CHAPTER 4

Chapter 4

Effect of magnetic field on buoyancy driven flow between parallel plane vertical walls

4.1. Introduction

Thermal convection in a fluid layer heated from below represents the simplest example of hydrodynamic instability and transition to turbulence in fluid system. In the idealized case of an infinitely extended layer with prescribed constant temperatures at the boundaries, the physical conditions are homogeneous and isotropic with respect to the horizontal dimensions.

The fluid is used as a coolant for industrial purposes. In most cases the coolant is forced over a hot surface which is cooled as a result. In other cases the fluid motion is generated because the fluid nearer the hotter region becomes higher and so gives rise to what is termed as natural or free convection. In both cases the underlying function of the moving fluid is to disperse the heat although the efficacy of such heat transfer is greatly enhanced with forced convection (Schmidt (1951)). Problems involving free convection in tubes have been studied by Lighthill (1953) and by Ostrach and Thornton (1958). Combined heat and mass transfer natural convection has previously been studied mostly for external flows.

A challenging problem faced by engineers and applied mathematicians working in a given field of study is to find solutions of the basic equations arising in that field while employing a minimum number of simplifying assumptions. Mostly, the basic equations expressing physical laws are partial differential equations. There are many problems in which solutions can not be found by usual classical methods (separation of variables, Laplace transforms etc.).

It is of interest to note that solutions of certain sets of partial differential equations occuring in applied fields can be found quite readily in spite of the failure of the more common classical methods to yield results. Notable among such solutions are those that have been obtained by employing transformations that reduce the system of partial differential equations. These solutions are generally designated as "similarity solutions".

Gebhart and Pera (1971) obtained similarity solutions for flow over a vertical plate with uniform temperature and concentration, and for point source or plume flows. Chen and Yuh (1979) obtained similarity solutions for inclined plates with either uniform heat or mass flux or uniform temperature and concentration boundary conditions.

Developing natural convection heat transfer between vertical parallel plates has been considered more extensively. Symmetric uniform wall temperature conditions were investigated experimentally by Elenbaas (1942) and numerically by Bodoia and Osterle (1962) and Aihara (1973). Miyatake and Fujii (1972), Sparrow *et al* (1984), Sparrow and Azevedo (1985) and Applebaum (1984) reported experimental results for uniform heat flux parallel plates in air for symmetric and asymmetric heating. Nelson and Wood (1989) presented a numerical analysis of developing laminar flow between vertical parallel plates for combined heat and mass transfer natural convection with uniform wall temperature and concentration boundary conditions. Garandet *et al* (1992) proposed an analytical solution to the equations of magnetohydrodynamics that can be used to model the effect of a transverse magnetic field on buoyancy driven convection in a two-dimensional cavity. Banks and Zaturska (1991) examined theoretically the two dimensional free convective flow in a parallel walled cell with vertical walls at non-uniform temperature.

In the present analysis, we extend the work of Banks and Zaturska (1991) to an electrically conducting fluid in the presence of a transverse magnetic field. We model the physical problem by specifying the non constant temperature on the walls. The temperature is presented in such a way that a similarity solution is possible which ultimately leads to a system of ordinary differential equations. Both symmetric and asymmetric boundary conditions are considered. We have investigated analytically the temporal stability of trivial steady symmetric solutions.

4.2. Mathematical formulation

We consider the two-dimensional unsteady flow of an incompressible electrically conducting fluid confined between two plane walls with velocity field $\vec{u}$ and temperature field T in an ambient magnetic field. We introduce a system of cartesian co-ordinates in such a way that the plane walls have equations $y = \pm h$, with the direction of Ox vertically upwards and the flow is assumed to be two- dimensional in the xy plane of the space Oxyz. Hence the governing system of equations of conservation of mass, momentum and energy is given by

$$\nabla . \vec{u} = 0 \tag{4.2.1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} . \nabla) \vec{u} = - \frac{\nabla p}{\rho} + \vec{F} + \nu \nabla^2 \vec{u} + \frac{\mu_m}{\rho} (\nabla \times \vec{H}) \times \vec{H} \tag{4.2.2}$$

$$\frac{\partial T}{\partial t} + (\vec{u} . \nabla) T = \alpha \nabla^2 T \tag{4.2.3}$$

where t denotes time, $\vec{F}$ is the body force and ν the kinematic viscosity of the fluid. In the equation of motion, Lorentz force is included among the other forces acting on the fluid. Viscous dissipation has been neglected in the heat equation. The fluid motion results solely from the buoyancy effects.

The Maxwell equations that express the interaction between the fluid motion and the magnetic fields are

$$\nabla . \vec{H} = 0 \tag{4.2.4}$$

$$\nabla \times \vec{H} = \vec{j} \tag{4.2.5}$$

$$\nabla \times \vec{E} = -\mu_m \frac{\partial \vec{H}}{\partial t} \qquad (4.2.6)$$

$$\vec{j} = \sigma(\vec{E} + \mu_m \vec{u} \times \vec{H}) \qquad (4.2.7)$$

where, in electrical units, $\vec{E}$ and $\vec{H}$ are the intensities of the electric and the magnetic fields, $\vec{j}$ is the current density, μ_m is the magnetic permeability and σ is the co-efficient of electrical conductivity. We have ignored the displacement current and charge density (Chandrasekhar (1981)).

From equations (4.2.5) - (4.2.7), we obtain the magnetic induction equation as

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) + \eta \nabla^2 \vec{B} \qquad (4.2.8)$$

where η is the resistivity.

Most of hydrodynamic stability theory is restricted to fluids of constant density for which a solenoidal velocity field can be assumed. Thermal convection is caused by temperature induced variations of density but the theoretical advantages of constant density fluids can be retained if the Boussinesq approximation of the Navier-Stokes equations of motion is used. In this approximation all the material properties are assumed to be constant with the exception of the temperature dependence of the density which is taken into account in the gravity term only. It is easy to satisfy the assumptions of Boussinesq approximation to a high degree of accuracy in laboratory experiments. To a lesser degree of accuracy, the theoretical description based on this approximation can be applied to convection in the atmosphere and other large-scale systems if the temperature difference across the convection layer is replaced by the excess over the adiabatic temperature difference.

By Boussinesq approximation, the body force $\vec{F}$ is taken as $\vec{F} = (g\beta(T-T_c), 0, 0)$ where T_c is a reference temperature and β is the volumetric expansion co-efficient. To eliminate pressure from equation (4.2.2), we take curl of (4.2.2) and get the vorticity equation as follows,

$$\frac{\partial \vec{w}}{\partial t} + (\vec{u} . \nabla)\vec{w} = (\vec{w} . \nabla)\vec{u} + g\beta \nabla \times T\hat{i} + \nu \nabla^2 \vec{w} + \frac{\mu_m}{\rho} \nabla \times ((\nabla \times \vec{H}) \times \vec{H}) \qquad (4.2.9)$$

where $\vec{w}$ is the curl of the velocity vector.

The fluid is confined between the planes $y = \pm h$; On these two planes certain boundary conditions must be satisfied. Regardless of the nature of these boundary surfaces, we must require

$$\begin{aligned} \vec{u} &= 0 && \text{on} \quad y = \pm h \\ T &= T_c(1 - h^{-1}x) && \text{on} \quad y = -h \\ T &= T_c(1 - \mu h^{-1}x) && \text{on} \quad y = h \\ \vec{H} &= 0 && \text{on} \quad y = \pm h \end{aligned} \qquad (4.2.10)$$

where μ is a non-dimensional parameter which represents the asymmetry of the imposed temperature on the walls which decrease with increasing x. The temperature conditions in (4.2.10) assume a more general form which includes a multiplicative factor of the term in x; this factor provides a dimensionless measure of the temperature variation.

To obtain dimensionless equations, it is convenient to introduce the thickness of the layer h as the length scale, and the thermal diffusion time h^2 / α as the time scale, where α is the thermal diffusivity. In addition, we shall use T_c and H_0 as scale of the temperature and magnetic field, respectively.

Hence the non-dimensional equation for the velocity vector $\vec{u}$ and heat equation for T and magnetic field $\vec{H}$ can be written in the form

$$\nabla . \vec{u} = 0 \qquad (4.2.11)$$

$$\frac{\partial T}{\partial t} + (\vec{u} . \nabla) T = \nabla^2 T \qquad (4.2.12)$$

$$Pr^{-1} \{ \frac{\partial \vec{w}}{\partial t} + (\vec{u} . \nabla) \vec{w} - (\vec{w} . \nabla) \vec{u} \}$$

$$= - Ra \nabla \times (T \hat{i}) + \nabla^2 \vec{w} + Q \nabla \times [\nabla \times \vec{H} \times \vec{H}] \qquad (4.2.13)$$

$$\frac{\partial \vec{H}}{\partial t} = \nabla \times (\vec{u} \times \vec{H}) + P_m \nabla^2 \vec{H} \qquad (4.2.14)$$

$$\nabla . \vec{H} = 0 \qquad (4.2.15)$$

The dependence of the problem on the material properties and the exterior conditions has been reduced to four dimensionless parameters, the Prandtl number, the magnetic Prandtl number, Rayleigh number and the magnetic pressure number

$$\Pr = \nu / \alpha; \quad P_m = \eta / \alpha; \qquad Ra = \frac{g\beta T_0 h^3}{\nu \alpha}; \qquad Q = \frac{\mu_m H_0^2 h^2}{\rho \alpha \nu} \tag{4.2.16}$$

Here g, ν and β denote the acceleration of gravity, kinematic viscosity and the co-efficient of thermal expansion.

The particular choice (4.2.16) of the dimensionless parameters of the problem offers the advantage that the onset of convection depends only on the Rayleigh number. The Prandtl number measures the relative importance of the advection of momentum and heat and thus affects the nonlinear properties of convection strongly.

