Chapter 3

Thermal Instability in a Brinkman Porous Medium

3.1 Introduction

Stability analysis of natural convective flow in a horizontal porous layer heated uniformly and nonuniformly either from below or from above has gained considerable attention in recent decades due to its promising applications in engineering and technology. The knowledge of thermal convection in mushy layers of mixed solid and liquid phases has become increasingly important in crystal growth, solidification of molten alloys and other related areas. For instance there are significant differences in the compositional homogeneity and structural perfection between space grown and ground grown crystals. The unfavourable buoyancy driven convection, sedimentation and hydrostatic pressure in the process of crystal growth can be suppressed under reduced gravity environment available in space laboratories. However, random fluctuations of the gravity field, both in magnitude and direction, experienced in the space laboratories significantly influence natural convection and thereby the product. Hence several other ways are being adopted to have a control on the buoyancy driven flow like rotation, agitation techniques, etc. The growing volume of work devoted to this area is well documented by the most recent review of Razi et al. (2005).
Existence of adverse density variations within a fluid and a body force are the necessary conditions to initiate natural convection. The idea of using mechanical vibration as a tool to improve the heat transfer rate has received much attention. Forbes et al. (1970) studied experimentally the influence of vibration on heat transfer and fluid flow behavior of a differentially heated vertical slot and found that there is a possibility of resonance, which leads to the heat transfer augmentation. Since then many studies are being carried out to understand the interaction between gravity induced convection and vibration induced convection in both fluid and fluid filled porous media (see for example Khallouf et al. (1996), Kim et al. (2002), Khaled and Vafai (2004)). Time dependent body forces occur in systems, with density gradients, subjected to vibrations. In such systems the influence on thermal convection depends not only on the orientation of the fluctuating body force with respect to the thermal stratification but also the vibrating amplitude and frequency. This type of body force can even alter the stable distribution of a stratifying agency under constant gravity environment and introduce parametric resonance under suitable conditions. Much work has been done in vertically modulated pure fluid layers with constant vertical density stratification, i.e., modulated Rayleigh-Bénard convection. This type of modulation in gravity may be realized by vertically oscillating a fluid layer in a constant gravitational field.

Most of the studies in porous media carried out in the past few decades are based on the Darcy flow model, which in turn is based on the assumption of creeping flow through an infinitely extended uniform porous medium. However non-Darcian characteristics become quite important for many practical applications. Different models have been developed for accounting and studying non-Darcian attributes like inertial effect, boundary effect, variable porosity effect, etc. An excellent review of studies on convection in porous media has been provided by Nield and Bejan (2006). In the present work we shall employ Brinkman's law for momentum balance to investigate the effect of vertical modulation on the onset of convection in a horizontal porous layer. The Brinkman model is valid for a highly porous medium and when the Darcy number is not small. It takes care of the boundary effects near the boundaries up to a thickness of order \( \sqrt{K} \), where
is the permeability and the potential nature of the Darcy model valid away from the bounding surfaces. This makes the model more flexible in the sense that it reduces to a form of Navier-Stokes equation as $K \to \infty$ and the Darcy’s law as $K \to 0$. The study of vibrating porous medium is comparatively recent origin. More recently, Razi et al. (2009) introduced the time averaged governing equations for Darcy-Brinkman model and they shown that there is a significant deviation from the Darcy model in determining the critical Rayleigh number. Saravanan and Purusothaman (2009) have carried out an investigation to find the effect of non-Darcian effects in an anisotropic porous medium and found that non-Darcian effects significantly affect the synchronous mode of instability. The present study investigates the effects of vibration on the onset of convection in a fluid saturated porous medium using the Darcy extended Brinkman model.

### 3.2 Mathematical Analysis

A fluid saturated shallow horizontal porous layer of infinite extent is considered. It is confined between the surfaces $z = 0$ and $z = h$ maintained at temperatures $T_1$ and $T_2$ respectively. This induces a vertical temperature gradient which is positive for the case of unstable equilibrium (heated from below) and is negative for the stable equilibrium (heated from above). The layer with its boundaries is subjected to vertical vibrations of arbitrary amplitude and frequency. In addition the following assumptions are made. The fluid is Newtonian and incompressible. The solid phase is homogeneous, nondeformable, isotropic and is locally in thermal equilibrium with the fluid phase. Brinkman’s law is used to model flow through it after neglecting inertial effects. The Oberbeck-Boussinesq approximation setting constant all physical properties except density in the buoyancy term, which varies linearly with temperature, is employed.

