m \parallel (K_m \circ K_n)^*,$$ for $(m, n) \notin \{(3, 2r + 1), (5, 2r + 1)\},$$

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\[ P_m \parallel (K_m \times K_n)^*, \text{ for all } m \geq 2, n \geq 2. \]

Finally, the non existence of \( P_{2s+1} \) in \((C_{2s+1} \circ \overline{K}_{2r+1})^*\) is also proved.

To prove our results we require the following:

**Theorem 4.1.1.** [68] For \( 2n \geq 8 \), \( \overline{C}_{2n} \parallel K_{2n}^* \).

**Note 4.1.** From Theorem 2.2.1, we have \( P_{2n} \parallel K_{2n} \) and by Theorem 4.1.1 we have \( \overline{P}_{2n-1} \parallel K_{2n-1}^* \) for \( 2n - 1 \geq 7 \). From the above we have \( \overline{P}_n \parallel K_n^* \) for \( n \neq 3, 5 \).

### 4.2 Gregarious Path Factorizations

**Lemma 4.2.1.** \( P_m \circ K_n \) admits a \( \hat{P}_m \)-factorization.

**Proof.** Let \( V(P_m \circ K_n) = \bigcup_{i=1}^{n} V_i \), where \( V_i = \{v_i^0, v_i^1, \ldots, v_i^{n-1}\} \). \( \hat{P}_m \)-factorization of \( P_m \circ R_m \) is constructed as follows: Let \( H_i = \alpha_i(V_1, V_2) \oplus \alpha_i(V_2, V_3) \oplus \ldots \oplus \alpha_i(V_{m-1}, V_m), 0 \leq i \leq n - 1 \). Clearly \( \{H_1, H_2, \ldots, H_n\} \) is a \( \hat{P}_m \)-factorization of \( P_m \circ R_n \).

**Note 4.2.** By orienting the edges of \( P_m \circ K_n \) downwards (that is, from \( V_i \) to \( V_j, i < j \)) we get \( \overline{P}_m \circ K_n \). Thus \( \hat{P}_m \)-factorization of \( P_m \circ K_n \) implies the \( \hat{P}_m \)-factorization of \( \overline{P}_m \circ K_n \).

**Lemma 4.2.2.** \( P_m \times K_n \) admits a \( \hat{P}_m \)-factorization.

**Proof.** Follows from the fact

\[
P_m \times K_n = \{P_m \circ K_n\} - \left\{ \bigcup_{V_i \in P_m} (\alpha_0(V_i, V_j)) \right\}
\]

and by the proof of Lemma 4.2.1.
Note 4.3. By orienting the edges of $P_m \times K_n$ downwards (that is, from $V_i$ to $V_j, i < j$) we get $\overrightarrow{P}_m \times K_n^*$. Thus $\overrightarrow{P}_m$-factorization of $P_m \times K_n$ implies that of $\overrightarrow{P}_m \times K_n^*$. □

Theorem 4.2.3. For even $m \geq 2$, $\overrightarrow{P}_m \parallel K_m \circ K_n$.

Proof. By Note 4.1, $P_m \parallel K_m$. Therefore,

$$K_m \circ K_n = (H_1 \oplus H_2 \oplus \ldots \oplus H_m) \circ K_n$$

$$= \oplus_{i=1}^{m} (H_i \circ K_n),$$

where $H_i \cong P_m$. By Lemma 4.2.1, $\overrightarrow{P}_m \parallel (H_i \circ K_n)$, for each $i$.

Thus $\overrightarrow{P}_m \parallel K_m \circ K_n$. □

Theorem 4.2.4. For even $m \geq 2$, $(K_m \times K_n)$ admits a $\overrightarrow{P}_m$-factorization.

Proof. Since $P_m \parallel K_m$ by Note 4.1, the result follows from Lemma 4.2.2. □

Lemma 4.2.5. For odd $m \geq 3$, $P_m \parallel K_m (2)$.

Proof. Let $H = v_1v_2v_3v_4\ldots v_{m+5/2}v_{m+1/2}v_{m+3/2}$ be a path on $m$ vertices. Let $\sigma = (123\ldots m)$ be the permutation on $m$ elements.