To eliminate the equations (4.2.11) and (4.2.15), we introduce two stream functions ψ and ψ_m such that

$$\vec{u} = \left(\frac{\partial \psi}{\partial y}, \; - \frac{\partial \psi}{\partial x}, \; 0 \right) \text{ and } \vec{H} = \left(\frac{\partial \psi_m}{\partial y}, \; - \frac{\partial \psi_m}{\partial x}, \; 0 \right)$$

Thus equations (4.2.12) - (4.2.14) become

$$\frac{\partial T}{\partial t} + \frac{\partial (T, \psi)}{\partial (x, y)} = \nabla^2 T \tag{4.2.17}$$

$$\Pr^{-1} \left\{ \frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial (\nabla^2 \psi, \psi)}{\partial (x, y)} \right\} = - Ra \frac{\partial T}{\partial y} + \nabla^2 \psi + Q \frac{\partial (\nabla^2 \psi_m, \psi_m)}{\partial (x, y)} \tag{4.2.18}$$

$$\frac{\partial \psi_m}{\partial t} - P_m \nabla^2 \psi_m - \frac{\partial (\psi, \psi_m)}{\partial (x, y)} = 0 \tag{4.2.19}$$

where

$$\frac{\partial (a, b)}{\partial (x, y)} = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial a}{\partial y} \frac{\partial b}{\partial x}$$

The boundary conditions become

$$\psi_y = \psi_x = 0 \quad \text{on } y = \pm 1$$

$$T = x \quad \text{on } y = -1 \text{ and } T = \mu x \text{ on } y = 1$$

$$\psi_{mx} = 1 \quad \text{on } y = \pm 1 \tag{4.2.20}$$

where the subscripts x and y denote partial differentiation with respect to x and y respectively.

Because of the special form of the boundary conditions we now seek a similarity solution of the Hiemenz type

$$\psi(x,y,t) = x\, f(y,t)$$

$$T(x,y,t) = x\, g(y,t)$$

$$\psi_m\,(x,y,t) = x\, \phi(y,t) \tag{4.2.21}$$

From equations (4.2.19) - (4.2.21), it is apparent that

$$g_t + f_y\, g - f\, g_y = g_{yy}$$

$$Pr^{-1}\, \{f_{yyt} + f_y\, f_{yy} - f\, f_{yyy}\} = -\, Ra\, g_y + f_{yyyy} + Q\{\phi_{yy}\, \phi_y - \phi_{yyy}\, \phi\}$$

$$\phi_t - \{\, f\, \phi_y - f_y\, \phi\,\} = P_m\, \phi_{yy} \tag{4.2.22}$$

Thus the boundary conditions require that

$$f = f_y = 0 \quad \text{on } y = \pm 1$$

$$g = 1 \quad \text{on } y = -1 \quad g = \mu \quad \text{on } y = 1$$

$$\phi = 1 \quad \text{on } y = \pm 1 \tag{4.2.23}$$

We may assume without loss of generality that μ lies in the range [−1,1]; a simple rescaling may be applied to put μ in that range.

Steady state solutions correspond to

$$G F' - F G' = G''$$

$$Pr^{-1} \{F' F'' - F F'''\} = - Ra\, G' + F^{iv} + Q\,(\Phi'' \Phi' - \Phi''' \Phi)$$

$$\Phi F' - \Phi' F = P_m \Phi'' \qquad (4.2.24)$$

with boundary conditions

$$F = F' = 0 \qquad \text{on } y = \pm 1$$

$$G = 1 \qquad \text{on } y = -1; \qquad G = \mu \qquad \text{on } y = 1$$

$$\Phi = 1 \qquad \text{on } y = \pm 1 \qquad (4.2.25)$$

where the temperature and stream functions are decomposed into two parts as

$$f(y,t) = F(y) + f_1(y,t)$$

$$g(y,t) = G(y) + g_1(y,t)$$

$$\phi(y,t) = \Phi(y) + \phi_1(y,t) \qquad (4.2.26)$$

$F(y)$, $G(y)$, $\Phi(y)$ denote the steady state solutions and $f_1(y,t)$, $g_1(y,t)$, $\phi_1(y,t)$ describe small perturbations superposed on the initial variables.

We analyse the problem on hand using normal mode approach. We neglect terms that are bilinear in f_1, g_1 and ϕ_1. We assume that the dependence of quantities describing the perturbation on t is of the form exp(st) where s is the stability parameter which may be complex.

In accordance with the remarks in the preceding paragraph we suppose that the perturbations f_1, g_1, ϕ_1 have the forms:

$$f_1(y,t) = e^{st}\,\theta(y)$$

$$g_1(y,t) = e^{st}\,\xi(y)$$

$$\phi_1(y,t) = e^{st}\,\lambda(y) \qquad (4.2.27)$$

Hence the equations for the perturbations are

$$Pr^{-1}\{s\,\theta'' + F''\,\theta' + F'\,\theta'' - \theta\,F''' - F\,\theta'''\}$$

$$= -\,Ra\xi' + \theta^{iv} + Q\,\{\lambda'\,\Phi'' + \Phi'\,\lambda'' - \lambda\,\Phi''' - \Phi\,\lambda'''\}$$

$$s\,\xi - F\xi' + F'\,\xi + G\theta' - G'\,\theta = \xi''$$

$$s\,\lambda - \theta\,\Phi' - F\lambda' + \Phi\,\theta' + F'\,\lambda = P_m\,\lambda'' \qquad (4.2.28)$$

Solutions of these equations are to be sought so as to satisfy the boundary conditions

$$\theta = \theta' = \lambda = \xi = 0 \quad \text{at } y = \pm 1 \qquad (4.2.29)$$

We have formulated the eigenvalue problem that governs the temporal stability of the two-dimensional free convective flow in a parallel-walled cell with vertical walls at non-uniform temperature. The physical problem is modelled in a particular way so that a similarity solution is possible which results in the Navier-stokes and temperature equations being reduced to a sixth-order system of ordinary differential equations. The problem involves four parameters Ra, the Rayleigh number, Pr, the Prandtl number, P_m, magnetic Prandtl number and μ, a parameter which measures the asymmetry of the imposed temperature on the cell walls.

When $\mu = 1$, the problem is symmetrical about the central plane of the cell. In the absence of magnetic field, it has been shown by Banks and Zaturska (1991) that when $\mu = 1$ in addition to the trivial symmetric solution, there exists a non-trivial symmetric solution for all Ra. We have presented analytical results concerning the frequency and velocity of the perturbed flow for small Ra. Also, the effects of a cell with asymmetrically heated walls corresponding to $\mu \neq 1$ will also be considered in the following sections.

4.3. Some analytical results

It can be seen from (4.2.25) that the boundary conditions are symmetric about the center plane of the channel corresponding to the special case $\mu = 1$. For the particular value of $\mu = 1$, equation (4.2.24) admits the trivial symmetric solution

$$F = 0, \quad G = 1, \quad \Phi = 1 \tag{4.3.1}$$

for all Prandtl and Rayleigh numbers. For Ra < 0, which is equivalent to the wall temperature increasing with height, we expect such a solution to be stable. However, for Ra > 0 which corresponds to wall temperature decreasing with height, we anticipate that this solution will become unstable at some value of Ra. The occurrence of such an instability leads us to expect other, possibly stable, solutions.

We restrict our attention to the special case Pr =1 =P_m. With the trivial steady basic flow given in (4.3.1), equation (4.2.28) will become

$$s\,\theta'' = -\,Ra\,\xi' + \theta^{iv} - Q\,\lambda''' \tag{4.3.2}$$

$$s\,\xi + \theta' = \xi'' \tag{4.3.3}$$

$$s\,\lambda + \theta' = \lambda'' \tag{4.3.4}$$

with boundary conditions

$$\theta = \theta' = \lambda = \xi = 0 \qquad \text{on } y = \pm 1.$$

We see from equations (4.3.2) - (4.3.4) that for $P_m = 1 = Pr$, perturbations ξ and λ of the temperature and magnetic field satisfy the same equations from which it can be deduced easily that ξ and λ are identical. Hence elimination of λ and ξ between the above equations leads to a single equation in θ as

$$\theta^{vi} - (2s + Q)\,\theta^{iv} + (s^2 - Ra)\,\theta'' = 0 \tag{4.3.5}$$

We observe that it follows from the evenness of the operator, the proper solutions of equation (4.3.5) fall into two non-combining groups of even and odd solutions.

It is evident that the general solution of equation (4.3.5) can be expressed as a superposition of solutions of the form

$$\theta = e^{\pm qz}$$

where q is a root of the equation

$$q^2\{q^4 - (2s+Q)\, q^2 + (s^2 - Ra)\} = 0 \qquad (4.3.6)$$

We find that the roots of equation (4.3.6) are given by

$$q^2 = 0, \qquad \text{and} \qquad q^2 = \frac{1}{2}\,[2s + Q \pm\sqrt{4sQ + Q^2 + 4Ra}]$$

or taking the square roots, we have the six roots

$$0,\ 0,\ \pm R_1,\ \pm R_2 \ \text{ where } R_{1,2} = \left(\frac{2s + Q \pm \sqrt{4sQ + Q^2 + 4Ra}}{2}\right)^{1/2} \qquad (4.3.7)$$

i) Even solutions

Considering the even solutions, we can clearly write

$$\theta = C_1 \cosh(R_1\, y) + C_2 \cosh(R_2\, y) + C_3$$

$$\xi = \lambda = C_2 \frac{R_2}{R_2^2 - s} \sinh(R_2\, y) + C_1 \frac{R_1}{R_1^2 - s} \sinh(R_1\, y) \qquad (4.3.8)$$

where C_1, C_2 and C_3 are arbitrary constants to be determined and R_1 and R_2 are given in (4.3.7).

Now, to determine the arbitrary constants we impose the boundary conditions that θ, θ' and λ vanish at the boundaries i.e.,

$$\theta = \theta' = \lambda = 0 \qquad \text{on } y = \pm 1 \qquad (4.3.9)$$

Thus the boundary conditions require

$$\begin{bmatrix} \cosh(R_1) & \cosh(R_2) & 1 \\ R_1\sinh(R_1) & R_2\sinh(R_2) & 0 \\ \frac{R_1}{R_1^2 - s}\sinh(R_1) & \frac{R_2}{R_2^2 - s}\sinh(R_2) & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = 0 \tag{4.3.10}$$

For a nonvanishing solution the determinant of the matrix in (4.3.10) must vanish. Hence the eigenvalues are given by

$$s = \pm \sqrt{Ra}$$

$$s = - [\frac{Q}{4} + \frac{Ra}{Q}]$$

$$\text{and} \quad s = - in\pi \pm \sqrt{Ra - Q\, in\pi} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{4.3.11}$$

We note that when $Ra < 0$, we have a pair of neutral mode and when $Ra > 0$, we have one stable mode and one unstable mode. It can also be noted that increasing Ra increases the region of stability.