As the frame of reference is attached to one of the boundaries the vibrations of the whole set up results in a time dependent gravitational field $g(t)$. The dimensional form of the governing equations for the present investigation following
Alazmi and Vafai (2000) are

\[
\frac{1}{\varphi} \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\varphi^2} \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p - \frac{\lambda}{K} \mathbf{v} + \nu \nabla^2 \mathbf{v} + \beta T g(t) \hat{k} \tag{3.1}
\]

\[
\kappa \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \chi \nabla^2 T \tag{3.2}
\]

\[
\nabla \cdot \mathbf{v} = 0 \tag{3.3}
\]

where \( \mathbf{v} = (v_1, v_2, v_3) \) is the filtration velocity, \( p \) the pressure, \( T \) the temperature, \( \varphi \) the porosity, \( \rho \) the density, \( \nu \) the kinematic viscosity, \( K \) the permeability, \( \beta \) the thermal expansion coefficient, \( \hat{k} \) the unit vector directed vertically upward, \( \kappa = (\rho c_p)_m/(\rho c_p)_f \) the heat capacity ratio of the porous medium and fluid and \( \chi \) the thermal diffusivity of the porous medium. For \( \lambda = 0, \varphi = 1 \) and \( \kappa = 1 \), the above equations reduce to the case of clear fluid (Markman and Yudovich (1972)) and \( \lambda = 1 \) corresponds to the present study. The gravitational field is taken to be \( g(t) = g_0 + A \Omega^2 f(\tau) \), where \( g_0 \) is the acceleration in an otherwise nonvibrating layer which is taken as the reference level, \( A \) the vibration amplitude, \( \Omega \) the vibration frequency and \( f(\tau) \) the 2\( \pi \)-periodic function with zero 2\( \pi \)-average.

The surfaces are assumed to be either rigid or flat and tangential stress free and hence obey the following boundary conditions for velocity

\[
v_3 = \frac{\partial v_1}{\partial z} = \frac{\partial v_2}{\partial z} = 0 \text{ at } z = 0 \text{ & } h \text{ for the stress free case} \tag{3.4}
\]

\[
v_3 = \frac{\partial v_3}{\partial z} = 0 \text{ at } z = 0 \text{ & } h \text{ for the rigid case} \tag{3.5}
\]

We seek a solution to the quasi-equilibrium basic state in the form \( \mathbf{v} = \mathbf{v}^0(z), T = T^0(z) \) and \( p = p^0(z,t) \). Thus Eqs.(3.1)-(3.3) together with the boundary conditions possess the following solution

\[
\mathbf{v}^0 = 0 \tag{3.6}
\]

\[
T^0 = T_1 - \frac{1}{h}(T_1 - T_2)z \tag{3.7}
\]
Chapter 3

\[ p^0 = \beta \rho g(t) \left( T_1 z - \frac{1}{2h} (T_1 - T_2) z^2 \right) \] (3.8)

We study the stability of this basic state using the method of small perturbations. Let us consider the motion

\[ v = v^0 + u, \quad p = p^0 + q, \quad T = T^0 + \theta \] (3.9)

where \( u, q \) and \( \theta \) are small unsteady perturbations. Dimensionless variables are defined in terms of the length scale \( h \), the time scale \( h^2/\nu \), the velocity scale \( \nu/h \), the pressure scale \( \rho \nu^2/K \) and the temperature scale \( Ch \) where \( C = (T_1 - T_2)/h \) is the basic quasi-equilibrium temperature gradient. Then the non-dimensional governing equations are

\[ c \frac{\partial u}{\partial t} = -\nabla q - \lambda u + Da \nabla^2 u + Gr (1 + \eta f''(\tau)) \theta \hat{k} \] (3.10)

\[ c \frac{\partial \theta}{\partial t} - u_3 = \frac{1}{Pr} \nabla^2 \theta \] (3.11)

\[ \nabla \cdot u = 0 \] (3.12)

where \( Da = \frac{K}{h^2} \) the Darcy number, \( Gr = \frac{\beta Ch^2 g_0 K}{\nu^2} \) the filtration Grashof number, \( Pr = \frac{\nu}{\chi} \) the Prandtl number, \( c = \frac{Da}{\varphi} \) the porosity-permeability parameter, \( \eta = \frac{A\Omega^2}{\varphi g_0} \) the nondimensional amplitude and \( \omega = \frac{Qh^2}{\nu} \) the nondimensional frequency of vibration. The nondimensional boundary conditions are

\[ u_3 = \frac{\partial^2 u_3}{\partial z^2} = \theta = 0 \text{ at } z = 0 \& 1 \text{ for stress free boundaries} \] (3.13)

\[ u_3 = \frac{\partial u_3}{\partial z} = \theta = 0 \text{ at } z = 0 \& 1 \text{ for rigid boundaries} \] (3.14)

In order to find the linear instability boundary for the system we first take curl curl of Eq.(3.10). The vertical component of the resulting equation has the following form

\[ \nabla^2 \left( c \frac{\partial u_3}{\partial t} + \lambda u_3 \right) = Da \nabla^4 u_3 + Gr (1 + \eta f''(\tau)) \nabla^2 \theta \] (3.15)
where $\nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}$ and $\nabla_2^2 = \nabla_1^2 + \frac{\partial^2}{\partial z^2}$ are the Laplacian operators. We then expand the vertical of velocity and the temperature in terms of normal modes as

$$(u_3, \theta) = (\tilde{u}_3(z, t), \tilde{\theta}(z, t)) e^{i(\alpha_1 x + \alpha_2 y)}$$ (3.16)

where $\alpha_1$ and $\alpha_2$ represent wavenumbers in the $x$ and $y$ directions respectively. Substituting this into Eq.(3.15) and Eq.(3.11), we obtain

$$\left[ c \frac{\partial}{\partial t} + \lambda \right] \left( D_2 - \alpha^2 \right) \tilde{u}_3 - Da(D_2 - \alpha^2)^2 \tilde{u}_3 + \alpha^2 Gr(1 + \eta f''(\tau)) \tilde{\theta} = 0$$ (3.17)

$$\kappa \frac{\partial \tilde{\theta}}{\partial t} - \frac{1}{Pr} (D_2 - \alpha^2) \tilde{\theta} - \tilde{u}_3 = 0$$ (3.18)

where $D = \frac{\partial}{\partial z}$ and $\alpha^2 = \alpha_1^2 + \alpha_2^2$ is overall horizontal wavenumber.

