Chapter 6

Thermal Instability in a
Viscoelastic Fluid Saturated
Porous Medium

6.1 Introduction

Convection in fluid layers in the presence of vibration is an important type of problem in heat transfer. Inhomogeneity of density in fluids occurs due to various causes like temperature gradients, the presence of a free surface or interface, the presence of solid or fluid inclusions and so on. As examples, we could mention the strong influence of vibrations on the convective instability of a nonuniformly heated fluid, on the generation of periodic structures on a fluid surface, or on stratification in disperse systems. Under certain conditions vibrations can also induce average flows in inhomogeneous media, even in the absence of other external forces. This phenomenon called vibrational convection has been studied for a fairly long time [see Gershuni and Lyubimov (1998)]. Vibrational convection in a fluid medium has received much attention than vibrational convection in a porous medium. Understanding different instability mechanisms and their interactions allows control of convective motions in various engineering applications like crystal growth, solidification of molten alloys, material processing, etc. In low gravity or microgravity
environments, we can expect that reduction or elimination of convective motion may enhance the properties and performance of materials such as crystals. The exploration of vibration at normal earth gravity and reduced gravity in a fluid saturated porous medium may provide a better understanding of certain physical processes and possibly may lead to the identification of new phenomena [see Razi et al. (2005)].

Polymeric solutions have high molecular weight and are viscoelastic in nature. The viscoelastic character of polymer solutions and melts gives rise to instabilities that are not seen in the flows of Newtonian fluids. Newtonian fluids respond instantaneously to an imposed deformation rate, whereas polymeric fluids respond on a macroscopically large time scale, known as the relaxation time. When subjected to deformation rates much larger than the inverse relaxation time, their behavior resembles that of elastic solids, whereas their response to deformation rates much smaller in magnitude than the inverse relaxation time resembles that of viscous liquids. For this reason, they are known as viscoelastic fluids. Oldroyd (1950) proposed a set of constitutive equations to explain the rheological behaviour of some viscoelastic fluids. It is now generally accepted that the Oldroyd model contains a set of parameters which can be adjusted to suit a large class of fluids. Vest and Arpaci (1969) have studied the conditions under which thermally induced overstability occurs in a viscoelastic liquid. It was found that oscillatory convection occurs at a possibly lower adverse temperature gradient. Rosenblat (1986) performed a stability analysis for the onset of convection in a fluid layer with free boundaries employing a very general constitutive relation that encompasses the Maxwell (1867) model, Oldroyd model (or the Jeffrey’s model) and Phan-Thien-Tanner (1977) model. He obtained analytical solutions of steady states for the case of exchange of stabilities and time periodic states for the case of overstability. Kolkka and Ierley (1987) extended this problem to the case of rigid boundaries and discussed the physical reasons for the stabilizing effect of Oldroyd-B fluid.

Basic understanding of natural convection in a non-Newtonian fluid saturated porous medium is of considerable importance in geophysics, geothermal energy modeling, bioengineering, thermal insulation material and solar receivers. Thermal
instability of Oldroyd-B type viscoelastic fluids in a horizontal porous medium was investigated by Kim et al. (2003). They found that overstability is the preferred mode for a certain parameter range and the onset of convection has the form of a supercritical and stable bifurcation independent of the values of the viscoelastic parameters. Yoon et al. (2004) theoretically analyzed the onset of buoyancy driven convection in a horizontal porous layer saturated with a viscoelastic liquid. They used Alisaev and Mirzadjanzade (1975) model to incorporate the relaxation time and showed that oscillatory instabilities can set in before the stationary ones. The same problem with a dynamical system approach was studied theoretically by Bertola and Cafaro (2006). They showed that the onset of the instability occurs at smaller values of the Rayleigh number compared to the case of a Newtonian fluid depending on the thermophysical properties of the fluid.

