Chapter 2

Mathematical Analysis

In this chapter the problem formulation, nondimensional parameters, linear stability analysis and solution methodology using continued fraction and Hill's infinite determinant methods are discussed.

2.1 Formulation of the Problem

A shallow horizontal porous layer saturated with an incompressible fluid confined between the surfaces \( z = 0 \) and \( z = h \) is considered (see Fig.(2.1)). There exists a vertical temperature gradient which is positive for the case of unstable equilibrium (heated from below) and is negative for the stable equilibrium (heated from above). Moreover the layer is subjected to vertical vibrations of arbitrary amplitude and frequency. The porous medium is homogeneous and isotropic and Darcy's law is used to model flow through it. The Oberbeck-Boussinesq approximation setting constant all physical properties except density in the buoyancy term, which varies linearly with temperature, is employed. Under these assumptions, the dimensional form of the governing equations are

\[
\frac{1}{\varphi} \frac{\partial \nu}{\partial t} = -\frac{1}{\rho} \nabla p - \frac{\nu}{K} \nu + \beta T g(t) \tilde{k}
\]  

(2.1)
\[ \kappa \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \chi \nabla^2 T \]  
\[ \nabla \cdot \mathbf{v} = 0 \]

where \( \mathbf{v} = (v_1, v_2, v_3) \) is the filtration velocity, \( p \) the pressure, \( T \) the temperature, \( \varphi \) the porosity, \( \rho \) the density, \( \nu \) the kinematic viscosity, \( K \) the permeability, \( \beta \) the thermal expansion coefficient, \( \mathbf{k} \) the unit vector directed vertically upward, \( \kappa = (\rho \varphi c_m)/(\rho \varphi c_f) \) the heat capacity ratio of the porous medium and fluid and \( \chi \) the thermal diffusivity of the porous medium. The time dependent gravitational field is taken to be \( g(t) = g_0 + \frac{A}{\Omega^2} f''(\tau) \), where \( g_0 \) is a reference acceleration level, \( A \) the vibration amplitude, \( \Omega \) the vibration frequency and \( f(\tau) \) a 2\pi-periodic function with zero 2\pi-average. The surfaces are assumed to be flat and impervious and obey the boundary conditions

\[ v_3 = 0, \quad T = T_1 \text{ at } z = 0 \]  
\[ v_3 = 0, \quad T = T_2 \text{ at } z = h \]

We seek a solution to the quasi-equilibrium basic state in the form \( \mathbf{v} = \mathbf{v}^0 \), \( T = T^0(z) \) and \( p = p^0(z, t) \). Thus Eqs.(2.1)-(2.3) together with the boundary conditions possess the following solution

\[ \mathbf{v}^0 = 0 \]  
\[ T^0 = T_1 - \frac{1}{h} (T_1 - T_2) z \]  
\[ p^0 = \beta \rho g(t) \left( T_1 z - \frac{1}{2h} (T_1 - T_2) z^2 \right) \]

We study the stability of this basic state using the method of small perturbations.
2.2 Linear Stability Analysis

Let us consider the motion
\[ \mathbf{v} = \mathbf{v}^0 + \mathbf{u}, \quad p = p^0 + q, \quad T = T^0 + \theta \] (2.9)
where \( u, q \) and \( \theta \) are small unsteady perturbations. Dimensionless variables are defined based on the length scale \( h \), the time scale \( h^2/\nu \), the velocity scale \( \nu/h \), the pressure scale \( \rho \nu^2/K \) and the temperature scale \( C \) where \( C = (T_1 - T_2)/h \) is the basic quasi-equilibrium temperature gradient. Then the nondimensional governing equations are

\[ \frac{\partial \mathbf{u}}{\partial t} = -\nabla q - \mathbf{u} + Gr(1 + \eta f''(\tau))\theta \hat{k} \] (2.10)

\[ \frac{\partial \theta}{\partial t} - u_3 = \frac{1}{Pr} \nabla^2 \theta \] (2.11)

\[ \nabla \cdot \mathbf{u} = 0 \] (2.12)

The nondimensional boundary conditions are

\[ u_3 = \theta = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1. \] (2.13)