### 3.2.1 Stress Free Boundaries

In this case, one can find the exact solution. Inserting the relation for $\tilde{u}_3(z, t)$ in terms of $\tilde{\theta}(z, t)$ with the help of Eq.(3.18) into Eq.(3.17) yields the following equation for $\tilde{\theta}(z, t)$

$$\left[ c \frac{\partial}{\partial t} + \lambda \right] \left( \frac{1}{Pr} D_1^4 - \frac{2\alpha^2}{Pr} D_2^2 - \kappa \frac{\partial}{\partial t} D_2^2 + \frac{1}{Pr} \alpha^4 + \kappa \alpha^2 \frac{\partial}{\partial t} \right) \tilde{\theta}$$

$$= \alpha^2 Gr \left( 1 + \eta f''(\tau) \right) \tilde{\theta} + Da \left( \frac{1}{Pr} D_1^6 - \frac{3\alpha^2}{Pr} D_2^4 - \kappa \frac{\partial}{\partial t} D_2^4 \right)$$

$$+ \frac{3\alpha^4}{Pr} D_2^2 + 2\kappa \alpha^2 \frac{\partial}{\partial t} D_2^2 - \kappa \alpha^4 \frac{\partial}{\partial t} - \frac{\alpha^6}{Pr} \right) \tilde{\theta}$$ (3.19)

Boundary conditions for this equation are

$$\tilde{\theta} = D_2 \tilde{\theta} = D_4 \tilde{\theta} = 0 \text{ at } z = 0 \text{ and } z = 1$$ (3.20)

From Eq.(3.19) the $z$-variable is separated by taking $\tilde{\theta}(z, t) = \sin(\pi z) \Theta(t)$ and the resulting equation is reduced to the canonical form of damped Mathieu's equation in the form

$$\Theta''(\tilde{t}) + c_1 \Theta'(\tilde{t}) + c_2 (R_0 - Ra(1 + \eta f''(\tau))) \Theta(\tilde{t}) = 0$$ (3.21)
where \( \tilde{t} = \frac{t}{\sqrt{Pr \kappa \varepsilon}} \), \( P = \sqrt{\frac{Pr \kappa}{c}} \), \( Ra = Gr \cdot Pr \) is the Rayleigh number which is positive for unstable equilibrium and negative for stable equilibrium. Also
\[
\begin{align*}
   c_1 &= \frac{\alpha^2 + \pi^2}{P} + \lambda P + Da P (\alpha^2 + \pi^2) \\
   c_2 &= \frac{\alpha_2}{\alpha^2 + \pi^2} \\
   R_0 &= \frac{(Da(\alpha^2 + \pi^2) + \lambda)(\alpha^2 + \pi^2)^2}{\alpha^2} \tag{3.22}
\end{align*}
\]
where \( R_0 \) is the Rayleigh number in the unmodulated situation. When \( \lambda = 1 \) the expression for \( R_0 \) reduces to the one given by Malashetty and Padmavathy (1997) corresponding to the monotonic instability in an unmodulated Brinkman porous layer.

Now, we set \( f(\tau) = \cos \tau \) in Eq.(3.21) and get
\[
\Theta''(\tilde{t}) + c_1 \Theta'(\tilde{t}) + c_2 (R_0 - Ra(1 - \eta \cos(\tilde{\omega} \tilde{t})))\Theta(\tilde{t}) = 0 \tag{3.23}
\]
where \( \tilde{\omega} = \omega \sqrt{Pr \kappa \varepsilon} \). For notational convenience, tilde from \( \tilde{t} \) and \( \tilde{\omega} \) will be subsequently omitted.

### 3.2.2 Rigid Boundaries

In this more realistic case, the exact solution is not possible. To solve the Eqs.(3.17) and (3.18) subject to the boundary conditions (3.14), we adopt the Galerkin method with of course first order. Accordingly we represent the amplitudes \( \tilde{u}_3 \) and \( \tilde{\theta} \) as:
\[
\begin{align*}
   \tilde{u}_3(z, t) &= a(t)F(z), & \tilde{\theta}(z, t) &= b(t)\Phi(z)
\end{align*} \tag{3.24}
\]
where the trial functions \( F(z) \) and \( \Phi(z) \) are given by
\[
\begin{align*}
   F(z) &= z^2(1 - z)^2, & \Phi(z) &= z(1 - z)(1 + z - z^2) \tag{3.25}
\end{align*}
\]
respectively satisfying the appropriate boundary conditions. Gershuni and Zhukhovitskii (1976) first used these trial functions in the Rayleigh-Bénard problem and
found that they are good approximations to determine the convection threshold. In choosing the approximation for $\Phi(z)$, the additional conditions $D^2\tilde{\theta} = 0$ at $z = 0$ and $z = 1$ were taken into account, provided by Eq.(3.18).

