Chapter 4

Global Attractivity for Nonlinear Functional Integral Equation with Deviating Arguments

4.1 Introduction

There are two approaches for dealing with the characteristics of solutions, namely the classical fixed point theorems involving hypotheses from analysis and topology and the fixed point theorems involving the use of measures of noncompactness. Each one of these approaches has some advantages and disadvantages over others.

In this chapter, we use the latter method to prove the existence results for a nonlinear functional integral equation with deviating arguments. Apart from this we prove the global attractivity results of solutions of the equation which was introduced by Hu and Yan [83]. The investigations are carried out in the Banach space of real, continuous and bounded functions defined on the real axis. The main tool used in this chapter is a variant of fixed point theorem of Krasnoselskii [87] which was proved recently by Burton [21].

The results obtained generalize and extend several ones obtained earlier in numerous papers concerning asymptotic stability and attractivity of solutions of some functional integral equations [20, 29, 30, 39, 64, 66, 83, 94, 112].
4.2 Preliminaries

Consider the Banach space $BC(R_+)$ consisting of all real functions which are continuous and bounded on $R_+$ and endowed with the standard norm $||x|| = \sup\{|x(t)| : t \geq 0\}$.

Let us put $X = BC(R_+)$ and let $\Omega$ be a nonempty subset of $X$. We need the following theorem for proving the main result.

**Theorem 4.2.1.** [33] Let $\Omega$ be a bounded subset of the space $BC(R_+)$. Assume that the following conditions are satisfied:

1° The functions belonging to $\Omega$ are locally equicontinuous on the interval $R_+$.

2° The functions of the set $\Omega$ vanish at infinity uniformly with respect to $\Omega$, that is, for each $\varepsilon > 0$, there exists $T > 0$ such that $|x(t)| \leq \varepsilon$ for every $t \geq T$ and for all $x \in \Omega$.

Then the set $\Omega$ is relatively compact in the space $BC(R_+)$. Consider the following nonlinear functional integral equation with deviating arguments

$$x(t) = f(t, x(\alpha_1(t)), ..., x(\alpha_n(t)))$$

$$+ \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s)))ds, \quad t \geq 0.$$  \hspace{1cm} (4.2.1)

We will investigate equation (4.2.1) under the following hypotheses:

(H1) The function $f : R_+ \times R^n \to R$ is continuous and there exist constants $k_i \in [0, 1)$ ($i = 1, 2, ..., n$) such that

$$|f(t, x_1, x_2, ..., x_n) - f(t, y_1, y_2, ..., y_n)| \leq \sum_{i=1}^n k_i|x_i - y_i|,$$

for all $t \in R_+$ and for all $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_n) \in R^n$.

(H2) The function $t \to f(t, 0, ..., 0)$ is bounded on $R_+$ with $F_0 = \sup\{|f(t, 0, ..., 0)| : t \in R_+\}$.

(H3) The functions $\alpha_i, \gamma_j : R_+ \to R_+$ are continuous and $\alpha_i(t) \to \infty$ as $t \to \infty$ ($i = 1, 2, ..., n; j = 1, 2, ..., m$).
(H4) The function $\beta : R_+ \to R_+$ is continuous.

(H5) The function $g : R_+^2 \times R^m \to R$ is continuous and there exist continuous functions $q : R_+^2 \to R_+$ and $a, b : R_+ \to R_+$ such that

$$|g(t, s, x_1, x_2, \ldots, x_m)| \leq q(t, s) + a(t)b(s)\sum_{i=1}^m |x_i|$$

for all $t, s \in R_+$ and $(x_1, x_2, \ldots, x_m) \in R^m$. Moreover we assume that

$$\lim_{t \to \infty} \int_0^{\beta(t)} q(t, s)ds = 0, \quad \lim_{t \to \infty} a(t) \int_0^{\beta(t)} b(s)ds = 0.$$ 

Let us observe that in view of assumption (H5) the functions $v_1, v_2 : R_+ \to R_+$ defined by the formulas

$$v_1(t) = \int_0^{\beta(t)} q(t, s)ds, \quad v_2(t) = a(t) \int_0^{\beta(t)} b(s)ds$$

are continuous and bounded on $R_+$. This implies that the constants $M_1$ and $M_2$ defined as

$$M_i = \sup\{v_i(t) : t \in R_+\} \quad (i = 1, 2)$$

are finite.