In order to find the linear instability boundary for the system we first take curl curl of Eq.(2.10). The vertical component of the resulting equation has the following form

\[ \nabla^2 \left( c \frac{\partial u_3}{\partial t} + u_3 \right) = Gr(1 + \eta f''(\tau))\nabla_1^2 \theta \] (2.14)

where \( \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) and \( \nabla^2 = \nabla_1^2 + \frac{\partial^2}{\partial z^2} \) are the Laplacian operators. We then expand the vertical component of velocity and the temperature in terms of normal modes as

\[ (u_3, \theta) = \left( \tilde{u}_3(z, t), \tilde{\theta}(z, t) \right) e^{i(\alpha_1 x + \alpha_2 y)} \] (2.15)

where \( \alpha_1 \) and \( \alpha_2 \) represent wavenumbers in the \( x \) and \( y \) directions respectively. Substituting this into Eq.(2.14) and Eq.(2.11) and inserting the relation \( \tilde{u}_3(z, t) \) in
terms of $\tilde{\theta}(z,t)$ yields the following equation for $\tilde{\theta}(z,t)$

$$
\left[ c \frac{\partial}{\partial t} + 1 \right] \left( \frac{1}{Pr} D^4 - \frac{2\alpha^2}{Pr} D^2 - \kappa \frac{\partial}{\partial t} D^2 + \frac{\alpha^4}{Pr} + \kappa \alpha^2 \frac{\partial}{\partial t} \right) \tilde{\theta}
= \alpha^2 Gr (1 + \eta f''(\tau)) \tilde{\theta} \tag{2.16}
$$

where $D \equiv \frac{\partial}{\partial z}$ and $\alpha^2 = \alpha_1^2 + \alpha_2^2$ is overall horizontal wavenumber. Boundary conditions for this equation are

$$
\tilde{\theta} = \frac{\partial^2 \tilde{\theta}}{\partial z^2} = 0 \quad \text{at} \quad z = 0 \text{ and } z = 1 \tag{2.17}
$$

From Eq.(2.16) the $z$-variable is separated by taking the following representation for $\tilde{\theta}(z,t)$:

$$
\tilde{\theta}(z,t) = \sin(\pi z) \Theta(t)
$$

The resulting equation is reduced to the canonical form of damped Mathieu’s equation in the form

$$
\Theta''(\tilde{t}) + \left( \frac{m^2}{P} + P \right) \Theta'(\tilde{t}) + \left( m^2 - \frac{\alpha^2}{m^2} Ra(1 + \eta f''(\tau)) \right) \Theta(\tilde{t}) = 0 \tag{2.18}
$$

where $\tilde{t} = \frac{t}{\sqrt{Pr xc}}$, $P = \sqrt{\frac{Pr xc}{c}}$, $m^2 = \alpha^2 + \pi^2$ and $Ra = Gr \cdot Pr$ is the Rayleigh number which is positive for unstable equilibrium and negative for stable equilibrium. Now, we set $f(\tau) = \cos \tau$ in Eq.(2.18) and get

$$
\Theta''(\tilde{t}) + \left( \frac{m^2}{P} + P \right) \Theta'(\tilde{t}) + \left( m^2 - \frac{\alpha^2}{m^2} Ra(1 - \eta \cos(\tilde{\omega} \tilde{t})) \right) \Theta(\tilde{t}) = 0 \tag{2.19}
$$

where $\tilde{\omega} = \omega \sqrt{Pr xc}$. For notational convenience, tilde from $\tilde{t}$ and $\tilde{\omega}$ will be subsequently omitted.