Then $\{H, \sigma (H), \sigma^2 (H), \ldots, \sigma^{m-1} (H)\}$ is a $P_m$-factorization of $K_m (2)$. Hence $P_m$ decomposes $K_m (2)$. □

Corollary 4.2.6. For odd $m \geq 3$, $P_m \parallel K_m (2\lambda)$. □

Theorem 4.2.7. For all even $n$ and odd $m \geq 3$, $\overrightarrow{P}_m \parallel K_m \circ K_n$.

Proof. Let $K_m \circ K_n = G$. Let the vertex set of $G$ be $V_0, V_1, \ldots, V_{m-1}$. Then the multigraph corresponding to $G$ is $M (G) \cong K_m (n)$. As $n$ is even, $K_m (n)$
is \( P_m \)-factorable by Corollary 4.2.6. But by the definition of multigraph, corresponding to each edge \( v_i v_j \) of \( P_m \) in the \( P_m \)-factorization of \( K_m(n) \), we have a 1-factor in \( G[V_i, V_j] \). Therefore corresponding to each \( P_m \)-factor in \( K_m(n) \) we have a \( \hat{P}_m \)-factor in \( G \). Thus, the \( P_m \)-factorizations of \( K_m(n) \) implies a \( \hat{P}_m \)-factorization of \( G \).

\textbf{Corollary 4.2.8.} For all even \( n \) and odd \( m \geq 3 \), \( \hat{P}_m \parallel (K_m \circ K_n)^* \).

\textbf{Remark 4.2.9.} The necessary condition for the existence of \( \hat{P}_m \)-factorization in \( K_m \circ K_n \) is \( n(m-1) \mid m(m-1)n^2/2 \). The above condition is not satisfied, when both \( m \) and \( n \) are odd.

\textbf{Theorem 4.2.10.} For all \( m \geq 2 \) and \( n \geq 1 \), \( \hat{P}_m \parallel (K_m \circ K_n) \) if and only if \( n(m-1) \mid m(m-1)n^2/2 \).

\textbf{Proof.} Necessity follows by counting the number of edges of a \( \hat{P}_m \)-factor and the graph \( K_m \circ K_n \). Sufficiency follows by Theorems 4.2.3 and 4.2.7.

From the Theorems 4.2.3, 4.2.7 and 4.2.10, we conclude the following:

\textbf{Note 4.4.} \( \hat{P}_m \parallel K_m \circ K_n \) if and only if \((m, n) \neq (2r + 1, 2s + 1) \).

\textbf{Theorem 4.2.11.} For \( m \geq 3 \) and odd \( n \geq 1 \), \( \hat{P}_m \parallel K_m \times K_n \).

\textbf{Proof.} Let \( K_m \times K_n = G \). Then \( M(G) \cong K_m(n-1) \). Rest of the proof follows from Corollary 4.2.6 and proof of Theorem 4.2.7.

\textbf{Remark 4.2.12.} The necessary condition for the existence of \( \hat{P}_m \)-factorization in \( K_m \times K_n \) is \( n(m-1) \mid m(m-1)n(n-1)/2 \). The above condition is not satisfied, when \( m \) is odd and \( n \) is even.
Theorem 4.2.13. \( \hat{P}_m \parallel K_m \times K_n \) if and only if \( n(m-1) \mid m(m-1)n(n-1)/2 \).

**Proof.** Necessity follows by counting the number of edges in a \( \hat{P}_m \)-factor and in the graph \( K_m \times K_n \). Sufficiency follows by Theorems 4.2.4 and 4.2.11 and Remark 4.2.12.

From the Theorems 4.2.4, 4.2.11 and 4.2.13, we conclude the following:

**Note 4.5.** \( \hat{P}_m \parallel K_m \times K_n \) if and only if \( (m, n) \neq (2r+1, 2s) \).

**Note 4.6.** \( \hat{P}_m \parallel (K_m \times K_n)^* \) for all \( (m, n) \neq (2r+1, 2s) \).

**Theorem 4.2.14.** For all \( m \geq 3 \) and odd \( n \geq 1 \), \( \hat{P}_m \parallel (K_m \circ K_n)^* \).

**Proof.** Let \( (K_m \circ K_n)^*(2) = G \). Then the multigraph \( M(G) \) of \( G \) is isomorphic to \( K_m(2n) \). By Theorem 4.2.3 and Corollary 4.2.8, \( P_m \parallel K_m(2n) \). Thus by the proof of Theorem 4.2.7, \( \hat{P}_m \parallel G \).