The use of this method yields a set of first order differential equations for $a(t)$ and $b(t)$ which can in turn be reduced to the damped Mathieu's equation (3.23) with the coefficients

$$
c_1 = \frac{1}{P} \left( \alpha^2 + \frac{306}{31} \right) + \lambda P + DaP \left( \frac{\alpha^4 + 24\alpha^2 + 504}{\alpha^2 + 12} \right)
$$

$$
c_2 = \frac{121\alpha^2}{124(\alpha^2 + 12)}
$$

$$
R_0 = \frac{4}{121\alpha^2} (\alpha^2 \lambda + 12\lambda + Da(\alpha^4 + 24\alpha^2 + 504))(31\alpha^2 + 306)
$$

where $R_0$ is the unmodulated Rayleigh number. It can be noticed from Eq.(3.26) that as $Da \to \infty$, we obtain $R_0c/Da = 1717.98$ and $\alpha_c = 3.14$ against the exact values $R_{ac} = 1707.76$ and $\alpha_c = 3.12$ of Chandrasekhar (1961) with a deviation of 0.6% which is anticipated within the framework of the first order approximation.

Both continued fraction and Hill’s infinite determinant methods were used to solve the Eq.(3.23). The expressions appearing in Eq.(2.21) and Eq.(2.29) corresponding to this problem are

$$
M_n = (\sigma + in\omega)^2 + c_1(\sigma + in\omega) + c_2(R_0 - Ra)
$$

and

$$
A = \frac{4}{\omega^2} (c_2(R_0 - Ra) - \zeta^2), \quad B = -\frac{4}{\omega^2} q, \quad \zeta = \frac{c_1}{2}.
$$

### 3.3 Results and Discussion

The effect of gravity modulation on the onset of convection in a horizontal porous layer using the Brinkman model is discussed in this section for two cases: (i) $Ra > 0$, corresponding to a layer heated from below and (ii) $Ra < 0$, corresponding a layer heated from above. Both stress free and rigid boundaries are considered.
Before discussing the instability behaviour, it is of interest to compare our results with previous ones. For $Da = 0$, Eq. (3.22) leads to $R_{qc} = 4\pi^2$ and $\alpha_c = \pi$, the well known criteria for the Darcy model. On the other extreme for $Da \rightarrow \infty$, we obtain $R_{qc}/Da = 27\pi^4/4$ and $\alpha_c = \pi/\sqrt{2}$ (Chandrasekhar (1961)), the results of classical Rayleigh-Bénard problem with stress free boundaries. The expression (3.23) contains, in particular, the corresponding result of Natalia (2008) for $Da = 0$, the Darcy model. Existing results in the open literature dealing with the same problem as we discuss are based on either high frequency or low amplitude assumption. They have used different scaling for non-dimensionalization and hence term by term comparison cannot be made with them. We have used kinematic viscosity for non-dimensionalization instead of thermal diffusivity which associates the Prandtl number in our nondimensional frequency. Justification for this and the dependence of $Pr$ in frequency has been clearly explained by Skarda (2001).

The marginal and critical curves are constructed as functions of modulation amplitude and frequency. The marginal curves for the stress free boundaries alone are presented as those of the rigid boundaries exhibited qualitatively similar topological changes. The parameter $c$, the ratio of nondimensional permeability (the Darcy number) to porosity, appearing in the time derivative of Eq. (3.10) is the inverse Vadasz number $Va = \varphi Pr/Da$ with $Pr = 1$ and it can take very small values in the range from $10^{-20}$ to $10^{-1}$ (Vadasz (1998) and Kaviany (1984)). However the time derivative term cannot be neglected and retaining this term has significant effects when there is gravity modulation. Hence in this study we examine the influence of $c$ as well. It is well known that there is no permeability-porosity relationship that can be applied to all porous materials. It is widely accepted however that the permeability is determined by microstructures such as pores and cracks which are connected. So one can suppose in general that increasing porosity results in more interconnected void spaces which in turn contributes to higher permeability. Having this in mind we fixed the porosities to lie in the range from 0.01 to 0.1 for $Da = 10^{-4}$ (Darcy regime) and from 0.1 to 1 for $Da = 10^{-1}$ (Brinkman regime). It should be noted that $Da$ assumes values less than $10^{-3}$ for the Darcy model and exceeds $10^{-3}$ for the Brinkman model (Walker and Homsy (1977)). The values of
the other parameters used in this study are \( Pr = 1 \) and \( \kappa = 1 \). This combination covers the following porous medium - fluid pairs. For air \( Pr \approx 1 \) and at 30\(^\circ\)C, \( Pr = 0.713 \) and specific heat capacity \( \rho C_f = 1005 \) \( J/KgK \). Hence concrete - air combination has \( \kappa = \varphi (\rho C)_f + (1 - \varphi) (\rho C)_s \) \( \rho C_f = 0.96 \) as for concrete \( \varphi = 0.05 \) and \( \rho C)_s = 960 \) \( J/KgK \). Similarly a closely packed bed of spherical glass beads of diameter 0.3 cm \( (\varphi = 0.4) \) filled with air has \( \kappa = 0.91 \). Low density foam made up of polyurethane \( (\varphi = 0.98) \) saturated with air has \( \kappa = 0.998 \). Moreover we note that the kinematic viscosity of air at 30\(^\circ\)C is \( \nu = 0.16 \times 10^{-4} m^2/s \) and the acceleration due to gravity is \( g_0 = 9.81 m/s^2 \). Hence a simple calculation shows that if the layer is vibrating with the frequency \( \Omega = 15 Hz \) and amplitude \( A = 0.15 m \) we obtain the non-dimensional frequency \( \omega \) as 23.438 and the non-dimensional amplitude \( \eta \) as 3.443, 34.438 and 344.387 for the porosity \( \varphi = 1, 0.1 \) and 0.01 respectively.