When Q approaches zero, for positive Ra, another family of eigenvalues occur in complex conjugate pairs whose real part is equal to $\pm \sqrt{Ra}$. Effect of the magnetic parameter Q on the stability characteristics are discussed numerically in Section (4.4.).

ii) Odd solutions

Considering next the odd solutions, we now have

$$\theta \;=\; C_4 \sinh(R_1 y) + C_5 \sinh(R_2 y) + C_6 y \tag{4.3.12}$$

$$\xi \;=\; \lambda \;=\; C_4 \frac{R_1}{R_1^2 - s} \cosh(R_1 y) + C_5 \frac{R_2}{R_2^2 - s} \cosh(R_2 y) - \frac{C_6}{s} \tag{4.3.13}$$

The characteristic determinant which follows from this solution is

$$\begin{vmatrix} \sinh(R_1) & \sinh(R_2) & 1 \\ R_1 \cosh(R_1) & R_2 \cosh(R_2) & 1 \\ \frac{R_1\, s}{R_1^2 - s}\cosh(R_1) & \frac{R_2\, s}{R_2^2 - s}\cosh(R_2) & -1 \end{vmatrix} \;=\; 0 \,. \tag{4.3.14}$$

and this leads to the equation

$$R_2^3\,[R_1 \cosh(R_1) - \sinh(R_1)] \cosh(R_2)\,(R_1^2 - s)$$
$$- \; R_1^3\,[R_2 \cosh(R_2) - \sinh(R_2)] \cosh(R_1)\,(R_2^2 - s) \;=0 \tag{4.3.15}$$

which expresses the stability parameter s as a function of Ra and Q.

The information contained in equation (4.3.15) is not obvious and can be obtained from numerical investigation.

Now we examine a particular case, θ identically zero. From the equations of λ and ξ, the eigenvalues are given by

$$s \;= - n^2\pi^2$$

$$\text{and} \qquad s \;= - \left[\frac{(2n+1)\,\pi}{2}\right]^2 \qquad n \;= \pm 1, \pm 2, \ldots \tag{4.3.16}$$

The corresponding eigenfunctions for these modes are given by

$$\xi = \lambda = a \sin(n\pi y)$$

$$\text{and} \quad \xi = \lambda = b \cos((2n+1)\pi y / 2)$$

where a and b are normalising factors. It may be observed that, as n increases, the two types of modes become very close.

4.3.1. The case $\mu \neq 1$, Ra small

Here, we discuss the effects of a cell with asymmetrically heated walls corresponding to $\mu \neq 1$. For small Ra, we seek series solution of the form

$$G = G_0(y) + Ra\, G_1(y) + \ldots$$

$$F = RaF_1(y) + Ra^2 F_2(y) + \ldots$$

$$\Phi = \Phi_0(y) + Ra\, \Phi_1(y) + \ldots \tag{4.3.17}$$

subject to the boundary conditions

$$G_0(-1) = 1 \qquad G_0(1) = \mu$$

$$\Phi_0(\pm 1) = 1 \qquad F_0(\pm 1) = 0$$

$$G_n(\pm 1) = \Phi_n(\pm 1) = F_n(\pm 1) = F_n'(\pm 1) = 0 \qquad n \geq 1$$

On solving we get

$$G_0(y) = -\left(\frac{1-\mu}{2}\right) y + \left(\frac{1+\mu}{2}\right)$$

$$F_1(y) = k_1 + k_2 \cosh(\sqrt{Q/P_m}\, y) - \frac{P_m}{Q}\left(\frac{\mu-1}{2}\right)\frac{y^2}{2}$$

$$G_1(y) = k_3 + k_4 y$$

$$+ k_2 \left(\frac{\mu-1}{2}\right)\sqrt{P_m/Q}\,[y \sinh(\sqrt{Q/P_m}\, y)$$

$$-\sqrt{P_m/Q}\, \cosh\sqrt{Q/P_m}(y)]$$

$$+ k_2 \left(\frac{1+\mu}{2}\right)\sqrt{P_m/Q}\, \sinh(\sqrt{Q/P_m}\, y) - \frac{3}{2}\frac{P_m}{Q}\left(\frac{\mu-1}{2}\right)^2 \frac{y^4}{12}$$

$$- \frac{P_m}{Q}\left(\frac{\mu^2-1}{4}\right)\frac{y^3}{6} + k_2 \left(\frac{\mu-1}{2}\right)\frac{y^2}{2}$$

$$\Phi_0(y) = 1$$

$$\Phi_1(y) = k_5 y + k_2 \frac{1}{\sqrt{P_m Q}} \sinh(\sqrt{Q/P_m}\, y) - \frac{1}{Q}\left(\frac{\mu-1}{2}\right)\frac{y^3}{6}$$

where

$$k_1 = \frac{P_m}{Q}\left(\frac{\mu-1}{2}\right)\left[\frac{1}{2} - \sqrt{P_m/Q}\coth(\sqrt{Q/P_m}\, y)\right]$$

$$k_2 = \left(\frac{P_m}{Q}\right)^{3/2}\left(\frac{\mu-1}{2}\right)\frac{1}{\sinh\sqrt{Q/P_m}}$$

$$k_3 = k_2 \left(\frac{\mu-1}{2}\right)\sqrt{P_m/Q}\left[\sqrt{P_m/Q}\, \cosh(\sqrt{Q/P_m}) - \sinh(\sqrt{Q/P_m})\right]$$

$$+ \frac{3}{2}\frac{P_m}{Q}\left(\frac{\mu-1}{2}\right)^2 \frac{1}{12} - \frac{1}{2} k_1\left(\frac{\mu-1}{2}\right)$$

$$k_4 = -\left(\frac{\mu+1}{2}\right) k_2\sqrt{P_m/Q}\ \sinh(\sqrt{Q/P_m}) + \frac{P_m}{Q}\left(\frac{\mu^2-1}{4}\right)\frac{1}{6}$$

$$k_5 = -\ k_2 \frac{1}{\sqrt{P_m Q}}\sinh(\sqrt{Q/P_m}) + \frac{1}{Q}\left(\frac{\mu-1}{2}\right)\frac{1}{6}$$

In the same manner we proceed to find the perturbed solutions in the form

$$F(y) = F_0(y) + Ra\ F_1(y) + \dots$$

where F stands for θ, λ, ξ and

$$s = -\ s_0 + Ra\ s_1 + Ra^2\ s_2 + \dots$$

Hence the equation governing the leading order term of the velocity profile θ_0 is given by

$$P_m\ \theta^{vi} + (s_0 + Pr^{-1}\ P_m\ s_0 - Q)\ \theta^{iv} + Pr^{-1}\ s_0^2\ \theta'' = 0 \qquad (4.3.19)$$

Hence the eigenfuntions consists of three families: There are modes that are antisymmetric, symmetric and asymmetric. These solutions are cited below:

When $\theta_0 = 0$

$$s_0 = P_m\ n^2\ \pi^2 \qquad n = \pm 1, \pm\ 2, \dots$$

$$\lambda_0 = a\ \sin(n\pi y)$$

$$\xi_0 = b\ \sin(\sqrt{P_m}\, n\pi y) \qquad \text{if } P_m = \left(\frac{m}{n}\right)^2 \qquad m = \pm 1, \pm\ 2, \dots$$

$$= b\ \cos(\sqrt{P_m}\, n\pi y) \qquad \text{if } P_m = \left(\frac{2m+1}{4\pi^2}\right)^2 \qquad n = \pm 1, \pm\ 2, \dots$$

$$= 0 \qquad \text{otherwise} \qquad (4.3.20)$$

where a and b are normalising factors.

Another family of eigenvalues and eigenfunctions corresponding to the solution $\theta_0 = 0$ is

$$s_0 = P_m \left(\frac{2n+1}{2}\right)^2 \pi^2 \qquad n = \pm 1, \pm 2, \ldots$$

$$\lambda_0 = a \cos\left(\left(\frac{2n+1}{2}\right) \pi y\right)$$

$$\xi_0 = b \cos\left(\sqrt{P_m} \left(\frac{2n+1}{2}\right) \pi y\right)$$

$$\text{if} \quad P_m = \left(\frac{2m+1}{2n+1}\right)^2 \qquad n = \pm 1, \pm 2, \ldots$$

$$= b \sin\left(\sqrt{P_m}\left(\frac{2n+1}{2}\right) \pi y\right)$$

$$\text{if} \quad P_m = \frac{4 m^2}{(2n+1)^2} \qquad n = \pm 1, \pm 2, \ldots$$

$$= 0 \qquad \text{otherwise} \tag{4.3.21}$$

We can note from the above solutions that the results obtained for general values of Pr and P_m are not fundamentally different from the results obtained on the assumption of unit Prandtl number and magnetic Prandtl number. Also, it may easily be noted that the above obtained modes are independent of Pr and the effect of P_m on the eigenvalues of the asymmetric modes is a multiplicative one.

Taking

$$\theta_0 = C_2 y + C_4 \sin(R_1 y) + C_6 \sin(R_2 y)$$

the leading order terms of the temperature and magnetic field perturbations are obtained as

$$\lambda_0 = \frac{C_2}{s_0} + \frac{R_1}{s_0 - R_1^2} C_4 \cos(R_1 y) + \frac{R_2}{s_0 - P_m R_2^2} C_6 \cos(R_2 y)$$

$$\xi_0 = a \sin(\sqrt{s_0} y) + \frac{C_2}{s_0} + \frac{R_1}{s_0 - R_1^2} C_4 \cos(R_1 y)$$
$$+ \frac{R_2}{s_0 - R_2^2} C_6 \cos(R_2 y) \qquad (4.3.22)$$

where

$$a = -\frac{1}{\cos(\sqrt{s_0})} \left[\frac{C_2}{s_0} + C_4 \frac{R_1}{s_0 - R_1^2} \cos(R_1) + C_6 \frac{R_2}{s_0 - R_2^2} \cos(R_2) \right]$$

This set of solution, on imposing boundary conditions yields the eigenvalue relation

$$(s_0 - P_m R_3^2) R_4^3 \cos(R_4) [\sin(R_3) - R_3 \cos(R_3)]$$
$$- (s_0 - P_m R_4^2) R_3^3 \cos(R_3) [\sin(R_4) - R_4 \cos(R_4)] = 0$$
$$(4.3.23)$$

Where

$$R_{3,4} = \left\{ \frac{-[s_0 + Pr^{-1} s_0 - Q] \pm \sqrt{[s_0 + Pr^{-1} s_0 - Q]^2 - 4Pr^{-1} s_0^2}}{2P_m} \right\}^{1/2}$$

Corresponding to the solution

$$\theta_0 = C_1 + C_3 \cos(R_1 y) + C_5 \cos(R_2 y)$$

we obtain

$$\lambda_0 = -C_3 \frac{R_1}{s_0 - R_1^2 P_m} \sin(R_1 y) - C_5 \frac{R_2}{s_0 - R_2^2 P_m} \sin(R_2 y)$$

From which the eigenvalues are determined as

$$s_0 = \frac{Q}{(1 \pm \sqrt{Pr^{-1} P_m})^2} \tag{4.3.24}$$

$$s_0 = \frac{1}{2\, Pr^{-1} P_m} \{n^2 \pi^2 P_m (1 + Pr^{-1} P_m)$$

$$\pm \sqrt{n^4 \pi^4 P_m^2 (1 + Pr^{-1} P_m)^2 - 4 Pr^{-1} P_m n^2 \pi^2 [n^2 \pi^2 P_m^2 - P_m Q]}\} \tag{4.3.25}$$

ξ_0 is given by

$$\xi_0 = a \sin(\sqrt{s_0}\, y) - C_3 \frac{R_1}{s_0 - R_1^2} \sin(R_1 y) - C_5 \frac{R_2}{s_0 - R_2^2} \sin(R_2 y)$$

where

$$a = \frac{1}{\sin(\sqrt{s_0})} [\, C_3 \frac{R_1}{s_0 - R_1^2} \sin(R_1) + C_5 \frac{R_2}{s_0 - R_2^2} \sin(R_2)\,]$$

4.4. Numerical analysis

We supplement the results obtained in the previous sections by some numerical results.