### 2.3 Solution Procedure

In this section we shall explain two powerful techniques, viz., Hill’s determinant method (Morse and Feshbach (1953)) and continued fraction method (Meshalkin and Sinai (1961)) used to solve Eq.(2.19).
2.3.1 Continued Fraction Method

Before applying this method first we convert Eq.(2.19) into a system of algebraic equations. Following the Floquet theory, we search the solution to Eq.(2.19) in the form

$$\Theta = e^{\sigma t} \sum_{n=-\infty}^{+\infty} a_n e^{i\omega t}$$  \hspace{1cm} (2.20)

where the complex growth rate $\sigma$ is the Floquet exponent which defines the behaviour of the perturbation with time and needs to be chosen in such a way that the solution (2.20) is nonzero. The set of all such values of $\sigma$ defines the Floquet spectrum of Eq.(2.19). The behaviour of the solution (2.20) and therefore the stability of the quasi-equilibrium basic state (2.6)-(2.8) of the original system, is determined by the distribution of the Floquet spectrum with respect to the imaginary axis in the complex $\sigma-$plane. If the whole spectrum is located in the left half plane, the basic state (2.6)-(2.8) is asymptotically stable. If at least one point of the spectrum is located in the right half plane, the basic state is unstable. The points of the spectrum where $Re(\sigma) = 0$ correspond to the neutral surfaces in the parameter space, which separate regions of stability and instability.

Substitution of Eq.(2.20) into Eq.(2.19) yields an infinite system of linear algebraic equations to determine the unknown coefficient $a_n$:

$$M_n a_n = -q(a_{n-1} + a_{n+1}), \hspace{0.5cm} n = \ldots, -2, -1, 0, 1, 2, \ldots$$  \hspace{1cm} (2.21)

where

$$M_n = (\sigma + in\omega)^2 + \left(\frac{m^2}{P} + P_+\right)(\sigma + in\omega) + \left(m^2 - \frac{\alpha^2}{m^2}Ra\right) \quad \text{and} \quad q = \frac{\alpha^2 Ra \eta}{2m^2}.$$  

Now we use the continued fraction method to solve the above linear system. Substituting $a_n = q^nd_n$ into Eq.(2.21), we obtain

$$M_n d_n = -(d_{n-1} + q^2d_{n+1}), \hspace{0.5cm} n = \ldots, -2, -1, 0, 1, 2, \ldots$$  \hspace{1cm} (2.22)

After one more substitution $\zeta_n = \frac{d_{n-1}}{d_n} \quad (d_n \neq 0)$, the system (2.21) becomes

$$M_n = -\left(\zeta_n + \frac{q^2}{\zeta_{n+1}}\right), \hspace{0.5cm} n = \ldots, -2, -1, 0, 1, 2, \ldots$$  \hspace{1cm} (2.23)
The validity of the transition from Eq.(2.21) to Eq.(2.23) was discussed by Markman and Yudovich (1972a) and they proved that none of the coefficients $a_n$ can become zero for a solution of the system (2.21). From Eq.(2.23), using complex fractions, we derive two different recurrence relations for the unknown $\zeta_n$ as

$$\zeta_n = -M_n - \frac{q^2}{\zeta_{n+1}} = \frac{-q^2}{M_{n-1} + \zeta_{n-1}}$$

which in turn yield two different continued fractions for $\zeta_n$:

$$\zeta_n = -M_n + \frac{-q^2}{-M_{n+1} + \frac{-q^2}{-M_{n+2} + \frac{-q^2}{-M_{n+3} + \cdots}}}$$

Assigning $n = 0$ in these leads to the following dispersion equation for the Floquet exponent $\mu$ in the explicit form

$$M_0 - \frac{q^2}{M_1 - \frac{q^2}{M_2 - \frac{q^2}{M_3 - \cdots}}} = \frac{q^2}{M_{-1} - \frac{q^2}{M_{-2} - \frac{q^2}{M_{-3} - \cdots}}}$$

from which we can determine the values of the Floquet exponent $\sigma$.

The Eq.(2.25) is simplified to the real form when $\sigma = 0$ corresponding to the synchronous (S) mode with period $2\pi/\omega$ and $\sigma = i\omega/2$ corresponding to the subharmonic (SH) mode with period $4\pi/\omega$. When $\sigma = 0$, the expression for $M_n$ simplifies with the symmetry $M_{-n} = \overline{M_n}$ (bar denotes the complex conjugate) as

$$M_n = -n^2\omega^2 + \left(m^2 - \frac{\alpha^2}{m^2}Ra\right) + i\omega\left(\frac{m^2}{P} + P\right)$$

and hence Eq.(2.25) reduces to

$$\text{Re} \left( \frac{q^2}{M_1 - \frac{q^2}{M_2 - \frac{q^2}{M_3 - \cdots}}} \right) = \frac{M_0}{2}$$