In order to formulate our last assumption let us denote $k = \sum_{i=1}^n k_i$, where the constants $k_i$ ($i = 1, 2, \ldots, n$) are involved in the assumption (H1).

(H6) $k + mM_2 < 1$.

Then we have the following theorem containing our main result.

### 4.3 Main Result

**Theorem 4.3.1.** Under assumptions (H1)-(H6), equation (4.2.1) has at least one solution in the space $BC(R_+)$. Moreover the solutions of equation (4.2.1) are globally attractive.
Proof. Consider the closed ball \( B_r = B(0, r) \) in \( X \) centered at zero \( 0 \) and with radius \( r \), where \( r = (F_0 + M_1)/[1 - (k + rM_2)] \). Next define two mappings \( A \) on \( X \) and \( B \) on \( B_r \) by putting

\[
(Ax)(t) = f(t, x(\alpha_1(t)), ..., x(\alpha_n(t))) ,
\]

\[
(Bx)(t) = \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s)))ds ,
\]

for \( t \in R_+ \). Then, equation (4.2.1) can be written equivalently in the form

\[
x(t) = (Ax)(t) + (Bx)(t), \quad t \in R_+ .
\]

Let us observe that in view of assumptions (H1)-(H3), the mapping \( A \) is well defined and, for arbitrarily fixed function \( x \in X \), the function \( Ax \) is continuous and bounded on \( R_+ \). Thus \( A \) is a self-mapping of the space \( X \). On the other hand, applying assumptions (H3)-(H5), we deduce that the function \( Bx \) is continuous and bounded on \( R_+ \) which implies \( B \) transforms the ball \( B_r \) into \( X \).

Now we show that operators \( A \) and \( B \) satisfy the assumptions imposed in Theorem 1.4.3. To this end, take \( x, y \in X \). Then, in virtue of assumption (H1), for a fixed \( t \in R_+ \), we get:

\[
|(Ax)(t) - (Ay)(t)| = |f(t, x(\alpha_1(t)), ..., x(\alpha_n(t))) - f(t, y(\alpha_1(t)), ..., y(\alpha_n(t)))|
\]

\[
\leq \sum_{i=1}^{n} k_i |x(\alpha_i(t)) - y(\alpha_i(t))|
\]

\[
\leq k ||x - y|| .
\]

This yields \( ||Ax - Ay|| \leq k||x - y|| \). Hence, keeping in mind assumption (H6), we infer that \( A \) is a contraction on \( X \).

Next we claim that \( B \) is completely continuous on the ball \( B_r \). In order to show that \( B \) is continuous on \( B_r \), take \( \varepsilon > 0 \) and \( x, y \in B_r \) such that \( ||x - y|| \leq \varepsilon \). Then taking into account our assumptions we obtain

\[
|(Bx)(t) - (By)(t)|
\]

\[
\leq \int_0^{\beta(t)} |g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s))) - g(t, s, y(\gamma_1(s)), ..., y(\gamma_m(s)))|ds
\]

\[
\leq \int_0^{\beta(t)} [||g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s)))|| + ||g(t, s, y(\gamma_1(s)), ..., y(\gamma_m(s)))||]ds
\]

\[
\leq \int_0^{\beta(t)} [||g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s)))|| + ||g(t, s, y(\gamma_1(s)), ..., y(\gamma_m(s)))||]ds
\]
By assumption (H5), we deduce that there exists $T > 0$ such that $v_1(t) + mrv_2(t) \leq \varepsilon/2$ for $t \geq T$. Linking this fact with equation (4.3.2), we conclude that

$$|(Bx)(t) - (By)(t)| \leq \varepsilon$$

(4.3.3)

for $t \geq T$.