In the past, the stability of flow was judged only by the criteria of linear theory. In some cases the predictions of the spectral problem of linearized theory would lead to good agreement between theory and experiments and in other cases the agreement between theory and experiment was less good or no good.

Now it is understood that a flow which is stable by the criteria of the linearized theory need not be stable at all. Understanding the main physical features of the instability of these flows require analysis of the nonlinear problem. Though a fully general non-linear theory is not known, we can learn something about the full problem by considering the kinds of solutions which can develop as a result of the instability.

Integrating the steady basic flow defined by equations (4.2.24) subject to the symmetric boundary conditions (4.2.25) with $\mu = 1$, we have presented in figures (4.1) - (4.3), $F''(-1)$ which is related to skin friction at the wall $y = -1$, at representative values of Ra, for Q = 0.0, 1.0, 5.0. From figures (4.1) - (4.3), it can be noted that the trivial solution $G = 1$, $F = 0$ and $\Phi = 0$ is represented by Ra - axis.

We see that, for the Rayleigh number around 100, the pair of asymmetric solutions that arise at $Ra = Ra_1$ coalsce at $Ra = Ra_{1,1}$ to form a single non- trivial solution there. In other words, a subcritical pitchfork bifurcation is formed at $Ra_{1,1}$.

It may also be noted that as $Ra \rightarrow 0$ $F''(-1) \rightarrow \infty$, so that there is blow-up in the skin friction. Also, it may be found from figures (4.1) - (4.3), that for $Ra < Ra_{1,1}$, no further bifurcations were seen. We also observe that increase in the magnetic field increases the skin friction.

Next we summarise some results that are related to eigenvalues. For symmetric boundary conditions, $\mu = 1$ we have three families of eigensolutions : symmetric, antisymmetric and asymmetric.

When we consider the symmetric solutions, there are more than one mode. As mentioned earlier, we have two modes, one stable and another unstable, whose frequency is $\sqrt{Ra}$. To study the effect of Q on s more clearly, we have expressed

$$s = -\left[\frac{Q}{4} + \frac{Ra}{Q}\right]$$

as a function of Q in figure (4.4) for various values of Rayleigh number, Ra = 1.0, 5.0, 10.0. We see that as Q increases, s becomes constant. Here we have stable modes.

When $s = -in\pi \pm \sqrt{Ra - Qin\pi}$, real part of s as a function of Q is plotted in figure (4.5). We can easily observe that we have got one stable mode and another unstable mode. In both the cases, it can be noted that increasing Ra increases the value of the real part of eigen values and hence increase the region of stability.

In the case of asymmetric modes, we find from the θ equation the eigenvalues

$$s = -n^2\pi^2 \qquad \text{for } n = 1, 2, \ldots \qquad \text{and}$$

$$s = -\left[\frac{(2n+1)\pi}{2}\right]^2 \qquad \text{for } n = 1, 2, \ldots$$

and we note as n increases, both the modes become very close.

In the above mentioned solutions, the analysis is made simple by assuming the special value 1 for Prandtl number and magnetic Prandtl number. The case for general values of Pr and P_m is not fundamentally different and we give some results here for the case of asymmetrically heated cell i.e. $\mu \neq 1$.

In the case of vanishing Rayleigh numbers, we have presented the numerical results corresponding to the families of eigensolutions which are symmetric in figures (4.6) - (4.10). In the symmetric case, we have four families of eigenvalues. From figures (4.6) and (4.7), it can be elucidated that increasing values of Pr and P_m increase the region of stability of the eigenvalues given by

$$s = -\frac{Q}{(1+\sqrt{Pr^{-1}P_m})^2}$$

As can be expected from the above equation s varies linearly with Q. In figure (4.8), variation of s as a function of P_m is illustrated. In figures (4.9) and (4.10), we have dealt with the dependence of the eigenvalue

$$s = -\frac{Q}{(1-\sqrt{Pr^{-1}P_m})^2}$$

on the Prandtl number and magnetic Prandtl number. Magnitude of s decreases with increasing Pr and P_m.

In figures (4.11) - (4.18), we have presented the eigenvalues related to the modes given by (4.3.22). We have two pairs of real eigenvalues. The frequency of the waves given by

$$s = -\frac{1}{2\,Pr^{-1}\,P_m}\{n^2\pi^2 P_m\,(1 + Pr^{-1}\,P_m) + \sqrt{n^4\pi^4P_m^2\,(1 + Pr^{-1}\,P_m)^2 - 4Pr^{-1}P_m n^2\pi^2\,[n^2\pi^2 P_m^2 - P_m\,Q]}\}$$

increases as P_m increases and decreases as Q increases. In figures (4.13) and (4.14), we have depicted the variation of s as a function of P_m and Pr respectively at a fixed Q (= 5.0).

We have a family of unstable waves corresponding to the eigenvalues

$$s = -\frac{1}{2\,Pr^{-1}\,P_m}\{n^2\pi^2 P_m\,(1 + Pr^{-1}\,P_m) - \sqrt{n^4\pi^4P_m^2\,(1 + Pr^{-1}\,P_m)^2 - 4Pr^{-1}P_m n^2\pi^2\,[n^2\pi^2 P_m^2 - P_m\,Q]}\}$$

From figures (4.15) and (4.16), it may be understood that increasing P_m and Pr destabilizes the system. In figure (4.17), we observe that when Pr is small, s value increases by a increase in P_m. When Pr is large, the effect of P_m on s is negligible.

Similarly, from figure (4.18), we can observe that when P_m is small, s increases with Pr whereas for a large P_m s remains unaffected by Pr.

We note that the asymmetric modes are independent of Pr which is consistant with the result obtained by Banks and Zaturska (1991). We also note that the effect of P_m on the eigenvalues is a multiplicative one.

4.5. Conclusion

Two-dimensional unsteady flow of an incompressible, electrically conducting fluid confined in a parallel walled cell with vertical walls at non-uniform temperature was examined. Also, the variation in temperature has been taken in such a way that the similarity solution for the flow and temperature fields are possible. We have considered both symmetric and asymmetric boundary conditions. Corresponding to symmetric conditions, we have found eigensolutions related to trivial steady solutions. In the case of asymmetric boundary conditions, eigenfunctions and eigenvalues are found for the case of vanishing Ra. The analytical results obtained are discussed with the help of numerical analysis.

The steady state solutions were examined numerically for symmetric boundary conditions. It was observed that the presence of magnetic field enhances the skin friction at the walls.

When $\mu = 1$, increasing Ra increases the region of stability. The effect of magnetic field is to increase the frequency of stable and unstable waves.

When $\mu \neq 1$, for symmetric modes we have two pairs of eigenvalues which increase as P_m increases and decreases as Q increases.

Asymmetric modes are independent of Pr and the effect of P_m is a multiplicative one.