Previous works on the onset of Rayleigh-Bénard convection show that the critical limits for a layer with Rivlin-Ericksen fluid and Newtonian fluid coincide. With an additional effect like rotation, double diffusion, magnetic effect, etc., the oscillatory convection is possible for a certain range of parameter values. The non-Newtonian fluids modelled by the Rivlin-Ericksen constitutive equations (Rivlin-Ericksen (1955)) have the Cauchy stress tensor $\mathbf{T}$ given by

$$
\mathbf{T} = -p\mathbf{I} + \mu \mathbf{A}_1 + \overline{\Gamma} \mathbf{A}_2 + \mu_1 \mathbf{A}_3^2
$$

(6.1)

where $p$ is the pressure, $\mu$ the dynamic viscosity, $\overline{\Gamma}$ the viscoelasticity and $\mu_1$ the cross viscosity. $\mathbf{A}_1$ and $\mathbf{A}_2$, the Rivlin-Ericksen tensors denote respectively the rate of strain and acceleration and are defined by

$$
\mathbf{A}_1 = \nabla \mathbf{v} + (\nabla \mathbf{v})^T
$$

$$
\mathbf{A}_2 = \frac{d\mathbf{A}_3}{dt} + \mathbf{A}_1 \cdot \nabla \mathbf{v} + (\nabla \mathbf{v})^T \cdot \mathbf{A}_1
$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the filtration velocity and $\frac{d}{dt}$ the material time derivative. The viscoelastic fluids modelled by Rivlin-Ericksen constitutive equations are termed as second grade fluids. A detailed account on the characteristics of second grade fluids has been well documented by Dunn and Rajagopal (1995). They showed that it is necessary to have $\mu \geq 0$, $\overline{\Gamma} \geq 0$ and $\overline{\Gamma} + \mu_1 = 0$ for the
model to be consistent with thermodynamics. According to Brinkman’s extension of Darcy’s law the usual viscous term in the equation of the Rivlin-Ericksen fluid motion is replaced by

\[
\left(1 + \frac{\Gamma}{\partial t}\right) \left[ \nu \nabla^2 \mathbf{v} - \frac{\nu}{K} \mathbf{v} \right]
\]

where \( \nu \) and \( K \) are the kinematic viscosity and permeability respectively.

Now, let we see some recent works associated with Rivlin-Ericksen second grade fluid available in the literature. Sharma and Kumar (1997) studied hydromagnetic stability of two Rivlin-Ericksen elastico-viscous superposed conducting fluids. It was found that the stability criterion is independent of the effects of viscosity and viscoelasticity and is dependent on the orientation and magnitude of the magnetic field. Sunil et al. (2001) investigated thermosolutal instability of a Rivlin-Ericksen fluid in a porous medium in the presence of uniform vertical magnetic field and the Hall effect. They found that these effects introduce the oscillatory onset which is nonexistent in their absence. Siddheshwar and Srikrishna (2002) have made a weakly nonlinear analysis of convection in a second order fluid and concluded that onset of chaotic motion is possible. Recently, Xu and Yang (2007) studied thermosolutal convection in a Rivlin-Ericksen fluid saturated porous layer using a modified Darcy-Brinkman’s model. They proved that the necessary and sufficient conditions for stability coincide for a certain range of system parameters.

There are only few studies available in the literature on time dependent convection in a viscoelastic fluid layer. Yang (1997) studied numerically the instability of Maxwellian fluid layer heated from below in a modulated gravitational field. He reported that subharmonic disturbances are found to enhance the stabilizing effect at small Debroah numbers and destabilization effect at large Debroah numbers in the intermediate range of frequency. Recently, the effect of time-periodic temperature/gravity modulation at the onset of magneto-convection in a weak electrically conducting second order liquid was investigated by Mahabaleswar (2008). He concluded that the combined modulation is most destabilizing and stabilizing in the case of in-phase and out-of phase modulation respectively than individual modulations of temperature and gravity. More recently, Siddheshwar et al. (2010)
extended Yang’s (1997) work by considering Oldroyd-B model to constitute the viscoelastic effects. They obtained the results for Maxwell, Rivlin-Ericksen and Newtonian fluids as particular cases. Nonetheless, there seems to have been no work on the stability analysis of thermovibrational convection in a viscoelastic fluid saturated porous medium. Therefore, we perform an investigation in a second grade fluid saturated porous medium subjected to vibrations.