Further, fix arbitrarily $t \in [0, T]$. Then, evaluating similarly as above, we have

$$|(Bx)(t) - (By)(t)|$$

$$\leq \int_0^{\beta(t)} |g(t, s, x_1(s), ..., x_m(s)) - g(t, s, y_1(s), ..., y_m(s))|ds$$

$$\leq \int_0^{\beta(t)} \omega^T_r(g, \varepsilon)ds$$

$$\leq \beta_T \omega^T_r(g, \varepsilon) ,$$

(4.3.4)

where we denoted

$$\beta_T = \sup\{\beta(t) : t \in [0, T]\}$$

and

$$\omega^T_r(g, \varepsilon) = \sup\{|g(t, s, x_1, x_2, ..., x_m) - g(t, s, y_1, y_2, ..., y_m)| : t \in [0, T], s \in [0, \beta_T], x_i, y_i \in [-r, r], |x_i - y_i| \leq \varepsilon (i = 1, 2, ..., m)\} .$$

Obviously $\beta_T < \infty$ and in view of the uniform continuity of the function $g(t, s, x_1, x_2, ..., x_m)$ on the set $[0, T] \times [0, \beta_T] \times [-r, r]^m$, we deduce that $\omega^T_r(g, \varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, keeping in mind equation (4.3.3), equation (4.3.4) and the above established fact, we conclude that the operator $B$ is continuous on the ball $B_r$.

Next we show that the set $B(B_r)$ is bounded in the space $BC(R+)$. Indeed, taking an arbitrary function $x \in B_r$ and using assumptions (H4) and (H5), for arbitrarily
fixed $t \in \mathbb{R}_+$, we get:

$$|(Bx)(t)| \leq \int_0^{\beta(t)} |g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s)))| ds$$

$$\leq \int_0^{\beta(t)} \left[ q(t, s) + a(t)b(s) \sum_{i=1}^{m} |x(\gamma_i(s))| \right] ds$$

$$= \int_0^{\beta(t)} q(t, s) ds + a(t) \int_0^{\beta(t)} b(s) \left( \sum_{i=1}^{m} |x(\gamma_i(s))| \right) ds$$

$$\leq v_1(t) + a(t) \int_0^t rmb(s) ds = v_1(t) + rmv_2(t) . \quad (4.3.5)$$

This estimate yields that $||Bx|| \leq M_1 + rmM_2$ which shows that the set $B(B_r)$ is bounded.

Now we prove that the set $B(B_r)$ satisfies other assumptions of Theorem 4.2.1. To do this, fix arbitrarily a number $\varepsilon > 0$. Next, invoking assumption (H5), let us choose a number $T > 0$ such that $v_1(t) + rmv_2(t) \leq \varepsilon$ for $t \geq T$. Then, for an arbitrary function $x \in B_r$ and for $t \geq T$, in view of estimate (4.3.5), we infer that

$$|(Bx)(t)| \leq \varepsilon . \quad (4.3.6)$$

Further take arbitrary numbers $t, \tau \in [0, T]$ such that $|t - \tau| \leq \varepsilon$. Then, keeping in mind our assumptions, we obtain

$$|(Bx)(t) - (Bx)(\tau)|$$

$$\leq \left| \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s))) ds - \int_0^{\beta(\tau)} g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s))) ds \right|$$

$$+ \left| \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s))) ds - \int_0^{\beta(\tau)} g(\tau, s, x(\gamma_1(s)), ..., x(\gamma_m(s))) ds \right|$$

$$\leq \left| \int_0^{\beta(t)} |g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s)))| ds \right|$$

$$+ \left| \int_0^{\beta(\tau)} |g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s))) - g(\tau, s, x(\gamma_1(s)), ..., x(\gamma_m(s)))| ds \right|$$

$$= \left| \int_0^{\beta(t)} |g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s)))| ds \right|$$

$$+ \left| \int_0^{\beta(\tau)} |g(t, s, x(\gamma_1(s)), ..., x(\gamma_m(s))) - g(\tau, s, x(\gamma_1(s)), ..., x(\gamma_m(s)))| ds \right|$$
\[ \beta_T = \max \{ \beta(t) : 0 \leq t \leq T \} \]

and

\[ \omega_T^T(g, \varepsilon; r) = \sup \{|g(t, s, x_1, ..., x_m) - g(\tau, s, x_1, ..., x_m)| : t, \tau \in [0, T], |t - \tau| \leq \varepsilon, s \in [0, \beta_T], |x_1| \leq r, ..., |x_m| \leq r \} . \]