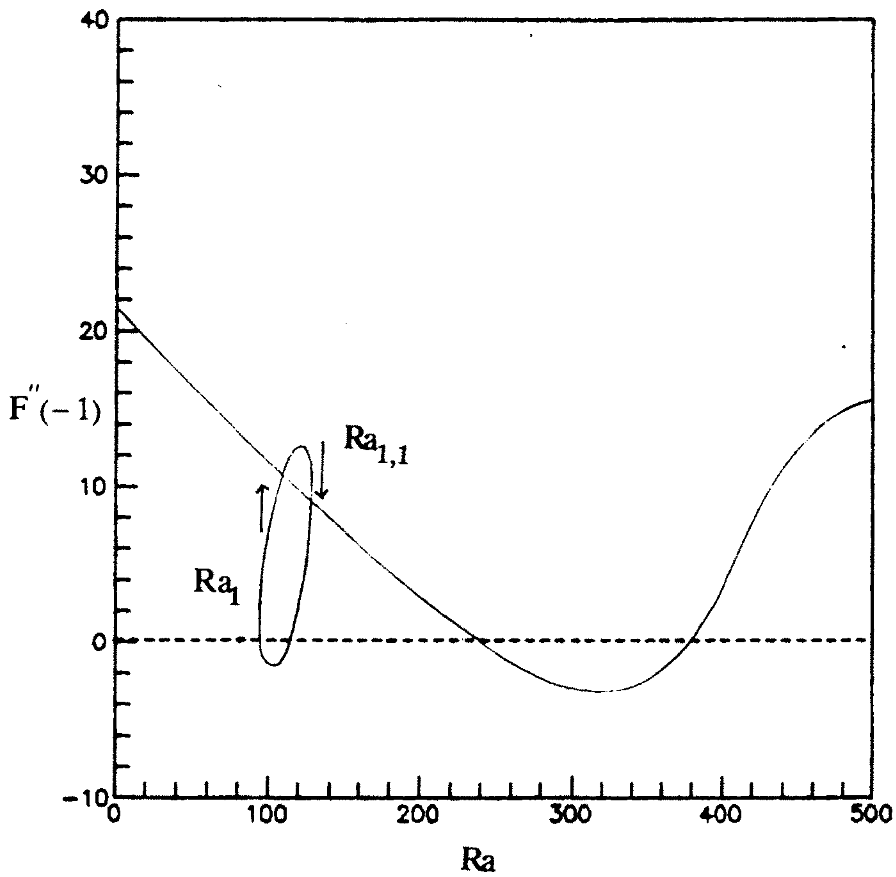

Fig. 4.1. Skin friction F″(–1) versus Ra for Q = 0.0

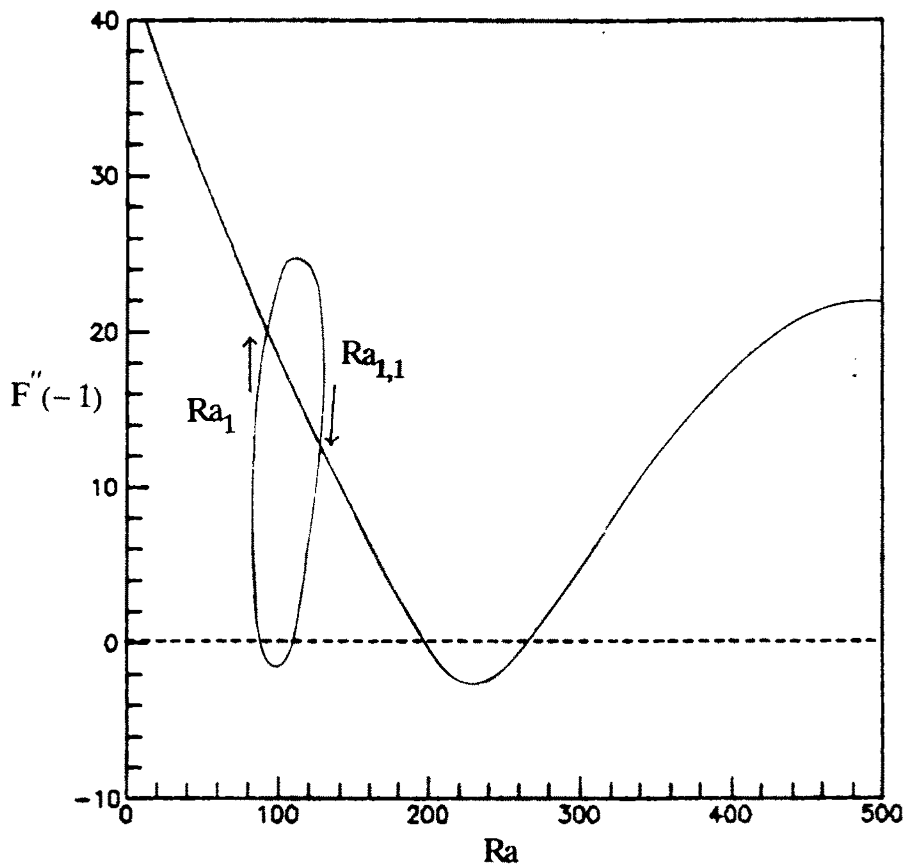

Fig. 4.2. Skin friction F''(–1) versus Ra for Q = 0.1

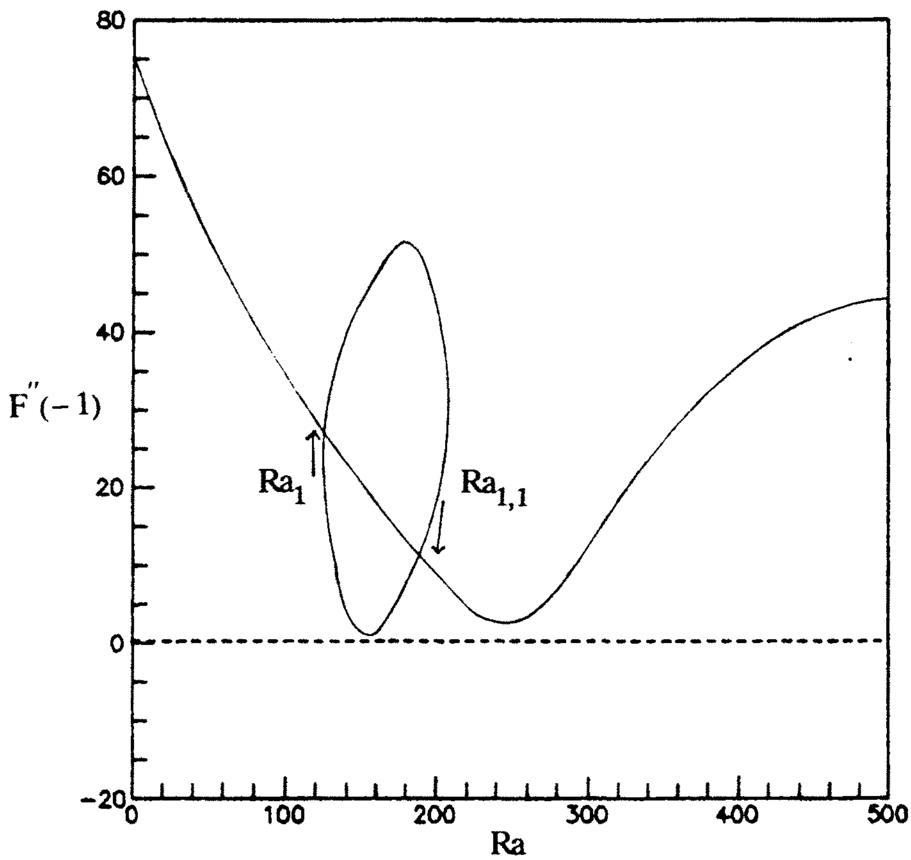

Fig. 4.3. Skin friction F''(−1) versus Ra for Q = 5.0

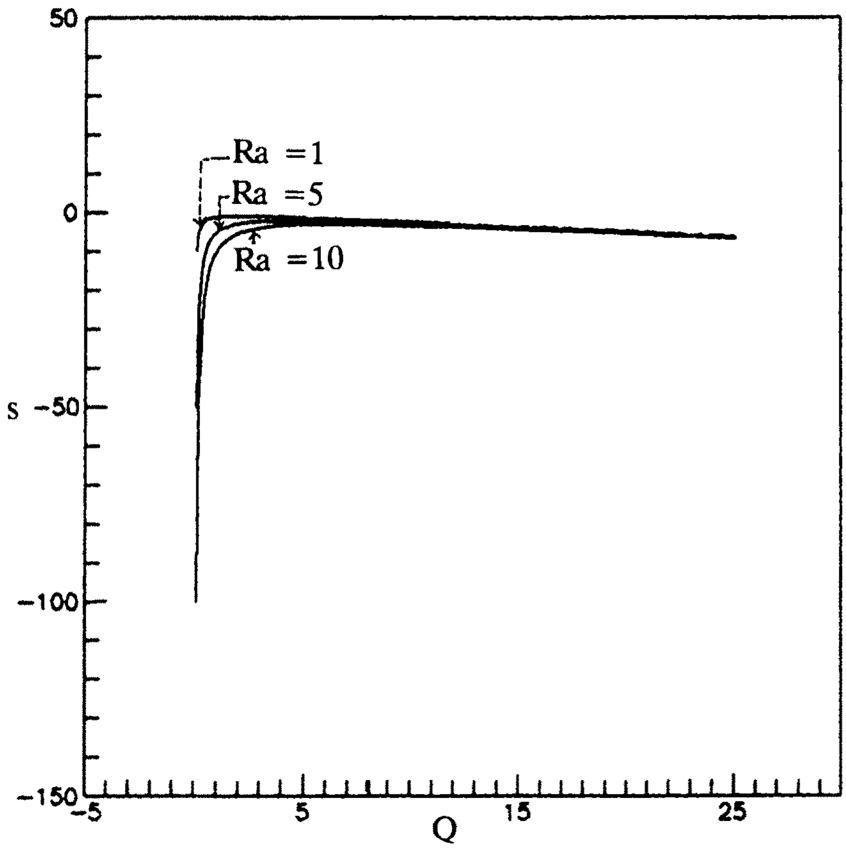

Fig. 4.4. Contours of real parts of eigenvalues for various values of Ra (Pr = P_m = 1,symmetric solutions)

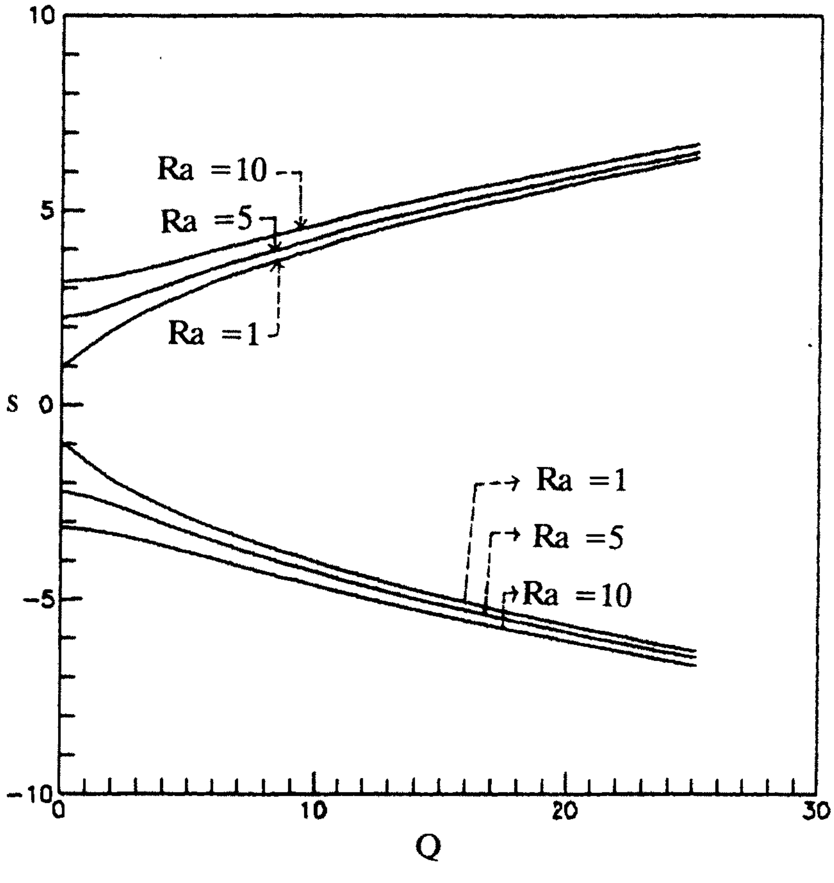

Fig. 4.5. Contours of real parts of eigenvalues for various values of Ra (Pr = P_m = 1,symmetric solutions)