6.2 Mathematical Analysis

We consider an infinite horizontal Rivlin-Ericksen fluid saturated porous medium confined between two surfaces at a distance \( h \) apart. A Cartesian coordinate system is taken with the \( xy \)-plane coinciding the lower surface and the \( z \)-axis pointing vertically upwards. The lower surface at \( z = 0 \) and the upper surface at \( z = h \) are maintained at constant temperatures \( T_1 \) and \( T_2 \) respectively. Moreover the layer is subjected to externally imposed vertical vibrations of arbitrary amplitude and frequency. The porous medium is homogeneous and isotropic and Brinkman’s law is used to model flow through it. The dimensional equations governing the second grade fluid flow within the porous medium under Oberbeck-Boussinesq are

\[
\begin{align*}
\frac{1}{\varphi} \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\varphi^2} \mathbf{v} \cdot \nabla \mathbf{v} &= -\frac{1}{\rho} \nabla p + \left( 1 + \frac{\Theta}{\partial t} \right) \left[ \nu \nabla^2 \mathbf{v} - \frac{\nu}{K} \mathbf{v} \right] + \beta T g(t) \mathbf{k} & (6.2) \\
\varphi \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T &= \chi \nabla^2 T & (6.3) \\
\nabla \cdot \mathbf{v} &= 0 & (6.4)
\end{align*}
\]

where \( T \) is the temperature, \( \varphi \) the porosity, \( \rho \) the density, \( \beta \) the thermal expansion coefficient, \( \mathbf{k} \) the unit vector directed vertically upward, \( \varpi = (\rho c_p)_m/(\rho c_p)_f \) the heat capacity ratio of the porous medium and fluid and \( \chi \) the thermal diffusivity of the porous medium. The time dependent gravitational field is taken to be \( g(t) = g_0 + \frac{A}{\varphi} \Omega^2 f''(\tau) \), where \( g_0 \) is a reference acceleration level, \( A \) the vibration amplitude, \( \Omega \) the vibration frequency and \( f(\tau) \) the \( 2\pi \)-periodic function with zero \( 2\pi \)-average. The surfaces are assumed to be flat and stress-free and obey the
boundary conditions

\begin{align*}
  v_3 = \frac{\partial v_1}{\partial z} = \frac{\partial v_2}{\partial z} &= 0, \quad T = T_1 \text{ at } z = 0 \\
  v_3 = \frac{\partial v_1}{\partial z} = \frac{\partial v_2}{\partial z} &= 0, \quad T = T_2 \text{ at } z = h
\end{align*}

(6.5) (6.6)

We seek a solution to the quasi-equilibrium basic state in the form \( v = v^0, \ T = T^0(z) \) and \( p = p^0(z,t) \). Thus Eq.(6.2)-(6.4) together with the boundary conditions possess the following solution

\begin{align*}
  v^0 &= 0 \\
  T^0 &= T_1 - \frac{1}{h}(T_1 - T_2)z \\
  p^0 &= \beta \rho g(t) \left( T_1 z - \frac{1}{2h} (T_1 - T_2) z^2 \right)
\end{align*}

(6.7) (6.8) (6.9)

We study the stability of this basic state using the method of small perturbations. Let us consider the motion

\begin{align*}
  v = v^0 + u, \quad p = p^0 + q, \quad T = T^0 + \theta
\end{align*}

(6.10)

where \( u, q \) and \( \theta \) are small unsteady perturbations. Dimensionless variables are defined in terms of the length scale \( h \), the time scale \( h^2/\nu \), the velocity scale \( \nu/h \), the pressure scale \( \rho \nu^2/K \) and the temperature scale \( C h \) where \( C = (T_1 - T_2)/h \) is the basic quasi-equilibrium temperature gradient. Then the non-dimensional governing equations are

\begin{align*}
  c \frac{\partial u}{\partial t} &= -\nabla q + \left( 1 + \Gamma Pr \frac{\partial}{\partial t} \right) [D u \nabla^2 u - u] + Gr(1 + \eta f''(\tau)) \partial h \kappa \\
  \kappa \frac{\partial \theta}{\partial t} - u_3 &= \frac{1}{Pr} \nabla^2 \theta \\
  \nabla \cdot u &= 0
\end{align*}

(6.11) (6.12) (6.13)
where \( Da = \frac{K}{h^2} \) the Darcy number, \( Gr = \frac{\beta Ch^2 g_0 K}{\nu^2} \) the filtration Grashof number, 
\( \Gamma = \frac{\Gamma X}{h^2} \) the viscoelastic or strain suppression parameter, \( Pr = \frac{\nu}{\chi} \) the Prandtl number, 
\( c = \frac{Da}{\varphi} \) the porosity-permeability parameter, \( \eta = \frac{A\Omega^2}{\varphi g_0} \) the nondimensional amplitude of vibration and \( \omega = \frac{\Omega h^2}{\nu} \) the nondimensional frequency of vibration.