(2.26)
When $\sigma = i\omega/2$, the expression for $M_n$ simplifies with the symmetry $M_n = \overline{M_{n-1}}$ as

$$M_n = -\omega^2 \left(n + \frac{1}{2}\right)^2 + \left(m^2 - \frac{\alpha^2}{m^2}Ra\right) + i\omega \left(n + \frac{1}{2}\right) \left(\frac{m^2}{P} + P\right)$$

and hence Eq.(2.25) becomes

$$M_n = -\omega^2 \left(n + \frac{1}{2}\right)^2 + \left(m^2 - \frac{\alpha^2}{m^2}Ra\right) + i\omega \left(n + \frac{1}{2}\right) \left(\frac{m^2}{P} + P\right)$$

The transcendental equations (2.26) and (2.27) are solved then to obtain the marginal curves of $Ra$ against $\alpha$. Prior to that convergence of the continued fractions was verified numerically and the continued fractions were truncated once the desired precision is achieved.

### 2.3.2 Hill’s Infinite Determinant Method

As a first step we rewrite Eq.(2.19) as

$$\Theta''(\tilde{t}) + 2\zeta \Theta'(\tilde{t}) + \left(m^2 - \frac{\alpha^2}{m^2}Ra(1 - \eta \cos(\omega \tilde{t}))\right) \Theta(\tilde{t}) = 0 \quad (2.28)$$

where $2\zeta = \left(\frac{m^2}{P} + P\right)$. Then introducing $\Theta(t) = e^{-\zeta t}F(t)$ and $2\tau = \omega t$ we arrive at the canonical form of Mathieu’s equation

$$F'' + [A - 2B \cos(2\tau)]F = 0 \quad (2.29)$$

where $A = \frac{4}{\omega^2} \left(m^2 - \frac{\alpha^2}{m^2}Ra - \zeta^2\right)$ and $B = -\frac{4}{\omega^2}q$. Following the Floquet theory, the general solution of Eq.(2.29) is of the form $F(\tau) = e^{\sigma \tau}G(\tau)$ where $\sigma$ is the Floquet exponent and $G(\tau)$ is a periodic function with period $\pi$. Then the solution of Eq.(2.29) is written as

$$\Theta(t) = e^{-\zeta t}F(t) = e^{(\frac{\sigma t}{\pi} - \zeta)}G(t) \quad (2.30)$$
The criterion for stability is $\frac{\sigma\omega}{2} \leq \zeta$. Hence the marginal stability condition corresponding to the periodic solutions is given by $\frac{\sigma\omega}{2} = \zeta$.

We express the solutions of Eq.(2.29) in the form

$$F = e^{\sigma\tau} \sum_{n=-\infty}^{+\infty} a_n e^{i(2n+p)\tau}$$  \hspace{1cm} (2.31)

If $p = 0$ the solution is of S type with period $\pi$ and if $p = 1$ the solution is of SH type with period $2\pi$. Substitution of expression (2.31) corresponding to the S mode in Eq.(2.29) leads to the homogeneous linear algebraic system

$$\xi_n a_{n-1} + a_n + \xi_n a_{n+1} = 0, \quad n = \ldots, -2, -1, 0, 1, 2, \ldots$$  \hspace{1cm} (2.32)

where $\xi_n(\sigma) = \frac{B}{(2n - i\sigma)^2 - A}$. A necessary and sufficient condition for the existence of nontrivial solution is the vanishing of its characteristic determinant:

$$\Delta(i\sigma) = \begin{vmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \xi_{-2} & 1 & \xi_{-2} & 0 & 0 & 0 \\ 0 & \xi_{-1} & 1 & \xi_{-1} & 0 & 0 \\ 0 & 0 & \xi_0 & 1 & \xi_0 & 0 \\ 0 & 0 & 0 & \xi_1 & 1 & \xi_1 \\ 0 & 0 & 0 & 0 & \xi_2 & 1 & \xi_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0$$  \hspace{1cm} (2.33)

where $\Delta(i\sigma)$ is Hill’s infinite determinant. Following Morse and Feshbach (1953), this condition may be written as