3.3.1 Heated Below \( (Ra > 0) \)

The characteristic of solutions to Mathieu’s equation and the Floquet system exhibit marginal curves consisting of an array of loop-shaped branches which are completely different in shape from those of traditional stability problems (see Gershuni et al. (1970), Skarda (2001), Terrones and Chen (1993) and references therein). In Fig.(3.1), we have shown these loop-shaped regions as a function of wavenumber for \( \eta = 200, \omega = 10 \) and \( Da = c = 10^{-1} \). These regions indicate where the gravitational modulation destabilizes an otherwise stable configuration. Alternate regions of S and SH responses occur and the bottommost loop plays a decisive role in determining the onset of instability. The loops occurring at small wave numbers are thin and get widened and displaced upwards for an increase in \( \alpha \). Each loop has a minimum Rayleigh number at which the unstable region terminates. The global minimum of these minimum Rayleigh numbers is the critical Rayleigh number \( Ra_c \) and it occurs, in this case, at the bottommost loop which is SH.

In contrast to the multi-looped marginal curve displayed in Fig.(3.1), marginal curves possessing a single minimum are shown in Fig.(3.2) for the small amplitude
situation \( \eta = 1 \). All these marginal curves correspond to the S response alone, i.e., the frequency of the evolving convective pattern is same as the forcing frequency. We notice that in the limit \( \omega \to 0 \) we recover the unmodulated results \( Ra_c = 108.576 \) and \( \alpha_c = 2.472 \) as in the analysis of Govender (2004) and Saravanan and Purusothaman (2009) which is valid for small amplitudes. It is also clear from Fig.(3.2) that the marginal curves come down as \( \omega \) increases from 1 and approach the unmodulated results when \( \omega \) is as low as 25. In other words \( Ra_c \) and \( \alpha_c \) are away from those of the unmodulated case only in a narrow band of \( \omega \) for vibrations with small amplitude. The results reported in Bhadauria (2008) and Malashetty and Padmavathi (1997) who have also used small amplitude approximation are confined to this narrow band of \( \omega \) wherein the multi-looped marginal curves and considerable deviation of \( Ra_c \) proportional to \( \eta \), from those of the unmodulated ones do not arise. We shall concentrate on this aspect in the rest of the paper by lifting the constraint imposed on \( \eta \).