The nondimensional boundary conditions are

\[
\frac{\partial^2 u_3}{\partial z^2} = \theta = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1. \quad (6.14)
\]

In order to find the linear instability boundary for the system we first take curl curl of Eq.(6.11). The vertical component of the resulting equation has the following form

\[
c \frac{\partial}{\partial t} \nabla^2 u_3 = \left( 1 + \Gamma Pr \frac{\partial}{\partial t} \right) \left[ Da \nabla^4 u_3 - \nabla^2 u_3 \right] + Gr \left( 1 + \eta f''(\tau) \right) \nabla_1^2 \theta \quad (6.15)
\]

where \( \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) and \( \nabla^2 = \nabla_1^2 + \frac{\partial^2}{\partial z^2} \) are the Laplacian operators. We then expand the vertical of velocity and the temperature in terms of normal modes as

\[
(u_3, \theta) = \left( \tilde{u}_3(z, t), \tilde{\theta}(z, t) \right) e^{i(\alpha_1 x + \alpha_2 y)} \quad (6.16)
\]

where \( \alpha_1 \) and \( \alpha_2 \) represent wavenumbers in the \( x \) and \( y \) directions respectively. Substituting this into Eq.(6.15) and Eq.(6.12) and inserting the relation \( \tilde{u}_3(z, t) \) in terms of \( \tilde{\theta}(z, t) \) yields the following equation for \( \tilde{\theta}(z, t) \)

\[
\frac{c}{Pr} \frac{\partial}{\partial t} \left( \frac{1}{Pr} D^4 - \frac{2\alpha^2}{Pr} D^2 - \kappa \frac{\partial}{\partial t} D^2 + \frac{1}{Pr} \alpha_4 + \kappa \alpha^2 \frac{\partial}{\partial t} \right) \tilde{\theta} = \alpha^2 Gr \left( 1 + \eta f''(\tau) \right) \tilde{\theta} - \left( 1 + \Gamma Pr \frac{\partial}{\partial t} \right) \left( \frac{1}{Pr} D^4 - \frac{2\alpha^2}{Pr} D^2 \right) \frac{\partial}{\partial t} \tilde{\theta} \\
- \kappa \frac{\partial}{\partial t} D^2 + \frac{1}{Pr} \alpha_4 + \kappa \alpha^2 \frac{\partial}{\partial t} \tilde{\theta} + Da \left( 1 + \Gamma Pr \frac{\partial}{\partial t} \right) \left( \frac{1}{Pr} D^4 \right) \\
- \frac{3\alpha^2}{Pr} D^4 - \kappa \frac{\partial}{\partial t} D^4 + \frac{3\alpha^4}{Pr} D^2 + 2\kappa \alpha^2 \frac{\partial}{\partial t} D^2 - \kappa \alpha^4 \frac{\partial}{\partial t} - \frac{\alpha^6}{Pr} \tilde{\theta} \quad (6.17)
\]

where \( D = \frac{\partial}{\partial z} \) and \( \alpha^2 = \alpha_1^2 + \alpha_2^2 \) is overall horizontal wavenumber. Boundary conditions for this equation are

\[
\frac{\partial^2 \tilde{\theta}}{\partial z^2} = \frac{\partial^4 \tilde{\theta}}{\partial z^4} = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1 \quad (6.18)
\]
From Eq.(6.17) the \( z \)-variable is separated by taking \( \tilde{\theta}(z, t) = \sin(\pi z)\Theta(t) \) and the resulting equation is reduced to the canonical form of damped Mathieu’s equation in the form

\[
\Theta''(\tilde{t}) + \left( \frac{m^2}{P} + \frac{P + \text{Dam}^2 P + m^2 P_1 + \text{Dam}^4 P_1}{1 + P_1 P + \text{Dam}^2 P_1 P} \right) \Theta'(\tilde{t}) + \left( \frac{\text{Dam}^4 + m^2 - \frac{\alpha^2}{m^2} \text{Ra}(1 + \eta T''(\tau))}{1 + P_1 P + \text{Dam}^2 P_1 P} \right) \Theta(\tilde{t}) = 0 \tag{6.19}
\]