$$\cosh(\sigma\pi) = 1 - 2\Delta(0)\sin^2\left(\frac{\pi\sqrt{A}}{2}\right) = 1 + 2\Delta(0)\sinh^2\left(\frac{\pi\sqrt{-A}}{2}\right)$$  \hspace{1cm} (2.34)

where

$$\Delta(0) = \begin{vmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \xi_2 & 1 & \xi_2 & 0 & 0 & 0 \\ 0 & \xi_1 & 1 & \xi_1 & 0 & 0 \\ 0 & 0 & \xi_0 & 1 & \xi_0 & 0 \\ 0 & 0 & 0 & \xi_1 & 1 & \xi_1 \\ 0 & 0 & 0 & 0 & \xi_2 & 1 & \xi_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$
The advantage of the transformation (2.34) is that \( \Delta(0) \) can be computed from recurrence relations between the determinants of orders differing by two at each step, starting with the mid-determinant (Jordan and Smith (1987))

\[
\begin{align*}
\Delta_0 &= 1 \\
\Delta_1 &= 1 - 2\xi_0\xi_1 \\
\Delta_2 &= (1 - \xi_1\xi_2)^2 - 2\xi_0\xi_1(1 - \xi_1\xi_2) \\
&\quad \vdots \\
\Delta_{n+2} &= (1 - \xi_{n+1}\xi_{n+2})\Delta_{n+1} - \xi_{n+1}\xi_{n+2}(1 - \xi_{n+1}\xi_{n+2})\Delta_n + \xi_n^2\xi_{n+1}\xi_{n+2}\Delta_{n-1}.
\end{align*}
\] (2.35)

Convergence of \( \Delta(0) \) is rapid enough to produce data to two or three decimals in a very short time. Similarly for the SH mode, one obtains the corresponding characteristic equation

\[
cosh(\sigma\pi) = -1 + 2\Delta(0)\sin^2\left(\frac{\pi\sqrt{A}}{2}\right) = -1 - 2\Delta(0)\sinh^2\left(\frac{\pi\sqrt{A}}{2}\right)
\] (2.36)

In order to obtain the marginal curves in the parameter space \((Ra, \alpha)\), we fix values of all other parameters, and solve the characteristic equations (2.34) and (2.36) numerically corresponding to the S and SH modes respectively. For a fixed \( \omega \) and a fixed wavenumber \( \alpha \), we first calculate the Floquet exponent \( \sigma \) using the marginal stability condition and then determine the corresponding value of \( Ra \), verifying the characteristic equation.

The stability characteristics, viz., the critical Rayleigh number \( Ra_c \), obtained by minimizing marginal \( Ra \) against \( \alpha \) and the critical wavenumber \( \alpha_c \), the \( \alpha \) corresponding to \( Ra_c \) are then calculated by fixing the values of other parameters. These results obtained by the two different methods explained so far agree well throughout the parameter space of interest. One can see Tables (2.1), (2.2) and (2.3) for a comparison between the results for selected values of \( \eta \) and \( \omega \). The methods however consume different time durations to produce the results with a preassigned precision, \( 10^{-4} \) throughout this study. This is due to the difference in convergence rates between \( \{M_n, n = 0, 1, 2, \ldots\} \) of the continued fraction method and \( \{\Delta_n, n = 0, 1, 2, \ldots\} \) of Hill’s determinant method. On an average the continued fraction method requires 8 to 10 terms and 12 to 15 terms respectively for lower
and higher values of $\omega$. But in the case of Hill’s determinant method as high as 15 to 25 terms are required for lower values of $\omega$ whereas 2 to 3 terms are sufficient for higher values of $\omega$. Hence we found that the convergence of continued fraction method is faster at lower frequency range whereas that of Hill’s determinant method is better at intermediate and higher frequency ranges. A combination of these two methods depending on the frequency was adopted throughout this work.

2.4 Nondimensional Quantities

Reducing the governing equations to a nondimensional form is a well established procedure that allows one to simplify the problem formulation, in particular, this procedure allows one to rewrite the equations in terms of nondimensional group numbers. These numbers play a decisive role in predicting the nature of flow. The following nondimensional numbers are appear in this thesis.