**Corollary 4.2.15.** For all \( m \geq 3 \) and odd \( n \geq 1 \),
\( \hat{P}_m \parallel (K_m \circ K_n)^*(2\lambda) \).

### 4.3 Gregarious Directed Path Factorizations

**Lemma 4.3.1.** If \( (m, n) \notin \{(3, 2r+1), (5, 2r + 1)\} \),
then \( \hat{P}_m \parallel (K_m \circ K_n)^* \).

**Proof.** By Note 4.1, we have \( \overrightarrow{P}_m \parallel K_m^* \) for all \( m \neq 3, 5 \). Therefore \( (K_m \circ K_n)^* = (K_m^* \circ K_n) = \oplus_{i=1}^{m} (H_i \circ K_n) \), where \( H_i \cong \overrightarrow{P}_m \), for \( m \neq 3, 5 \). By Note 4.2, \( \hat{P}_m \parallel H_i \circ K_n \) for each \( i = 1, 2, \ldots, m \).

Thus \( \hat{P}_m \parallel (K_m \circ K_n)^* \) for \( m \neq 3, 5 \). By Corollary 4.2.8, \( \hat{P}_m \parallel (K_m \circ K_n)^* \) for all \( (m, n) \in \{(3, 2r), (5, 2r)\} \). Thus \( \hat{P}_m \parallel (K_m \circ K_n)^* \) for \( (m, n) \neq \{(3, 2r+1), (5, 2r + 1)\} \).
Lemma 4.3.2. $\hat{P}_3 \parallel (K_3 \times K_2)^\ast$.

\textbf{Proof.} Let $V(K_3 \times K_2)^\ast = \bigcup_{i=1}^{3} V_i$, where $V_i = \{v_i^0, v_i^1\}$. Now $\hat{P}_3$-factors of $(K_3 \times K_2)^\ast$ are constructed as follows: Let

$$
\overline{H}_1 = (v_0^0 v_1^0 v_2^0) \oplus (v_1^1 v_0^1 v_2^1),
\overline{H}_2 = (v_1^0 v_0^0 v_2^1) \oplus (v_2^0 v_0^1 v_1^0) \quad \text{and}
\overline{H}_3 = (v_2^0 v_1^0 v_0^1) \oplus (v_0^0 v_1^0 v_2^1).
$$

Clearly each $\overline{H}_i$ is a $\hat{P}_3$-factor of $(K_3 \times K_2)^\ast$. Thus $\{\overline{H}_1, \overline{H}_2, \overline{H}_3\}$ is a $\hat{P}_3$-factorization of $(K_3 \times K_2)^\ast$. Hence the result. \hfill \Box

Theorem 4.3.3. $\hat{P}_3 \parallel (K_3 \times K_{2n})^\ast$.

\textbf{Proof.}

$$(K_3 \times K_{2n})^\ast = [K_3 \times (F_1 \oplus \cdots \oplus F_{2n-1})]^\ast
= (K_3 \times F_1)^\ast \oplus \cdots \oplus (K_3 \times F_{2n-1})^\ast
= \oplus_{i=1}^{2n-1} (K_3 \times F_i)^\ast,$$

where $F_i$ is a 1-factor of $K_{2n}$. As each $(K_3 \times F_i)^\ast \cong n (K_3 \times K_2)^\ast$, the result follows by Lemma 4.3.2. \hfill \Box

Lemma 4.3.4. $\hat{P}_5 \parallel (K_5 \times K_2)^\ast$.

\textbf{Proof.} Let $V(K_5 \times K_2)^\ast = \bigcup_{i=1}^{5} V_i$, where $V_i = \{v_i^0, v_i^1\}$. Now $\hat{P}_5$-factors of $(K_5 \times K_2)^\ast$ are constructed as follows: Let

$$
\overline{H}_1 = (v_0^0 v_1^0 v_2^0 v_3^0 v_4^0) \oplus (v_1^1 v_2^1 v_3^1 v_4^1 v_0^1),
\overline{H}_2 = (v_1^0 v_2^0 v_3^0 v_4^0 v_1^1) \oplus (v_2^0 v_3^0 v_4^0 v_0^1 v_1^1),
\overline{H}_3 = (v_2^0 v_3^0 v_1^0 v_4^1 v_0^1) \oplus (v_3^0 v_0^1 v_1^0 v_2^1 v_4^1).
$$
Clearly, each $\overrightarrow{H}_i$ is a $\hat{P}_5$-factor of $(K_5 \times K_2)^*$. Thus $\hat{P}_5 \parallel (K_5 \times K_2)^*$. □