where \( \tilde{t} = \frac{t}{\sqrt{Prxc}} \), \( P = \Gamma \sqrt{\frac{Pr}{c\nu}} \), \( m^2 = \alpha^2 + \pi^2 \) and \( \text{Ra} = Gr \cdot Pr \) is the Rayleigh number which is positive for unstable equilibrium and negative for stable equilibrium. Now, we set \( f(\tau) = \cos \tau \) in Eq.(6.19) and get

\[
\tilde{\Theta}''(\tilde{t}) + \left( \frac{m^2}{P} + \frac{P + \text{Dam}^2 P + m^2 P_1 + \text{Dam}^4 P_1}{1 + P_1 P + \text{Dam}^2 P_1 P} \right) \tilde{\Theta}'(\tilde{t}) + \left( \frac{\text{Dam}^4 + m^2 - \frac{\alpha^2}{m^2} \text{Ra}(1 - \eta \cos(\tilde{\omega} \tilde{t}))}{1 + P_1 P + \text{Dam}^2 P_1 P} \right) \tilde{\Theta}(\tilde{t}) = 0 \tag{6.20}
\]

where \( \tilde{\omega} = \omega \sqrt{Prxc} \). For notational convenience, tilde from \( \tilde{t} \) and \( \tilde{\omega} \) will be subsequently omitted.

Both continued fraction and Hill’s infinite determinant methods were used to solve the Eq.(6.20). The expressions appearing in Eq.(2.21) and Eq.(2.29) are

\[
M_n = (1 + PP_1 + \text{Dam}^2 PP_1)(\sigma + i\omega)^2 + \left( \frac{P + m^2}{P} + \text{Dam}^2 P + m^2 P_1 + \text{Dam}^4 P_1 \right)(\sigma + i\omega) + m^2 + \text{Dam}^4 - \frac{\alpha^2}{m^2} \text{Ra}
\]

and

\[
\begin{align*}
A &= \frac{4}{\omega^2} \left( \frac{m^2 + \text{Dam}^4 - \frac{\alpha^2}{m^2} \text{Ra}}{1 + PP_1 + \text{Dam}^2 PP_1} - \zeta^2 \right), \\
B &= -\frac{4}{\omega^2} \left( \frac{q}{1 + PP_1 + \text{Dam}^2 PP_1} \right),
\end{align*}
\]
2z = \left( \frac{P + \frac{m^2}{P} + Dam^2 P + m^2 P_i + Dam^4 P_i}{1 + PP_i + Dam^2 PP_i} \right) .

6.3 Results and Discussion

We investigate the effect of vertical harmonic vibration on the stability characteristics of a second grade fluid saturated porous medium. We shall consider Case (i): a porous layer heated from below and Case (ii): a porous layer heated from above. We obtain the relation for Rayleigh number corresponding to steady onset of convection in a nonvibrating medium from Eq.(2.31) by setting \( \sigma = \eta = n = 0 \) as \( R_0 = (Dam^2 + 1)m^4/\alpha^2 \) (see Malashetty and Padmavathy (1997)). We note that this expression is independent of the viscoelastic parameter and hence applicable for Newtonian fluids as well. For \( Da = 0 \) it leads to \( R_{0c} = 4\pi^2 \) and \( \alpha_c = \pi \), the well known criteria for the Darcy model. In the limiting case \( Da \to \infty \) we get \( R_{0c}/Da = 27\pi^4/4 \) and \( \alpha_c = \sqrt{\pi}/2 \), the results of Rayleigh-Bénard problem with stress free boundaries. The expression (6.20) contains, in particular, the corresponding result of Chapter 3 for \( T = 0 \). The results were obtained for physically feasible values of the parameters under consideration. The value of \( \Gamma \) for dilute polymeric solution is most likely in the range \([0.1, 2]\). A typical value of Prandtl number for non-Newtonian fluids is \( Pr = 10 \).