A sequence of marginal curves with S and SH loops in the parameter space \((Ra, \alpha)\) are shown in Fig.(3.3) for \( \omega = 10, 100 \) and 1000 and \( \eta = 2, 20 \) and 200. Though there exist multiple-loops, as in Fig.(3.1), we have displayed only the two bottommost loops, which are relatively important. It is clear that the global minimum in each marginal curve occurs at the bottommost loop which is either S or SH. But this trend is not permanent; varying \( \eta \) or \( \omega \) may transfer the critical value to the adjacent loop which occurs just above the bottommost loop. We notice that modulations of small amplitude results in a marginal curve with the bottommost loop being S whereas those of large amplitude results in a marginal curve with the bottommost loop being SH. However as \( \omega \) increases the S loop develops as the critical one and extends downwards until it reaches the unmodulated value (see Fig.(3.3)a(ii), b(ii) and c(ii)). Also it is evident that the difference between the minima of the two loops becomes very small for modulations with low frequency and very large for modulations with high frequency. In general the loops grow along the \( \alpha \) direction for an increase in \( \omega \). We also observe that as \( \omega \) decreases and \( \eta \) increases, the unstable region corresponding to the S and SH modes shrinks.
The critical Rayleigh number and the critical wavenumber against modulation frequency are shown in Fig.(3.4) and Fig.(3.5) respectively for different values of $\eta$ and $Da = 10^{-1}$. Results for both rigid and stress free cases are presented. The influence of $c$ corresponding to two different porosities are considered. $c$ destabilizes the system for lower values of $\eta$ and stabilizes the system for higher values of $\eta$ and its effect becomes insignificant as $\omega$ increases. Now let us see the behaviour of modulation frequency on the onset criteria for the stress free case. For $\eta = 2$ and $\omega > 20$, $Ra_c$ and $\alpha_c$ approach those of the unmodulated case with S mode being critical throughout the frequency range. For $\eta = 20$, the onset of instability is of SH type at low frequencies until $\omega$ reaches a transition frequency 132.3 beyond which the type of instability changes to S mode. Here $Ra_c$ increases with increasing $\omega$ for the SH mode and decreases for the S mode. It is interesting to discuss the mode transition in terms of the marginal curves. When $\omega$ increases from 10 the bottommost loop (SH) in the marginal curve becomes big (see the second graph of Fig.(3.3)b). As $\omega$ increases nearer to the transition frequency, a thin and narrow S loop appears at low wavenumber region. This loop grows and its minimum reaches the level of adjacent SH loop when $\omega$ becomes 132.3. This represents the bicritical situation, where the wavenumber experiences a catastrophic jump from SH to S mode. Hence we have an $Ra_c$ with two different $\alpha_c$ for this transition frequency. For further increase in $\omega$ beyond this transition point the loop representing the S mode precedes and becomes critical (see the second graph of Fig.(3.3)c). We observe that when $\eta = 200$ the SH mode spreads over a wider range of $\omega$. We also see that in general modulation amplitude produces two different effects; $\eta$ favours convective motion for lower range of $\omega$, inhibits it for intermediate range of $\omega$ with both ranges depending on $\eta$ and leaves the critical boundaries corresponding to the unmodulated layer undisturbed for higher $\omega$. A similar result has been observed in the temperature modulation on Marangoni-Bénard convection and is shown in Fig.(3.3)a of Or and Kelly (2002). In Fig.(3.5), the critical wavenumber $\alpha_c$ increases with $\omega$ for both S and SH modes and undergoes a sudden drop whenever there is a transition from SH to S mode. We observe that the critical boundaries for the rigid boundaries experience almost similar changes except an additional mode transition at low frequencies in the parameter range under consideration. This is
because the transitions between the modes occur at relatively higher frequencies in the case of rigid boundaries. We notice that the corresponding $Ra_c$ and $\alpha_c$ are always greater than those of the stress free ones.

The influence of gravity modulation on the onset criteria for the Darcy model ($Da = 10^{-4}$) is shown in Figs.(3.6) and (3.7). The stability boundaries maintain the trend similar to the Brinkman model except admitting a few cusps in $Ra_c$ and discontinuities in $\alpha_c$ corresponding to mode transitions at low frequencies. A comparison of $Ra_c$ for $c = 10^{-1}$ of Figs.(3.4) and (3.5) with that for $c = 10^{-2}$ of Figs.(3.6) and (3.7) show that $Da$ produces a stabilizing effect with a decrease in $\alpha_c$ similar to the results obtained in Saravanan and Purusothaman (2009). This stabilizing behaviour of $Da$ is due to increased resistance to the fluid motion near the boundaries of the layer. Also the effect of $c$ becomes prominent for the Darcy model. Its effect on the Darcy model is opposite to that in the Brinkman model; it stabilizes the system for lower values of $\eta$ and destabilizes the system for higher values of $\eta$ and its effect becomes insignificant as $\omega$ increases. Thus $c$ has two different effects depending on $Da$ as discussed by Bhadauria (2008). Now let us discuss the Darcy model in detail. For $\eta = 20$, the system gets stabilized via the SH mode as $\omega$ increases until 87.1 and then destabilized via the S mode beyond that and the critical limits approach the constant unmodulated values $Ra_c = 4\pi^2$ and $\alpha_c = \pi$ as $\omega \to \infty$. A similar behaviour is seen when $\eta = 200$ with the critical limits reaching the unmodulated results at a proportionately higher $\omega$. Also we note that the jumps in $\alpha_c$ are small at low frequencies and they become large at high frequencies. Comparing Figs.(3.6) and (3.4) we observe that $Ra_c$ reaches the unmodulated result even at a lower $\omega$ in the Darcy model implying that $Da$ makes the system more sensitive to modulations. We also note that the onset of instability in both the Brinkman and the Darcy models occurs via the S mode for sufficiently large $\omega$ similar to the clear fluid case (see Terrones and Chen (1993)).
3.3.2 Heated Above ($Ra < 0$)

In this case also we observed alternate loop shaped regions of S and SH responses in the marginal stability curves and hence the two bottommost loops alone are shown in the marginal stability curves depicted in the $(-Ra, \alpha)$ parameter space (Fig.(3.8)). We see that an increase in either $\omega$ or $\eta$ makes the loops to grow along the $\alpha$ direction. However we notice that the effect of $\omega$ is prominent. The loops in the lower part of the $(-Ra, \alpha)$ plane become wider for an increase in $\omega$ whereas they become narrower for an increase in $\eta$. Unlike the case $Ra > 0$, the critical condition is determined by the SH mode except when both $\eta$ and $\omega$ assume lower values.