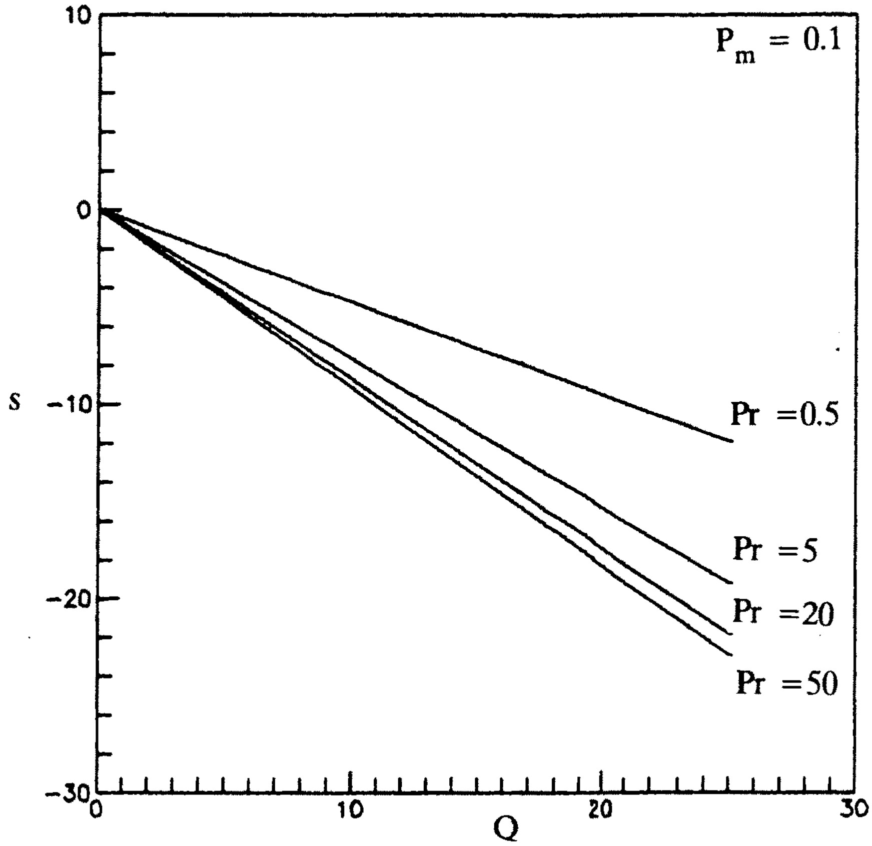

Fig. 4.6. Rate of decay as functions of Q'

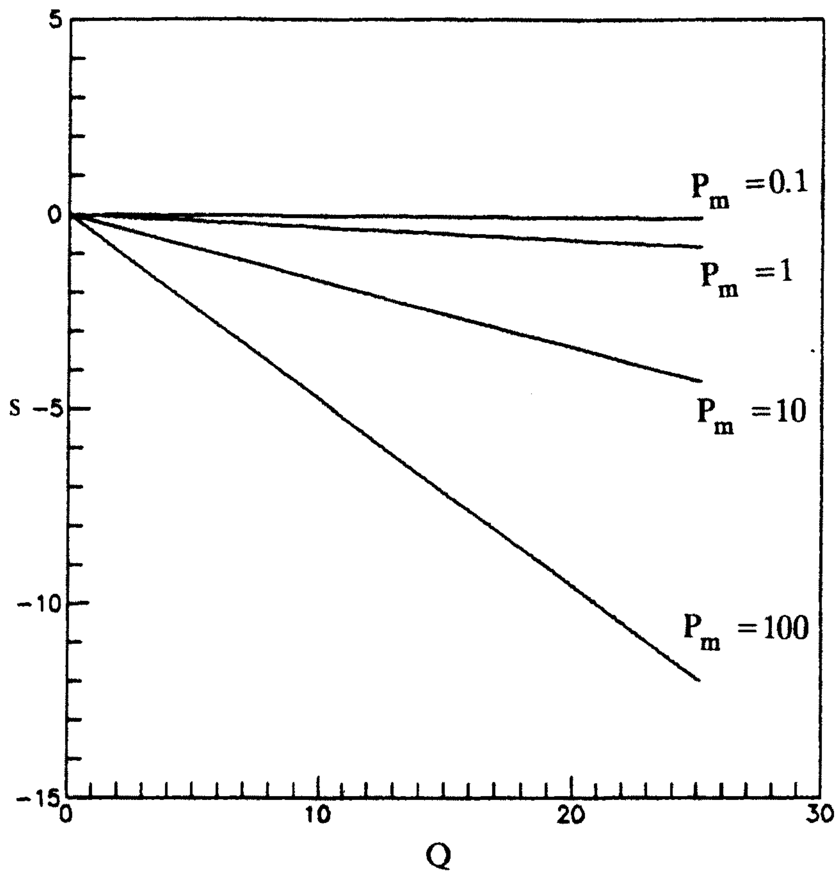

Fig. 4.7. Rate of decay as functions of Q (Pr = 0.5)