**Theorem 4.3.5.** $\hat{P}_5 \parallel (K_5 \times K_{2n})^*$.

**Proof.**

\[
(K_5 \times K_{2n})^* = [K_5 \times (F_1 \oplus \cdots \oplus F_{2n-1})]^*
\]

\[
= (K_5 \times F_1)^* \oplus \cdots \oplus (K_5 \times F_{2n})^* \oplus (K_3 \times F_{2n-1})^*
\]

\[
= \oplus_{i=1}^{2n-1} (K_5 \times F_i)^* .
\]

As each $(K_5 \times F_i)^* \cong n (K_5 \times K_2)^*$, the result follows by Lemma 4.3.4. □

**Theorem 4.3.6.** $\hat{P}_m \parallel (K_m \times K_n)^*$ for all odd $m \geq 7$.

**Proof.** We know that $(K_m \times K_n)^* = K_m^* \times K_n^* = (\oplus \overrightarrow{P}_m) \times K_n^* = \oplus (\overrightarrow{P}_m \times K_n^*) = \oplus \overrightarrow{P}_m$, by Note 4.1 and 4.3. □

**Theorem 4.3.7.** $\hat{P}_m \parallel (K_m \times K_n)^*$ for all $m, n \geq 2$.

**Proof.** Follows from Theorems 4.3.3, 4.3.5, 4.3.6 and Note 4.6. □
4.4 Non-Existence of a Path Factorization

The following theorem proves the non-existence of a path factorization in product graphs.

**Theorem 4.4.1.** For \( r, s \geq 1 \), \( \hat{P}_{2s+1} \nmid (C_{2s+1} \circ \overline{K}_{2r+1})^* \).

\[ \begin{align*}
V_0 & \quad V_1 & \quad V_2 & \quad V_0 \\
V_1 & \quad V_2 & \quad V_0 & \\
V_2 & \quad V_0 & \quad V_1 & \\
V_0 & \quad V_1 & \quad V_2 & \\
\hline
G_1 & & & G_2
\end{align*} \]

**Figure 4.1:** \( G_1 \oplus G_2 = C_3^* \circ \overline{K}_3 \).

**Proof.** Let \( G^* = (C_{2s+1} \circ \overline{K}_{2r+1})^* = C_{2s+1}^* \circ \overline{K}_{2r+1} \). As there are \( 2s + 1 \) symmetric pair of arcs in \( C_{2s+1}^* \circ \overline{K}_{2r+1} \), total number of arcs in upward (respectively downward) direction in \( C_{2s+1}^* \circ \overline{K}_{2r+1} \) is \( (2s + 1)(2r + 1)^2 \), for example see Figure 4.1. One can observe that, combination of both upward and downward arcs cannot be used in the construction of \( \hat{P}_{2s+1} \) in \( G^* \), since the paths are directed gregarious, that is, each \( \hat{P}_{2s+1} \) can use either downward arcs or
upward arcs. Therefore $2s$, the number of arcs in a $\hat{P}_{2s+1}$ must divide $(2s + 1)(2r + 1)^2$, the total number of arcs in upward (respectively downward) direction, if a $\hat{P}_{2s+1}$-decomposition exists in $G^*$. But $2s \nmid (2s + 1)(2r + 1)^2$. Thus $\hat{P}_{2s+1} \nmid (C_{2s+1} \circ \overline{K}_{2r+1})^*$. □

**Remark 4.4.2.** Even though the number of arcs of $\hat{P}_{2s+1}$ divides the number of arcs in $(C_{2s+1} \circ \overline{K}_{2r+1})^*$, Theorem 4.4.1 shows that the obvious necessary condition is not sufficient. □

**Remark 4.4.3.** Lemma 4.3.1, Theorems 4.3.7 and 4.4.1 completes our requirement. □

**Question:** Does there exists a $\hat{P}_5$-factorization in $(K_5 \circ \overline{K}_{2r+1})^*$?

**Remark 4.4.4.** Positive answer to the above question implies the $\hat{P}_m$-factorization of $(K_m \circ \overline{K}_n)^*$, for all $m > 3$. □