The variation of stability characteristics for different values of $\eta$ and $Da = 10^{-1}$ are shown in Figs.(3.9) and (3.10) for both stress free and rigid cases. We observed that $-Ra_c \rightarrow \infty$ for both $\omega \rightarrow 0$ and $\omega \rightarrow \infty$ which can be seen for a certain extent for $\eta = 2$. At very low modulation frequencies it is impossible to destabilize a thermally stable porous layer and this is a convincing physical explanation for the unboundedness of $Ra_c$ near $\omega = 0$. On the other extreme as the modulation frequency takes higher values the porous layer starts behaving as if it is not forced to oscillate and hence the effect of gravity modulation disappears, similar to the case $Ra > 0$. Moreover the SH mode always remains the preferred mode of instability for $\omega > 10^2$. Hence we have restricted the upper limit of $\omega$ to $10^3$. Unlike the case $Ra > 0$, the system is always more prone to instability which manifests in the form of smaller circulation patterns for an increase in $\eta$ for all $\omega$. Moreover the competition between the two modes of instability exists over a wider range of $\omega$ for lower values of $\eta$ and hence more nesting is visible in between them. This produces more number of cusps in $-Ra_c$ which are associated with jumps in the corresponding $\alpha_c$. The marginal curves near these cusps undergo dramatic changes affecting their topology as elaborated earlier. A new thin loop shaped marginal curve emerges and develops in the low wavenumber region and becomes dominant at the frequency where the jump occurs. We also observe that that the effect of $c$ is unique unlike the case of a fluid layer heated from below. It always inhibits the onset of instability and magnifies the convective cells ensuing at the
threshold. Finally we notice that $-Ra_c$ and $\alpha_c$ for the rigid case are greater than their stress free analogues. But in contrast to $Ra > 0$, we observe more number of transitions between the S and SH modes for the rigid case prevailing over a wider range of $\omega$.

Similar to Figs.(3.9) and (3.10), Figs.(3.11) and (3.12) illustrate the stability behaviour for the Darcy model ($Da = 10^{-4}$). We find both $\eta$ and $c$ favour the onset of convection. But the effect of $c$ is more significant at low frequencies and is seen in a wider region of the parameter space compared to the Brinkmann model. Moreover $c$ significantly affects both S and SH modes and its effect can be clearly seen for $\eta = 2$. It actually suppresses the competition between the two modes and restricts it to a narrower range of $\omega$ near $\omega = 0$. The corresponding $\alpha_c$ experience sharper changes at transition frequencies compared to the other cases. This shows that the heat transfer characteristics of the Darcy porous layer which is heated from above can be well adjusted by a proper choice of $c$. When the layer heated from above, the mode transitions occur only at lower frequencies compared to the heated from below case, beyond that the SH mode is permanent as $\omega \to \infty$. An increase in $\eta$ restricts the mode transitions to the region very nearer to $\omega = 0$. It is also evident that nesting of the two modes is denser for low amplitudes. The S and SH branches become increasingly small and closely packed as $\omega$ approaches zero. Comparing Fig.(3.4) with Fig.(3.9) and Fig.(3.6) with Fig.(3.11), it is noticed that $|Ra_c|$ is almost the same for $\eta = 200$. Thus we arrive at a significant result that vibrations with higher amplitude have the same effect whether the layer is heated from below or from above.

3.3.3 The Effect of $Pr$

Finally the dependence of stability characteristics on $Pr$ in the case a layer heated from below is illustrated in Figs.(3.13) and (3.14) for both types of boundaries. It is seen that for very low and high Prandtl number limits, $Ra_c$ and $\alpha_c$ approach the unmodulated values. The S mode remains critical for the entire range of $Pr$ for the Darcy model. This trend gets affected and the SH mode becomes dominant.
for intermediate values of $Pr$ in the case of the Brinkman model. In other words an increase in $Da$ can make the convective pattern to oscillate subharmonically depending on $Pr$. The corresponding $\alpha_c$ reaches a maximum in that intermediate range of $Pr$. This shows the existence of a critical $Pr$ corresponding to which the cells ensuing at the critical condition reach their smallest size. The results for the rigid case follow the patterns of its stress free counterpart, as seen earlier in all cases except in the range $10^{-4} < Pr < 10^{-3}$. Here the $\alpha_c$s exhibit opposite trends. Moreover $Ra_c$ of the stress free case exceeds that of the rigid case near in the neighbourhood of $Pr = 160$. We guess that these exceptions might have arisen due to the first order approximation which we have employed for the rigid case. Also we observed that the SH mode extends to a still wider range of $Pr$ and lowers its $Ra_c$ for higher values of $\eta$. Figs.(3.15) and (3.16) show the corresponding stability characteristics for $Ra < 0$. The SH mode is dominant for intermediate values of $Pr$ with a corresponding maximum $\alpha_c$ whereas both the modes alternate each other and approach the unmodulated results for very low and high values of $Pr$.