From the estimate (4.3.7), we get

\[ |(Bx)(t) - (Bx)(\tau)| \leq q_T \omega_T^T(\beta, \varepsilon) + m r a_T b_T \omega_T^T(\beta, \varepsilon) + \beta_T \omega_T^T(g, \varepsilon; r) , \] (4.3.8)

where we denote

\[ q_T = \max \{q(t, s) : t \in [0, T], s \in [0, \beta_T]\} , \]
\[ a_T = \max \{a(t) : t \in [0, \beta_T]\} , \]
\[ b_T = \max \{b(t) : t \in [0, \beta_T]\} . \]

Moreover, \( \omega_T^T(\beta, \varepsilon) \) denotes the modulus of continuity of the function \( \beta \) on the interval \([0, T]\), that is

\[ \omega_T^T(\beta, \varepsilon) = \sup \{|\beta(t) - \beta(\tau)| : t, \tau \in [0, T], |t - \tau| \leq \varepsilon\} . \]

Now let us observe that keeping in mind the uniform continuity of the function \( \beta = \beta(t) \) on the interval \([0, T]\) and the uniform continuity of the function \( g = g(t, s, x_1, x_2, ..., x_m) \) on the set \([0, T] \times [0, \beta_T] \times [-r, r]^m\), we infer that \( \omega_T^T(\beta, \varepsilon) \to 0 \) and \( \omega_T^T(g, \varepsilon; r) \to 0 \) as \( \varepsilon \to 0 \). Thus, taking into account the boundedness of the set \( B(B_r) \) and the estimates (4.3.6), (4.3.8), in view of Theorem 4.2.1, we conclude that the set \( B(B_r) \) is relatively compact in the space \( BC(R_+) \).

The above established assertions allow us to infer that the operator \( B \) is completely continuous on the ball \( B_r \).
Next fix arbitrarily \( x \in BC(R_+) \) and assume that the equality \( x = Ax + By \) holds for some \( y \in B_r \). Then, utilizing our assumptions, for a fixed \( t \in R_+ \), we get

\[
|x(t)| \leq |(Ax)(t)| + |(By)(t)|
\]

\[
\leq |f(t,x(\alpha_1(t)),...,x(\alpha_n(t)))| + \int_0^{\beta(t)} |g(t,s,y(\gamma_1(s)),...,y(\gamma_m(s)))|ds
\]

\[
\leq |f(t,x(\alpha_1(t)),...,x(\alpha_n(t)))) - f(t,0,...,0)| + |f(t,0,...,0)|
\]

\[
+ \int_0^{\beta(t)} q(t,s)ds + m||y||\int_0^{\beta(t)} b(s)ds
\]

\[
\leq \sum_{i=1}^n k_i|x(\alpha_i(t))| + F_0 + v_1(t) + mrv_2(t)
\]

\[
\leq k||x|| + F_0 + M_1 + mrM_2.
\]

The above inequality yields

\[
||x||(1-k) \leq F_0 + M_1 + mrM_2.
\]

Since \( 1-k > 0 \), this implies

\[
||x|| \leq \frac{F_0 + M_1 + mrM_2}{1-k}.
\]

On the other hand, we have that

\[
\frac{F_0 + M_1 + mrM_2}{1-k} = r.
\]

Hence \( ||x|| \leq r \) or equivalently \( x \in B_r \). This shows that assumption (c) of Theorem 1.4.3 is satisfied.

Finally, combining all above established facts and applying Theorem 1.4.3 we conclude that there exists at least one solution \( x = x(t) \) of equation (4.2.1) in the space \( BC(R_+) \) belonging to the ball \( B_r \).