6.3.1 Heated Below (\( Ra > 0 \))

Figures (6.1) and (6.2) show \( Ra_c \) and \( \alpha_c \) as functions of \( \omega \) for small vibration amplitude \( \eta = 2 \). \( \Gamma \) and \( \omega \) leave almost no effect on the stability characteristics for the Darcy model; one can see that \( Ra_c \approx 4\pi^2 \) and \( \alpha_c \approx \pi \) for all \( \omega \). On the other hand the effect of vibration is visible in the Brinkman model. When the fluid is Newtonian the effect of vibration is less, as seen in previous Chapters. However the onset of convection in a non-Newtonian fluid saturated porous medium due to vibration can be well controlled in the narrow range \( 1 \leq \omega \leq 10 \). We see that
the effect of increasing \( \Gamma \) is to destabilize the system except at low \( \omega \). \( Ra_c \) and \( \alpha_c \) become independent of \( \omega \) when either \( \Gamma \) or \( \omega \) takes higher values and approach the respective nonvibrating values \( Ra_c = 108.576 \) and \( \alpha_c = 2.472 \) in the Brinkman model. In the presence of vibration, the onset of instability can appear either in the form of S mode or SH mode. In Fig.(6.1), \( Ra_c \) exhibits a cusp at a particular \( \omega \), called transition frequency for \( \Gamma = 0.1 \). We find that the onset of instability is of SH type up to this \( \omega \) and of S type beyond this \( \omega \). This cuspidal point in \( Ra_c \) curve is in fact the intersection of an SH curve with an S curve, both of which can be continued beyond the intersection. Thus in the region above an SH curve there can be S responses as well, but disturbances with SH mode is expected to be more unstable. Similarly, above an S curve there may be SH responses, but disturbances with S mode are more unstable. Hence we are interested only in the most unstable mode through which the instability sets in first. The critical wavenumber corresponding to the cuspidal point (Fig.(6.2)) suffers a discontinuity only because we do not extend \( Ra_c \) curves for the SH and S modes beyond their intersection. We now explain this transition in terms of marginal curves. The solution to Eq.(6.20) through the Floquet analysis revealed marginal curves in the form of a group of loop shaped branches, as illustrated in previous Chapters. The bottommost loop was SH for very small \( \omega \) and the minimum of this loop determined the onset condition. An additional thin S loop emerged and developed in the low wavenumber region for an increase in \( \omega \) and at one stage its critical value reached the level of the SH loop. Hence \( Ra_c \) for both the S and SH modes were equal at the transition frequency with two different \( \alpha_c \) values. For a further increase in \( \omega \) beyond this transition frequency the S mode grew further and became critical.

Figures (6.3) and (6.4) show the stability characteristics against \( \omega \) for an increased vibration amplitude \( \eta = 20 \) and various values of \( \Gamma \). It is found that increasing \( \Gamma \) inhibits the onset of convection in the low \( \omega \) region through the SH response whereas augments it in the high \( \omega \) region through the S response. The observed behaviour is similar to that of \( \eta = 2 \) for larger values of \( \omega \). The corresponding \( \alpha_c \) for the SH mode increases against \( \Gamma \) at low \( \omega \) and becomes invariant as \( \omega \) increases further. Moreover the transition point in \( Ra_c \) gets shifted to lower
frequency region when $\Gamma$ increases. In general, the effect of $Da$ is to delay the onset of convection. However it could promote the onset for some intermediate values of $\omega$ in the case of a non-Newtonian fluid. Comparing Figs.(6.1) and (6.3) we observe that this is possible only for large amplitude vibration. The corresponding $\alpha_c$ maintains the general trend for all $\omega$, i.e., $\alpha_c$ for the Darcy model is always greater than that of the Brinkman model. In Fig.(6.3) we also notice that mode transfer in $Ra_c$ for the non-Newtonian fluid is scarce. On the other hand Newtonian fluid introduces a severe competition between the S and SH modes over a wider range of $\omega$. Thus the S and SH branches become increasingly small and closely packed as $\omega \to 0$. The corresponding $\alpha_c$ plotted in Fig.(6.4) exhibits several discontinuities and maintains an increasing trend corresponding to each branch. We notice that the jumps in $\alpha_c$ are small for $\Gamma \neq 0$ and large for $\Gamma = 0$. Also it is clear to see that $Ra_c$ approaches the nonvibrating value even when $\omega = 1$ in the case of a Newtonian fluid. But the effect of vibration is also significant for $\omega < 1$ in the case of a non-Newtonian fluid. One has to notice that the final transition frequency for $\Gamma = 0$ is almost ten times greater than that for $\Gamma = 2$ implying that non-Newtonian fluid shifts the vibration effect to lower frequency range. At the same time the stability limits are substantially modified at low $\omega$ as $\Gamma$ allows to reduce the critical Rayleigh number lower than that of the Newtonian case.