### 3.3.4 Validation

Before closing this section we present a validation of the present stress free results in the presence of modulation. In Figs.(3.17) and (3.18) we have plotted $Ra_c$ and $\alpha_c$ of both S and SH modes against $\omega \in (10^0, 10^3)$ for $Da = Pr = c = 1$, $\lambda = 0$ and $\eta = 10$, corresponding to a clear fluid. They reproduce qualitatively the curves of Markman and Yudovich (1972a,b) obtained through continued fractions method. Moreover we can observe that the maximal critical Rayleigh number $Ra_{c, max} = 10928.84$ and its corresponding critical wavenumber $\alpha_{c, max} = 8.94$ occur when $\omega = 140.461$. Markman and Yudovich (1972a,b) reported the corresponding results as $Ra_{c, max} = 10929$ and $\alpha_{c, max} = 9$ at $\omega^* = 140.462$ showing a quantitative agreement of the present results as well. It is also clear from Fig.(3.17) that the S mode occurs at high frequencies and the SH mode at intermediate frequencies. For $\omega < 10$, $Ra_c$ for the two modes are closer to each other and their respective critical curves intersect at several places accompanied by discontinuous $\alpha_c$ as discussed
by Or and Kelly (2002) in the problem of Marangoni-Bénard convection. We observed the same behaviour in all other cases and this is the reason why we fixed the left extreme of $\omega$ range as 10 in all our previous Figures. We have plotted the S branch of $Ra_c$ against the parameter $\eta/\omega$ in Fig. (3.19) for the clear fluid limit when $Pr = 1$. We see that at finite $\omega$ the critical curves are determined by $\eta$ as well as $\omega$. With increasing $\omega$ these curves approach and merge rapidly with that for $\omega \to \infty$. This indicates that $\eta/\omega$ is the only parameter characterising modulations and determines $Ra_c$ in the high frequency limit confirming an equivalent result reported earlier by Gershuni and Zhukovskii (1976) in their high frequency analysis of gravity modulation.
Figure 3.1 Brinkman model ($Da = 10^{-1}$). Marginal curve with S and SH resonant loops for $c = 10^{-1}$, $\eta = 200$ and $\omega = 10$.

Figure 3.2 Brinkman model ($Da = 10^{-1}$). Marginal curve with S for the case of low amplitude ($\eta = 1$) with $c = 10^{-1}$.
Figure 3.3 Brinkman model ($Da = 10^{-1}$). Marginal curves with S and SH for different values of $\eta$ and $\omega$ with $c = 10^{-1}$. 

(a) $\omega = 10$

(b) $\omega = 100$

(c) $\omega = 1000$
Figure 3.4 Brinkman model \((Da = 10^{-1})\). \(Ra_c\) against \(\omega\) for different \(\eta\) with \(c = 10^{-1}, 1\).

Figure 3.5 Brinkman model \((Da = 10^{-1})\). \(\alpha_c\) against \(\omega\) for different \(\eta\) with \(c = 10^{-1}, 1\).
Figure 3.6 Darcy model ($Da = 10^{-4}$). $Ra_c$ against $\omega$ for different $\eta$ with $c = 10^{-3}, 10^{-2}$.

Figure 3.7 Darcy model ($Da = 10^{-4}$). $\alpha_c$ against $\omega$ for different $\eta$ with $c = 10^{-3}, 10^{-2}$. 
Figure 3.8 Brinkman model ($Da = 10^{-1}$). Marginal curves with S and SH for different values of $\eta$ and $\omega$ with $c = 10^{-1}$. 
Figure 3.9 Brinkman model ($Da = 10^{-1}$). $-Ra_c$ against $\omega$ for different $\eta$ with $c = 10^{-1}, 1$.

Figure 3.10 Brinkman model ($Da = 10^{-1}$). $\alpha_c$ against $\omega$ for different $\eta$ with $c = 10^{-1}, 1$. 
Figure 3.11 Darcy model \((Da = 10^{-4})\). \(-Ra_c\) against \(\omega\) for different \(\eta\) with \(c = 10^{-3}, 10^{-2}\).

Figure 3.12 Darcy model \((Da = 10^{-4})\). \(\alpha_c\) against \(\omega\) for different \(\eta\) with \(c = 10^{-3}, 10^{-2}\).
Figure 3.13 $Ra_c$ against $Pr$ for $Da = 10^{-1}$, $c = 10^{-1}$ and $Da = 10^{-4}$, $c = 10^{-2}$ with $\eta = 20$ and $\omega = 100$.

Figure 3.14 $\sigma_c$ against $Pr$ for $Da = 10^{-1}$, $c = 10^{-1}$ and $Da = 10^{-4}$, $c = 10^{-2}$ with $\eta = 20$ and $\omega = 100$. 
Figure 3.15 $-Ra_c$ against $Pr$ for $Da = 10^{-1}$, $c = 10^{-1}$ and $Da = 10^{-4}$, $c = 10^{-2}$ with $\eta = 20$ and $\omega = 100$.

Figure 3.16 $\alpha_c$ against $Pr$ for $Da = 10^{-1}$, $c = 10^{-1}$ and $Da = 10^{-4}$, $c = 10^{-2}$ with $\eta = 20$ and $\omega = 100$. 
Figure 3.17 $Ra_c$ against $\omega$ for $Da = Pr = c = 1$, $\lambda = 0$ (Clear fluid) and $\eta = 10$.

Figure 3.18 $\alpha_c$ against $\omega$ for $Da = Pr = c = 1$, $\lambda = 0$ (Clear fluid) and $\eta = 10$. 
Figure 3.19 $Ra_c/R_0$ against $\eta/\omega$ for $Da = Pr = c = 1$ and $\lambda = 0$ (Clear fluid).