(i) **Filtration Grashof number** ($Gr$) is an important physical quantity appearing in problems involving natural convection in a porous media. It is defined as

$$Gr = \frac{\beta Ch^2 g_0 K}{\nu^2} \quad (2.37)$$

and it is proportional to the ratio of buoyancy force to viscous force.

(ii) **Prandtl number** ($Pr$), the material property of a fluid defined as

$$Pr = \frac{\nu}{\chi} \quad (2.38)$$

is the ratio of momentum and thermal diffusions.

(iii) **Rayleigh number** ($Ra = Gr \cdot Pr$) measures the ratio of buoyancy force to dissipation forces of viscous and thermal origin. It is defined as

$$Ra = \frac{\beta Ch^2 g_0 K}{\nu \chi} \quad (2.39)$$

and this number is the eigenvalue of the present study.
(iv) **Darcy number** \((Da)\) measures the ratio of viscous force to the Darcy resistance in a porous medium and is defined as

\[
Da = \frac{K}{h^2}
\]

When the permeability is very high the resistance of the flow becomes effectively controlled by ordinary viscous resistance. In that case, convection phenomenon is similar to that in an ordinary fluid layer. However in the most of the problems, either the viscous force is negligible or is of comparable order to the Darcy resistance.

(v) **Porosity-permeability parameter** \((c)\) arises due to the presence of time derivative in the momentum equation and is defined as

\[
c = \frac{Da}{\varphi}
\]

It is the ratio of dimensionless permeability (the Darcy number) to porosity and an important parameter when one considers gravity modulation.

(vi) **Heat capacity ratio** \((\varkappa)\) of a porous medium and fluid is defined as

\[
\varkappa = \frac{(\rho c_p)_m}{(\rho c_p)_f}
\]

where \((\rho c_p)_m = \varphi(\rho c_p)_f + (1 - \varphi)(\rho c_p)_s\) is the relative heat capacity of the porous medium.

(vii) **Anisotropic parameters**

The mechanical and thermal anisotropy parameters are defined respectively as

\[
K_r = \frac{K_x}{K_z}
\]

\[
\chi_r = \frac{\chi_x}{\chi_z}
\]

(viii) **Inter-phase heat transfer coefficient** \((H)\) and **porosity modified conductivity ratio** \((\gamma)\) are used while measuring the fluid-to-solid heat exchange and
are defined by

\[ H = \frac{\bar{H}h^2}{\varphi k_f} \]  
\[ \gamma = \frac{k_f \varphi}{k_s(1 - \varphi)} \]  

(2.44)

These parameters arise in the absence of local thermal equilibrium in porous media.

(x) **Viscoelastic (Strain suppression) parameter** \( \Gamma \) is the nondimensional retardation time which measures the increase in deformation with time under constant stress. It is defined by

\[ \Gamma = \frac{\bar{\Gamma}_x}{h^2} \]  

(2.45)

(xi) **Vibration parameters**

The nondimensional amplitude \( \eta \) and the nondimensional frequency \( \omega \) are the vibration parameters. They are defined as

\[ \eta = \frac{A \Omega^2}{\varphi g_0} \]  
\[ \omega = \frac{\Omega h^2}{\nu} \]  

(2.46)

where \( \eta \) and \( \omega \) measure the length and speed of oscillation respectively.
Figure 2.1 Schematic diagram
Table 2.1 Stress free case for the Brinkman model. Comparison of the numerical results obtained through continued fraction and Hill's determinant methods for $Pr = 1$ and $c = 0.01$.

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<th>$\omega$</th>
<th>$\eta$</th>
<th>$Ra_c$</th>
<th>$\alpha_c$</th>
<th>$Ra_c$</th>
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Table 2.2 Rigid case for the Brinkman model. Comparison of the numerical results obtained through continued fraction and Hill’s determinant methods for $Pr = 1$ and $c = 0.01$.

<table>
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<th>$Ra_c$</th>
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Table 2.3 Anisotropic porous medium. Comparison of the results obtained through continued fraction and Hill's determinant methods for $Pr = 1$ and $Da = c = 0.1$. 

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