Next, the stability behaviour for large vibration amplitude $\eta = 200$ is shown in Figs.(6.5) and (6.6). The stability boundaries maintain the same trend as in $\eta = 20$ except a drop in $Ra_c$ at low $\omega$ and a rise in it at high $\omega$. We note that $Ra_c$ for the non-Newtonian fluid reaches the nonvibrating value faster for sufficiently large $\omega$. It is interesting to note that in the Brinkman model $\alpha_c$ for the SH mode in the non-Newtonian case tends to a particular value as $\omega$ increases. But in the Darcy model it maintains a monotonically increasing behaviour. A comparison of Figs.(6.1)-(6.6) show that the viscoelastic parameter $\Gamma$ has almost equal effect for $\eta = 20$ and 200 whereas it produces a different response for $\eta = 2$. 
6.3.2 Heated Above \((Ra < 0)\)

Figures (6.7) and (6.8) show the onset criteria for different values of \(\Gamma\) and small vibration amplitude \(\eta = 2\). They show that convective instability is possible even for heating from above. We observed that \(-Ra_c \rightarrow \infty\) for both \(\omega \rightarrow 0\) and \(\omega \rightarrow \infty\). A substantial reason for the unboundedness of \(-Ra_c\) as \(\omega \rightarrow 0\) is that it would be almost impossible to destabilize a porous layer which is heated from above. This interesting behaviour again occurs at \(\omega \rightarrow \infty\). Unlike the case of \(Ra > 0\) for \(\eta = 2\) (see Figs.(6.1) and (6.2)), the effect of \(\Gamma \neq 0\) is found to stabilize the system. It is clear that \(Ra_c\) for \(\Gamma = 0\) is somewhat more complicated as the S and SH modes of instability frequently intersect at low frequencies. Hence a severe nesting is seen between them leading to a large number of cusps in \(-Ra_c\) which are associated with jumps in the corresponding \(\alpha_c\) (see Fig.(6.8)). Also we note that the jumps in \(\alpha_c\) are narrow at low frequencies and gradually widen for an increase in \(\omega\). Moreover an increase in \(\Gamma\) suppresses the competition between the two modes and restricts it to a narrower range of \(\omega\) near \(\omega = 0\). We notice that when the porous medium is of Darcy type there is no mode transition in \(Ra_c\) and SH mode is the preferred mode for all \(\omega\) under consideration. Similar to the case \(Ra > 0\), the corresponding \(\alpha_c\) for the SH mode increases against \(\Gamma\) at low \(\omega\) and becomes invariant as \(\omega \rightarrow \infty\), i.e., the effect of \(\Gamma\) is to constrict the convective cells at low \(\omega\) and sustain them as \(\omega \rightarrow \infty\). Moreover we observed that SH mode becomes the preferred mode of convective onset beyond \(\omega > 10^2\) and hence we considered \(\omega\) only up to \(10^3\).

In Figs.(6.9) and (6.10) we have shown a similar plot for \(\eta = 20\) for a porous layer heated from above. We observe a trend similar to Fig.(6.7) with a reduction in \(-Ra_c\) indicating the enhancement of convection. However we notice that the onset of instability is dictated solely by the SH mode for sufficiently higher \(\Gamma\). On the other hand when the porous medium is saturated with a nearly Newtonian fluid the instability is exhibited with a severe competition between the S and SH modes, as observed in Fig.(6.7). The corresponding critical wavenumbers shown in Fig.(6.10) are almost equal and preserve a trend similar to that in Fig.(6.8). Unlike the case \(Ra > 0\), we observe that \(-Ra_c\) for the Darcy model always exceed
that of the Brinkman model except at low $\omega$ for all $\eta$. Comparing Figs.(6.7)-(6.10), increasing $\eta$ and $\Gamma$ restricts the interaction between the two modes of instability to low frequencies substantially.