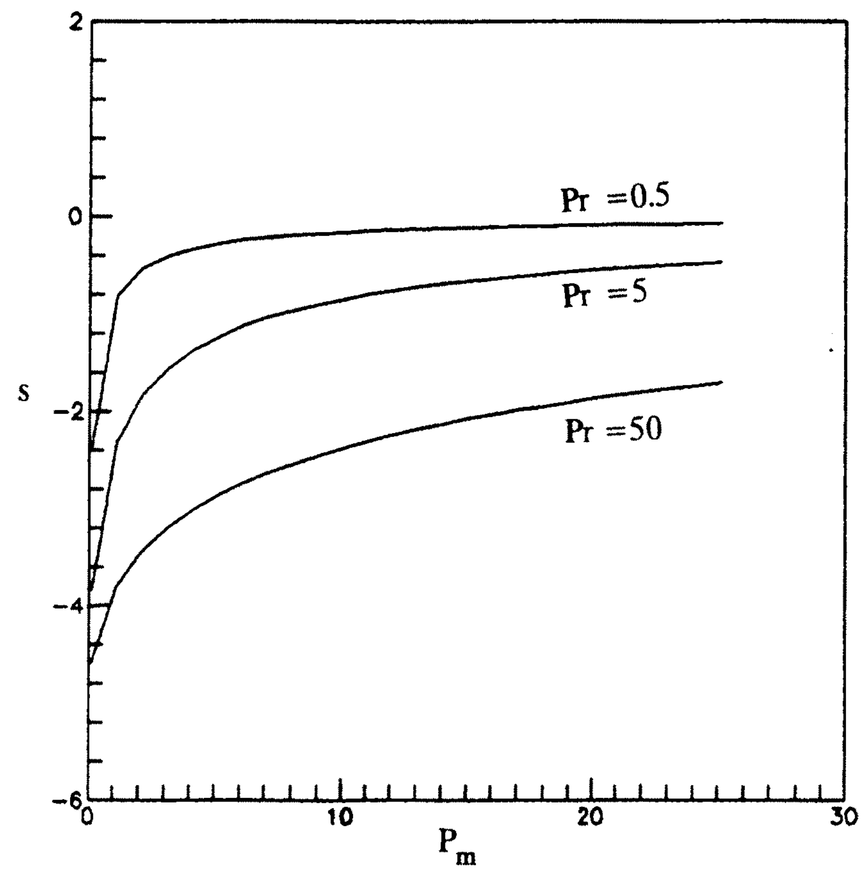

Fig. 4.8. Variation of growth rate with respect to P_m

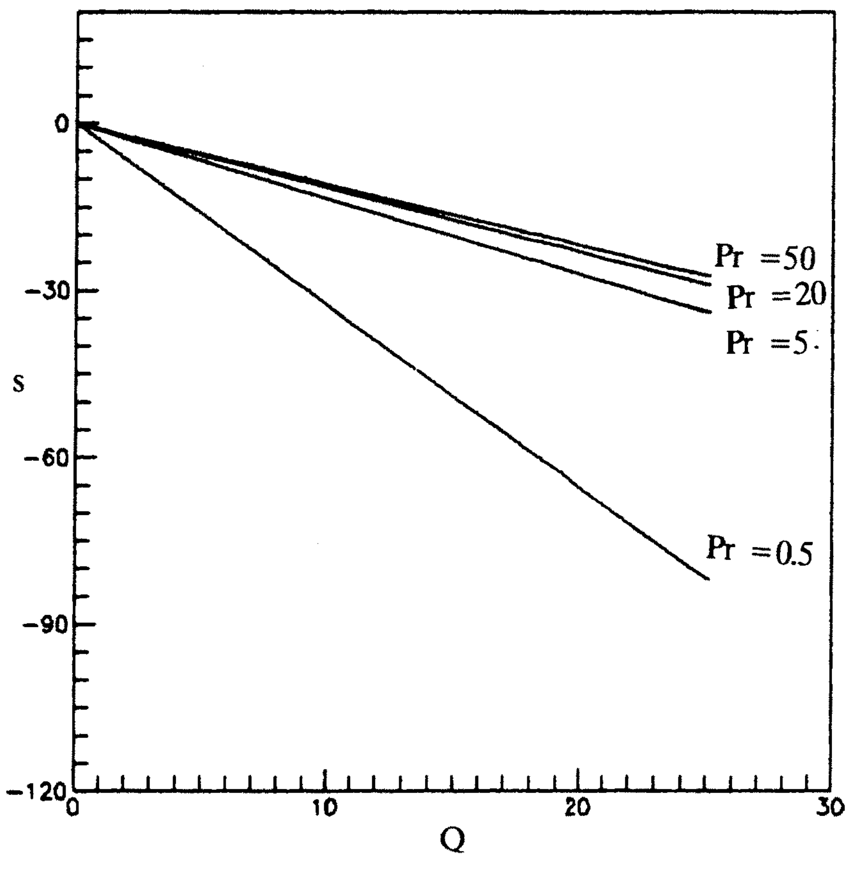

Fig. 4.9. Frequency for various values of Pr

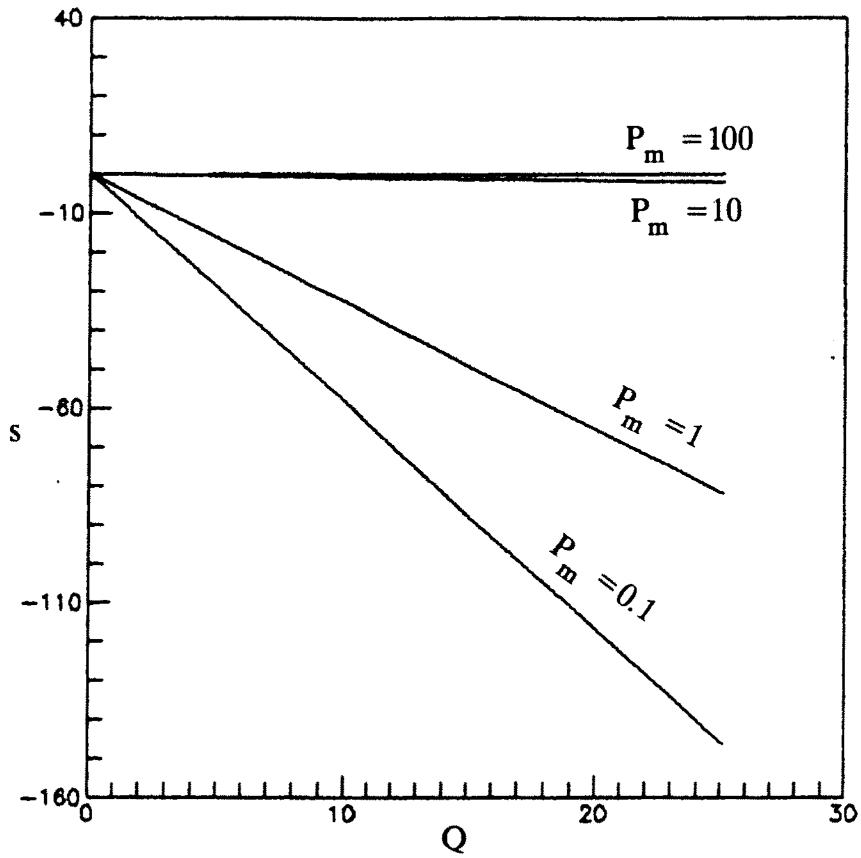

Fig. 4.10. Frequency as a function of Q

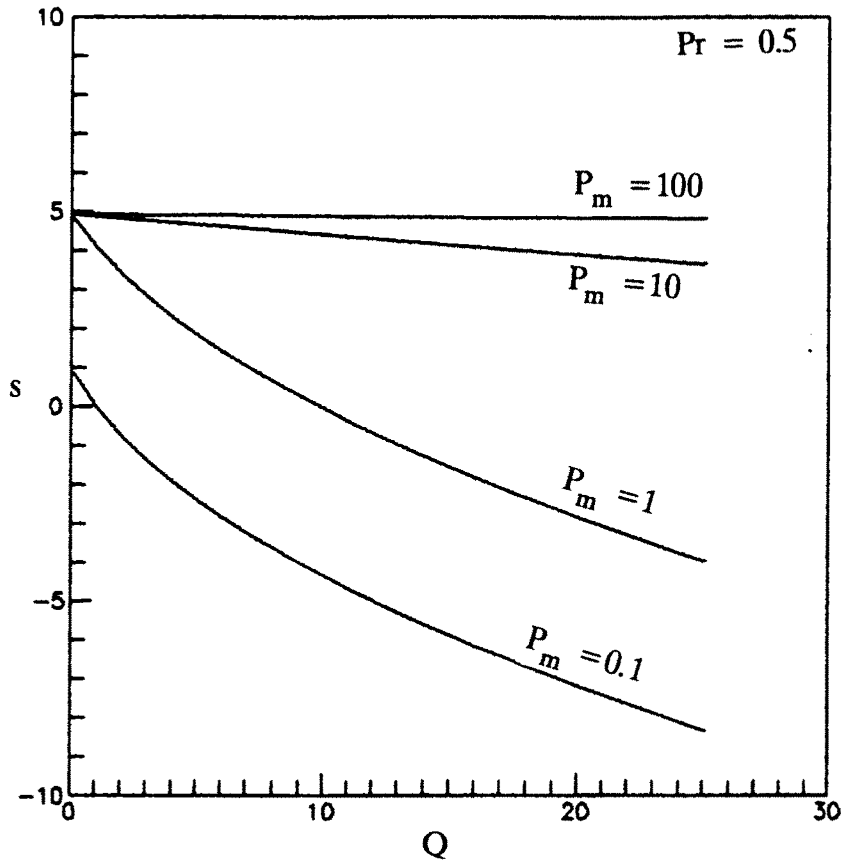

Fig. 4.11. Frequency as a variation of Q

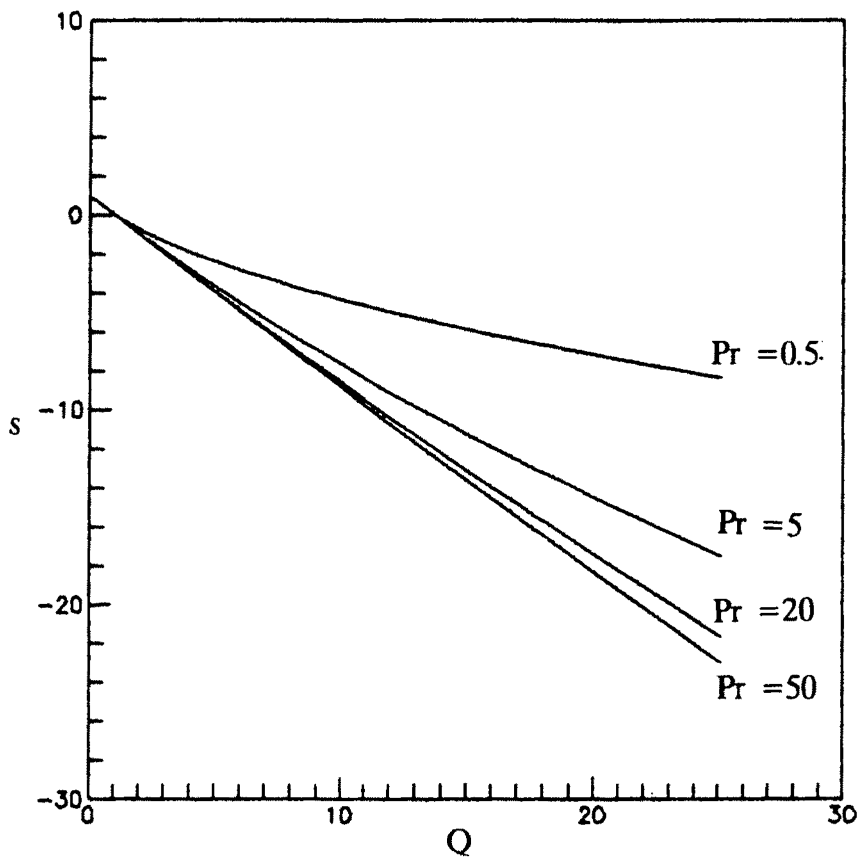

Fig. 4.12. Eigensolutions corresponding to the asymmetric solution ($P_m = 0.1$)

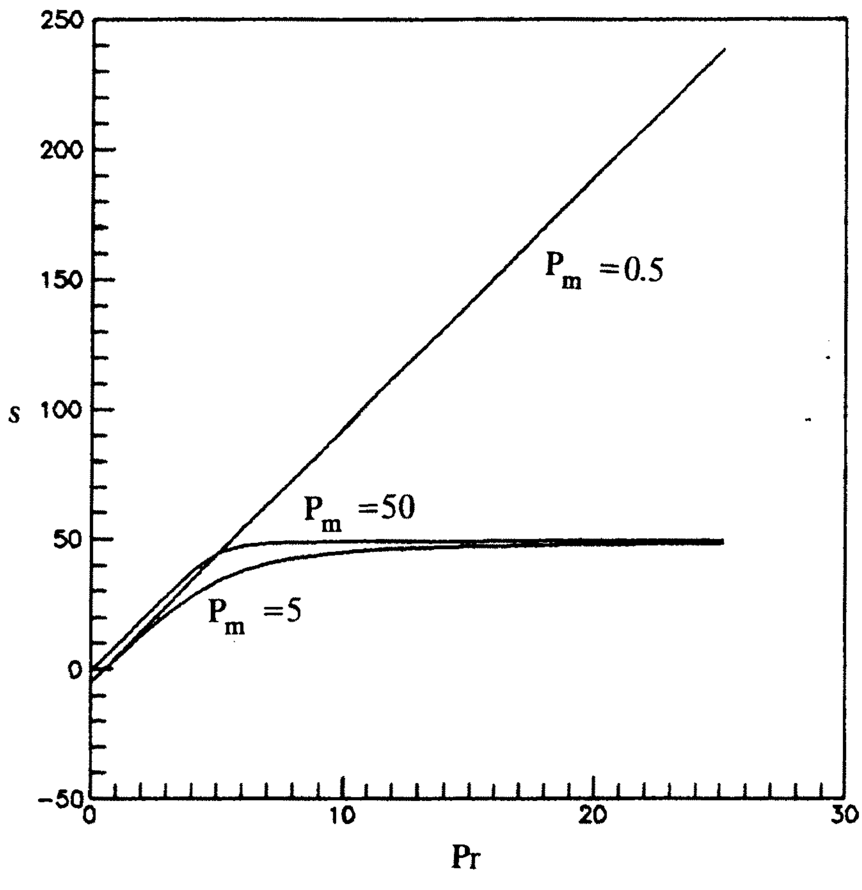

Fig. 4.13. Eigensolutions as functions of P_m (Q = 5.0)

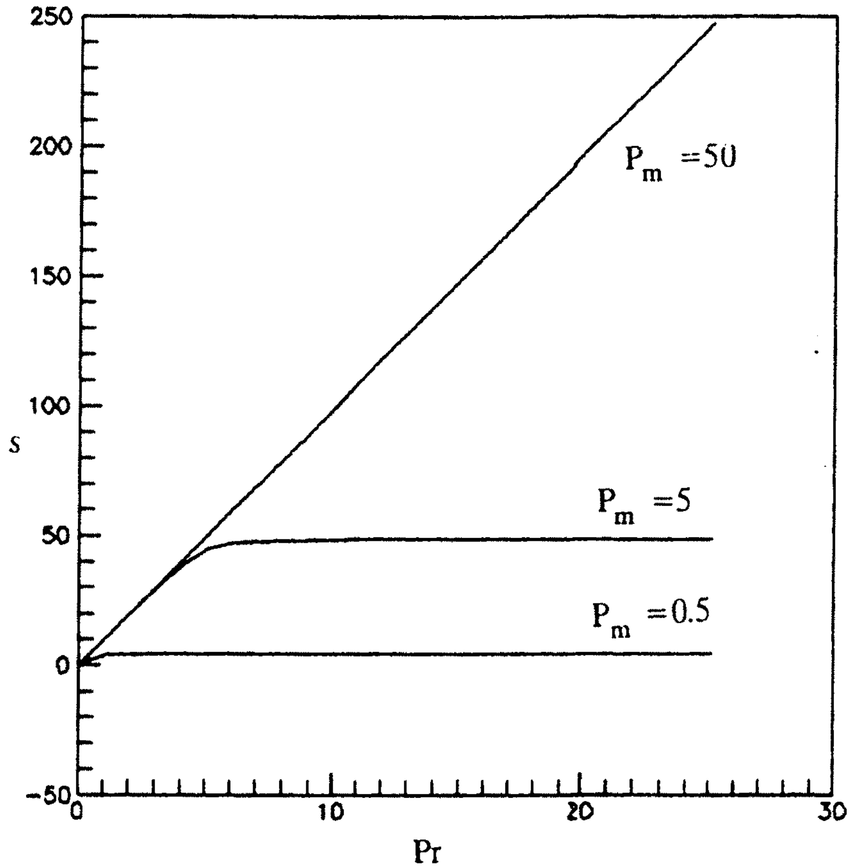

Fig. 4.14. Influence of Pr on the eigensolutions

(Q = 5.0)