In what follows we show that solutions of equation (4.2.1) are globally attractive in the sense of Definition 1.5.3. To this end, assume that the functions \( x = x(t), \ y = y(t) \) are solutions of equation (4.2.1) in the space \( BC(R_+) \). Then, invoking our assumptions, we have

\[
|x(t) - y(t)|
\]

\[
\leq |f(t,x(\alpha_1(t)),...,x(\alpha_n(t)))) - f(t,y(\alpha_1(t)),...,y(\alpha_n(t))))|
\]
\[ + \int_0^{\beta(t)} \left| g(t, s, x(\gamma_i(s)), ..., x(\gamma_m(s))) - g(t, s, y(\gamma_i(s)), ..., y(\gamma_m(s))) \right| ds \]
\[ \leq \sum_{i=1}^n k_i |x(\alpha_i(t)) - y(\alpha_i(t))| \]
\[ + \int_0^{\beta(t)} \left| g(t, s, x(\gamma_i(s)), ..., x(\gamma_m(s))) \right| + \left| g(t, s, y(\gamma_i(s)), ..., y(\gamma_m(s))) \right| ds \]
\[ \leq \sum_{i=1}^n k_i \max \{|x(\alpha_i(t)) - y(\alpha_i(t))| : i = 1, 2, ..., n\} + 2 \int_0^{\beta(t)} q(t, s) ds \]
\[ + a(t) \int_0^{\beta(t)} \left( b(s) \sum_{i=1}^m |x(\gamma_i(s))| \right) ds + a(t) \int_0^{\beta(t)} \left( b(s) \sum_{i=1}^m |y(\gamma_i(s))| \right) ds \]
\[ \leq k \max \{|x(\alpha_i(t)) - y(\alpha_i(t))| : i = 1, 2, ..., n\} \]
\[ + 2v_1(t) + m(||x|| + ||y||) v_2(t) . \]

Hence we obtain
\[ \limsup_{t \to \infty} |x(t) - y(t)| \leq k \max_{1 \leq i \leq n} \left\{ \limsup_{t \to \infty} |x(\alpha_i(t)) - y(\alpha_i(t))| \right\} \]
\[ + 2 \limsup_{t \to \infty} v_1(t) + m(||x|| + ||y||) \limsup_{t \to \infty} v_2(t) \]
\[ = k \limsup_{t \to \infty} |x(t) - y(t)| . \]

This implies that
\[ \limsup_{t \to \infty} |x(t) - y(t)| = \lim_{t \to \infty} |x(t) - y(t)| = 0 . \]

This means that solutions of equation (4.2.1) are globally attractive.

### 4.4 Examples

In this section we provide two examples illustrating our result contained in Theorem 4.3.1.
Example 4.4.1

Consider the following functional integral equation

\[ x(t) = \frac{t^2}{t^2 + 1} + \sum_{i=1}^{3} \frac{t}{(2 + i)t + 1} \ln \left(1 + \left|x\left(\frac{t}{2 + i}\right)\right|\right) + \int_{0}^{\sqrt{t}} \left[ \sin \left(\frac{s\sqrt{t}}{4 + t^4}\right) + se^{-\delta t} \sum_{i=1}^{3} \arctan \left(x\left(\frac{s}{i}\right)\right) \right] ds \]  

(4.4.1)

where \( t \geq 0 \) and \( \delta > 0 \) is a constant.

Observe that this equation is a special case of equation (4.2.1) if we put

\[ f(t, x_1, x_2, x_3) = \frac{t^2}{t^2 + 1} + \sum_{i=1}^{3} \frac{t}{(2 + i)t + 1} \ln(1 + |x_i|) \]

\[ g(t, s, x_1, x_2, x_3) = \sin \left(\frac{s\sqrt{t}}{4 + t^4}\right) + se^{-\delta t} \sum_{i=1}^{3} \arctan x_i \]

and \( \alpha_i(t) = \frac{t}{2 + i} \), \( \gamma_i(t) = \frac{\sqrt{t}}{i} \) \((i = 1, 2, 3)\), \( \beta(t) = t\sqrt{t}, m = 3, n = 3 \).