We illustrate in Figs.(6.11) and (6.12) the dependence of the stability characteristics on the Prandtl number $Pr$ for different values of the viscoelastic parameter $\Gamma$. We observe that $Ra_c \to R_{0c}$ for both $Pr \to 0$ and $Pr \to \infty$ via the S mode. Also we observe that the viscoelastic parameter $\Gamma$ has no effect at very low $Pr$ consistent with the physical reasoning. Moreover increasing $\Gamma$ stabilizes the system via the SH mode whereas destabilizes it via the S mode for intermediate values of $Pr$. However the SH mode shrinks to low $Pr$ region and S mode replaces it for an increase in $\Gamma$. We observe that $\alpha_c$ corresponding to the SH mode reaches a maximum at an intermediate $Pr$ for $\Gamma = 0$. This behaviour is not seen for an increased values of $\Gamma$. Comparing with $\Gamma \neq 0$, we observe that $\Gamma = 0$ has a wider SH response and it takes higher $Pr$ values to reach the nonvibrating limits. However the above trend is not seen in the Darcy model; only the S mode is observed to determine the critical condition throughout the range of $Pr$. Finally, a similar plot for the layer heated from above is shown in Figs.(6.13) and (6.14). The SH mode is dominant for intermediate values of $Pr$ and approach the nonvibrating results for very low and high values of $Pr$. This is more visible for the non-Newtonian fluid case. Similar to $Ra > 0$, $\Gamma$ has no effect at very low $Pr$ and the corresponding $Ra_c$ and $\alpha_c$ are almost identical. However $\alpha_c$ for intermediate values of $Pr$ decreases against $\Gamma$ and is invariant for further increase in $Pr$. 
Figure 6.1 $Ra_c$ against $\omega$ for different $\Gamma$ with $Da = 0.1 = c$ for Brinkman model, $Da = 10^{-4}$, $c = 10^{-2}$ for Darcy model and $\eta = 2$.

Figure 6.2 $\alpha_c$ against $\omega$ for different $\Gamma$ with $Da = 0.1 = c$ for Brinkman model, $Da = 10^{-4}$, $c = 10^{-2}$ for Darcy model and $\eta = 2$. 
Figure 6.3 $Ra_c$ against $\omega$ for different $\Gamma$ with $Da = 0.1 = c$ for Brinkman model, $Da = 10^{-4}, c = 10^{-2}$ for Darcy model and $\eta = 20$.

Figure 6.4 $\alpha_c$ against $\omega$ for different $\Gamma$ with $Da = 0.1 = c$ for Brinkman model, $Da = 10^{-4}, c = 10^{-2}$ for Darcy model and $\eta = 20$. 
Figure 6.7 $-Ra_c$ against $\omega$ for different $\Gamma$ with $Da = 0.1 = c$ for Brinkman model, $Da = 10^{-4}$, $c = 10^{-2}$ for Darcy model and $\eta = 2$.

Figure 6.8 $\alpha_c$ against $\omega$ for different $\Gamma$ with $Da = 0.1 = c$ for Brinkman model, $Da = 10^{-4}$, $c = 10^{-2}$ for Darcy model and $\eta = 2$. 
Figure 6.9 $-Ra_c$ against $\omega$ for different $\Gamma$ with $Da = 0.1 = c$ for Brinkman model, $Da = 10^{-4}, c = 10^{-2}$ for Darcy model and $\eta = 20$.

Figure 6.10 $\alpha_c$ against $\omega$ for different $\Gamma$ with $Da = 0.1 = c$ for Brinkman model, $Da = 10^{-4}, c = 10^{-2}$ for Darcy model and $\eta = 20$. 
Figure 6.11 $Ra_c$ against $Pr$ for different $\Gamma$ with $Da = 0.1 = c$ for Brinkman model, $Da = 10^{-4}$, $c = 10^{-2}$ for Darcy model and $\omega = 100$, $\eta = 20$.

Figure 6.12 $\alpha_c$ against $Pr$ for different $\Gamma$ with $Da = 0.1 = c$ for Brinkman model, $Da = 10^{-4}$, $c = 10^{-2}$ for Darcy model and $\omega = 100$, $\eta = 20$. 
Figure 6.13 $-Ra_c$ against $Pr$ for different $\Gamma$ with $Da = 0.1 = c$ for Brinkman model, $Da = 10^{-4}$, $c = 10^{-2}$ for Darcy model and $\omega = 100$, $\eta = 20$.

Figure 6.14 $\alpha_c$ against $Pr$ for different $\Gamma$ with $Da = 0.1 = c$ for Brinkman model, $Da = 10^{-4}$, $c = 10^{-2}$ for Darcy model and $\omega = 100$, $\eta = 20$. 