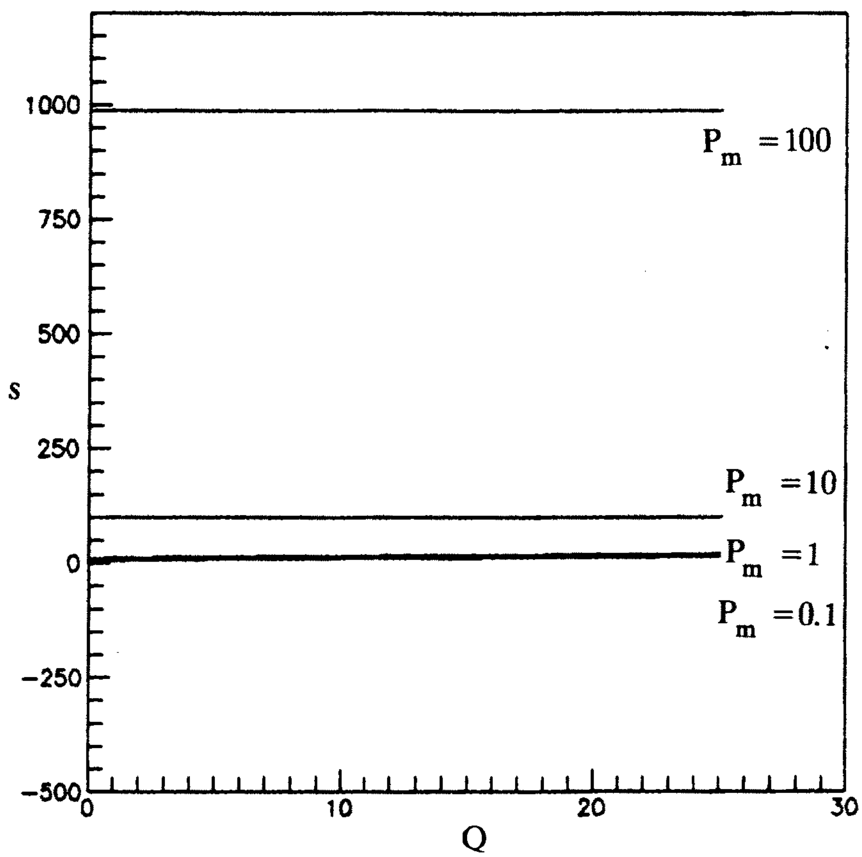

Fig. 4.15. Growth rate as a function of Q (P_m = 0.1)

Pr = 0.5

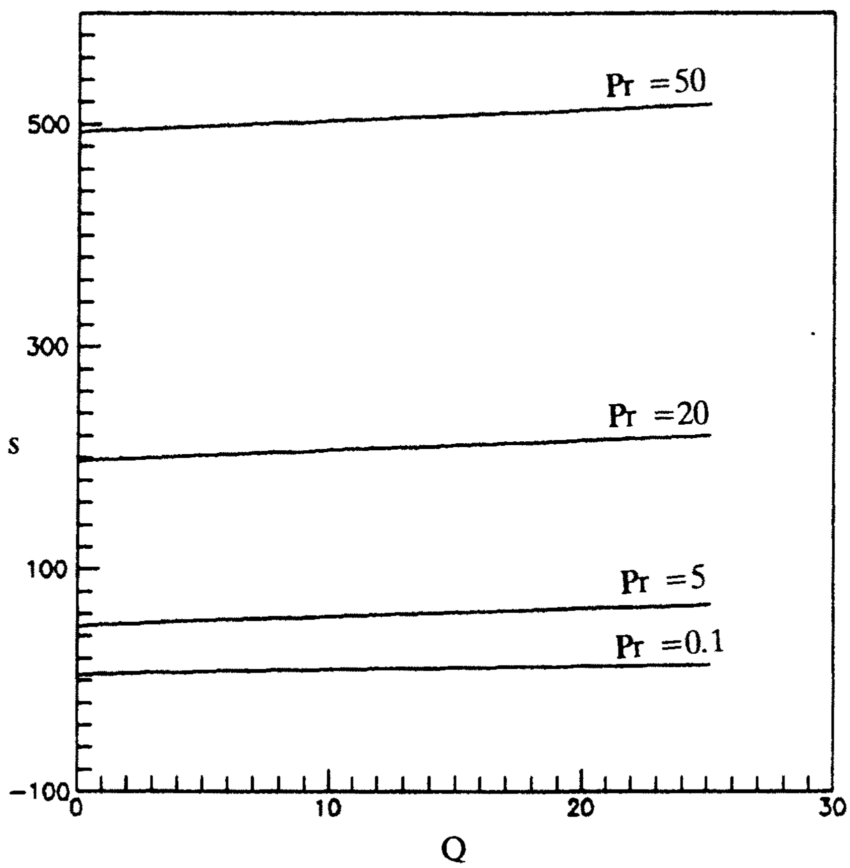

Fig. 4.16. Growth rate as a function of Pr

$P_m = 0.1$

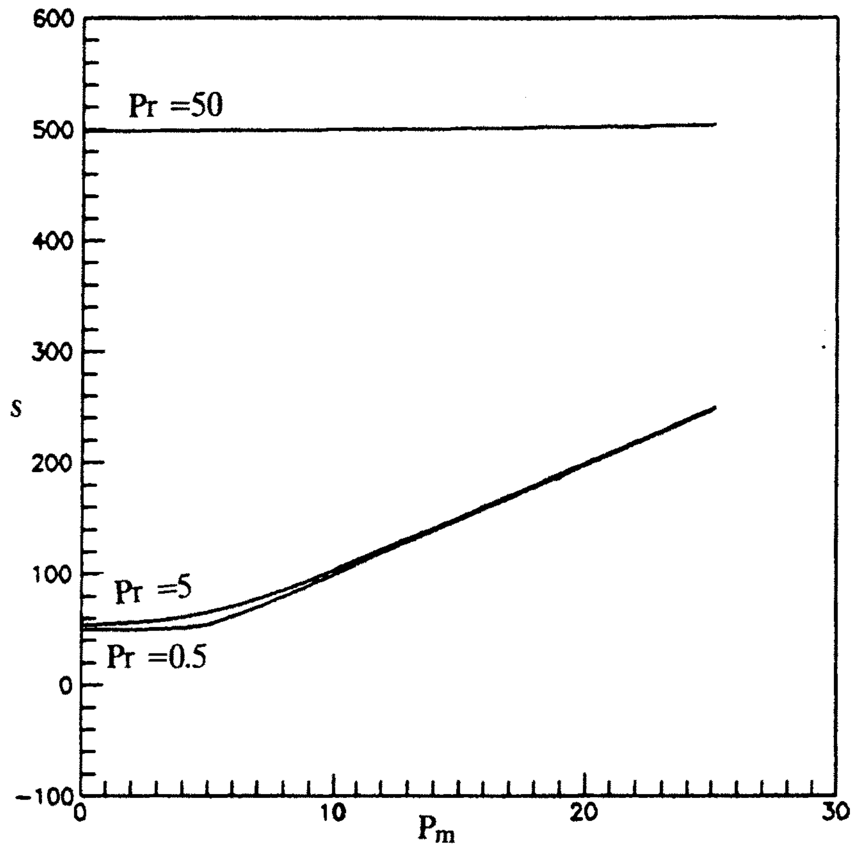

Fig. 4.17. Stability parameter as a function of P_m

(Q = 5.0)

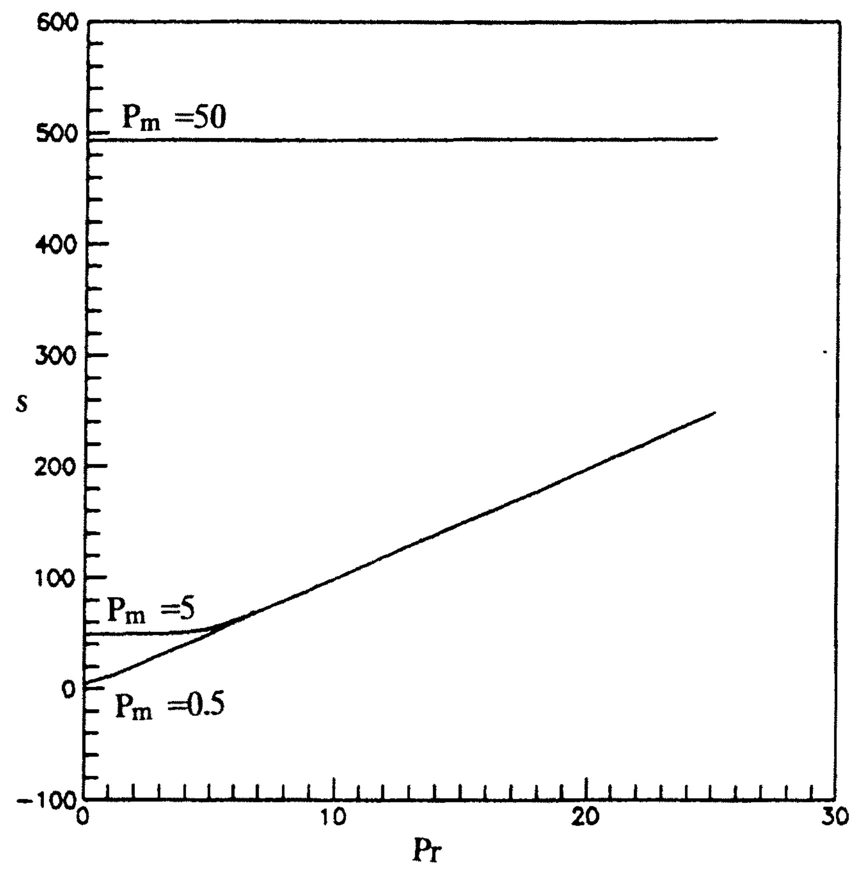

Fig. 4.18. Stability parameter as a function of P_m ($Q = 5.0$)