Let us notice that the functions involved in equation (4.4.1) satisfy assumptions of Theorem 4.3.1. Indeed, \( f : R_+ \times R^3 \rightarrow R \) is continuous and we have

\[ |f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| = \sum_{i=1}^{3} \frac{t}{(2 + i)t + 1} |\ln(1 + |x_i|) - \ln(1 + |y_i|)| \]

\[ \leq \sum_{i=1}^{3} \frac{t}{(2 + i)t + 1} |x_i - y_i| \]

\[ = \sum_{i=1}^{3} \frac{1}{2 + i} \frac{(2 + i)t}{(2 + i)t + 1} |x_i - y_i| \]

\[ \leq \sum_{i=1}^{3} \frac{1}{2 + i} |x_i - y_i| . \]

Thus the function \( f(t, x_1, x_2, x_3) \) satisfies assumption (H1) with \( k_i = 1/(2 + i) \) for \( i = 1, 2, 3 \). Hence \( k = k_1 + k_2 + k_3 = 47/60 \). Moreover we see that \( f(t, 0, 0, 0) = \frac{t^2}{t^2 + 1} \) is bounded and \( F_0 = \sup\{|f(t, 0, 0, 0)| : t \in R_+\} = 1 \).

Obviously the functions \( \alpha_i, \gamma_i \ (i = 1, 2, ..., 3) \) and \( \beta \) satisfy the assumptions (H3) and (H4).
Next let us observe that the function $g : \mathbb{R}^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous and we have

$$|g(t, s, x_1, x_2, x_3)| \leq \frac{s\sqrt{t}}{4 + t^4} + se^{-\delta t} \sum_{i=1}^{3} |x_i|.$$  

This yields that the function $g(t, s, x_1, x_2, x_3)$ satisfies assumption (H5) with $q(t, s) = \frac{s\sqrt{t}}{4 + t^4}$, $a(t) = e^{-\delta t}$ and $b(s) = s$.

Moreover we can also calculate that

$$v_1(t) = \int_0^{\beta(t)} q(t, s)ds = \int_0^{t\sqrt{t}} \frac{s\sqrt{t}}{4 + t^4}dt = \frac{t^3\sqrt{t}}{2(4 + t^4)}.$$  

This implies that

$$\lim_{t \to \infty} \beta(t) = 0.$$  

Apart from this, we have

$$M_1 = \sup\{v_1(t) : t \in \mathbb{R}_+\} = \frac{(28)^{7/8}}{64} = 0.288458... .$$  

On the other hand, we obtain

$$v_2(t) = a(t) \int_0^{\beta(t)} b(s)ds = \frac{1}{2} t^3 e^{-\delta t}.$$  

Hence we infer that

$$\lim_{t \to \infty} a(t) \int_0^{\beta(t)} b(s)ds = 0.$$  

Apart from this, we get

$$M_2 = \sup\{v_2(t) : t \in \mathbb{R}_+\} = \frac{27}{2\delta^3 e^3}.$$  

Consequently we obtain

$$k + mM_2 = \frac{47}{60} + \frac{81}{2\delta^3 e^3}.$$
Thus assumption (H6) is satisfied provided we assume that
\[ \delta > \frac{3}{e} \sqrt[90]{90/13} = 2.103422... \] (4.4.2)

Finally, on the basis of Theorem 4.3.1, we conclude that equation (4.4.1) has at least one solution \( x = x(t) \) in the space \( BC(R_+) \) provided the constant \( \delta \) satisfies the estimate (4.4.2). Moreover, from the proof of Theorem 4.3.1, we derive that \( x \in B_r \), where
\[ r = \frac{(F_0 + M_1)/[1 - (k + mM_2)]}{1 + \frac{(28)^{7/8}}{64}} \bigg/ \frac{13}{60} - \frac{81}{2e^35^3} \bigg). \]

For example, for \( \delta = 3 \) we have that \( r = 9.0745422... \). Apart from that, we also conclude that solutions of equation (4.4.1) are globally attractive.

**Example 4.4.2**

Let us take into account the functional integral equation having the form
\[
\begin{align*}
x(t) &= \sin t + px(t/2) + \frac{qx^2(2t)}{1 + x^2(2t)} \\
&\quad + \int_0^t \arctan \left[ \frac{t + s}{t^2 + 1} + e^{s-t-2}(x(s) + x(s^2) + x(s^3)) \right] ds , \tag{4.4.3}
\end{align*}
\]
where \( t \in R_+ \) and \( p, q \) are nonnegative constants.

Notice that equation (4.4.3) is a special case of equation (4.2.1). Indeed, let us put
\[
\begin{align*}
f(t, x_1, x_2) &= \sin t + px_1 + \frac{qx_2^2}{1 + x_2^2} , \\
g(t, s, x_1, x_2, x_3) &= \arctan \left[ \frac{t + s}{t^2 + 1} + e^{s-t-2}(x_1 + x_2 + x_3) \right]
\end{align*}
\]
and \( \alpha_1(t) = t/2, \alpha_2(t) = 2t, \gamma_1(t) = t, \gamma_2(t) = t^2, \gamma_3(t) = t^3, \beta(t) = \sqrt{t}, n = 2, m = 3. \)

It is easily seen that the function \( f \) satisfies assumptions (H1) and (H2) of Theorem 4.3.1 with \( k_1 = p \) and \( k_2 = \frac{9q}{8\sqrt{3}}. \) Thus \( k = k_1 + k_2 = p + q\frac{9}{8\sqrt{3}} \). Moreover, we have that \( F_0 = 1. \) Further let us observe that the functions \( \alpha_i(t)(i = 1, 2), \gamma_j(t)(j = 1, 2, 3) \) and \( \beta(t) \) satisfy assumptions (H3) and (H4). Moreover, the function \( g : R^2_+ \times R^3 \to R \)
is continuous on $R^2 \times R^3$. Next, for fixed $t, s \in R_+$ and for arbitrary $x_1, x_2, x_3 \in R$, we obtain

$$|g(t, s, x_1, x_2, x_3)| \leq \frac{t + s}{t^2 + 1} + e^{-t-2}e^s \sum_{i=1}^{3} |x_i|.$$ 

The above inequality yields that the function $g(t, s, x_1, x_2, x_3)$ satisfies the inequality from assumption (H5) with $q(t, s) = (t + s)/(t^2 + 1)$, $a(t) = e^{-t-2}$ and $b(s) = e^s$. Apart from this, we get

$$v_1(t) = \int_0^{\beta(t)} q(t, s) ds = \int_0^{\sqrt{t}} \frac{t + s}{t^2 + 1} ds = \frac{2t\sqrt{t} + t}{2(t^2 + 1)}.$$ 

Hence we infer that $\lim_{t \to \infty} v_1(t) = 0$. Moreover, we have

$$M_1 = \sup \{v_1(t) : t \in R_+\} \leq \frac{1 + \sqrt{27}}{4} = 0.819876... .$$ 

On the other hand, we obtain

$$v_2(t) = a(t) \int_0^{\beta(t)} b(s) ds = e^{-t-2} \int_0^{\sqrt{t}} e^s ds = e^{-t-2} \left(e^{\sqrt{t}} - 1\right).$$ 

This implies that $v_2(t) \to 0$ as $t \to \infty$. Apart from this we have that $v_2(t) \leq e^{\sqrt{t} - t - 2}$. Hence, using standard tools of differential calculus, we obtain that $M_2 \leq e^{-7/4}$. 

Finally let us consider the inequality from assumption (H6). In view of the above obtained estimates of the constants $k$ and $M_2$, we infer that the above mentioned inequality is satisfied if it satisfies the following one

$$p + q \frac{9}{8\sqrt{3}} + 3e^{-7/4} < 1. \quad (4.4.4)$$ 

It is easy to check that inequality (4.4.4) is satisfied if we put $p = q = 1/4$, for example.

In the light of the above proved facts and Theorem 4.3.1, we conclude that equation (4.4.3) has at least one solution $x = x(t)$ belonging to the space $BC(R_+)$ provided the constants $p$ and $q$ satisfy inequality (4.4.4). Apart from that we have that solutions of equation (4.4.3) are globally attractive.

☆☆☆☆☆