I. INTRODUCTION

1.1. SECOND ORDER INTEGRODIFFERENTIAL SYSTEMS

The study of second order integrodifferential equations in abstract spaces has emerged, in recent years, as an independent area of modern research because of its applications to many fields. A large class of scientific and engineering problems is modelled by partial differential equations, integral equations or coupled ordinary and partial differential equations which can be described as differential equations in infinite dimensional spaces using the semigroup theory. In particular, second order differential equations or integrodifferential equations serve as an abstract formulation of many partial integrodifferential equations which arise in problems connected with the transverse motion of an extensible beam, the vibration of hinged bars and many other physical phenomena. So it is quite significant to study the controllability problem for such systems in Banach spaces.

1.2. MOTIVATION

As the object of the thesis is to study the controllability of abstract second order nonlinear integrodifferential systems of various forms, we shall motivate our study briefly by giving the occurrence of these systems in different fields of study.

Suppose we have a stretched uniform string whose motion is described by

\[ \rho \frac{\partial^2 z}{\partial t^2}(x,t) - \alpha \frac{\partial^2 z}{\partial x^2}(x,t) = u(x,t), \]

\[ z(0,t) = 0, \quad z(1,t) = 0, \]

where \( z(x,t) \) is the displacement of the string at position \( x \) and time \( t \), \( \rho \) is the density of the string, \( \alpha \) is a scaled tensile parameter, and \( u \) is the control we can apply along the length of the string. An interesting question is whether we can choose \( u \) to bring the string to rest in finite time. In control theory, this is a controllability problem. In the remaining chapters, we discuss such controllability questions and give sufficient conditions for these.

Second order differential equations occur in diverse fields of study as electromagnetic theory, hydrodynamics, acoustics, elasticity and quantum theory. It
is important in these subjects because the solutions of these equations furnish a mathematical description of waves and propagation of effects. Physically the solutions may represent waves of electric or magnetic intensity, waves of acoustics pressure, transverse or longitudinal displacement waves in a solid or other phenomena.

Partial differential equations of second order arise frequently in mathematical physics. In fact, it is for this reason, the study of such equations is of great practical value. For the moment we shall briefly show how such equations arise most often in physics. Even though, in general, the mathematical models are non-linear, for clarity, we shall give only formulation of linear second order equations.

(a) Transverse Vibrations of a string

If a string of uniform linear density $\rho$ is stretched to a uniform tension $T$, and if, in the equilibrium position, the string coincides with $x$ axis, then when the string is disturbed slightly from its equilibrium position, the transverse displacement $y(x, t)$ satisfies the second order equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$$

where $c^2 = T/\rho$. At any point $x = a$ of the string which is fixed, $y(a, t) = 0$ for all values of $t$.

(b) Longitudinal Vibrations in a bar

If a uniform bar of elastic material of uniform cross section whose axis coincides with $Ox$ is stretched in such a way that each point of a typical cross section of the bar takes the same displacement $\xi(x, t)$, then

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \xi}{\partial t^2}$$

where $c^2 = E/\rho$, $E$ being the Young's modulus and $\rho$ the density of the material of the bar. The stress at any point in the bar is

$$\sigma = E \frac{\partial \xi}{\partial x}.$$

Suppose that the velocity of the end $x = 0$ of the bar $0 \leq x \leq a$ is prescribed to be $v(t)$, say, and that the other end $x = a$ is free from stress. Suppose further
that at that time $t = 0$ the bar is at rest. The longitudinal displacement of sections of the bar are determined by the second order equation (1.1) and the boundary and initial conditions are:

(i) $\frac{\partial \xi}{\partial t} = v(t)$ for $x = 0$;

(ii) $\frac{\partial \xi}{\partial x} = 0$ for $x = a$;

(iii) $\xi = \frac{\partial \xi}{\partial t} = 0$ at $t = 0$ for $0 \leq x \leq a$.

(c) Longitudinal Sound Waves

If plane waves of sound are being propagated in a horn whose cross section for the section with abscissa $x$ is $A(x)$ in such a way that every point of that section has the same longitudinal displacement $\xi(x,t)$, then $\xi$ satisfies the partial differential equation

$$\frac{\partial}{\partial x} \left\{ \frac{1}{A} \frac{\partial (A \xi)}{\partial x} \right\} = \frac{1}{c^2} \frac{\partial^2 \xi}{\partial t^2} \quad (1.2)$$

which reduces to the one-dimensional wave equation (1.1) in the case in which the cross section is uniform. In equation (1.2)

$$c^2 = \left( \frac{dp}{d\rho} \right)_0 \quad (1.3)$$

where the suffix 0 denotes that we take the value $dp/d\rho$ in the equilibrium state. The change in pressure in the gas from the equilibrium value $p_0$ is given by the formula

$$p - p_0 = -\frac{1}{c^2 \rho_0} \frac{\partial \xi}{\partial x}$$

where $\rho_0$ is the density of the gas in the equilibrium state. For instance, if we are considering the motion of the gas when a sound wave passes along a tube which is free at each of the ends $x = 0$, $x = a$, then we must determine solutions of equation (1.2) which are such that $\partial \xi/\partial x = 0$ at $x = 0$ and $x = a$. 

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(d) Sound Waves in Space

Suppose that because of the passage of a sound wave the gas at the point 
\((x, y, z)\) at time \(t\) has the velocity \(v = (u, v, w)\) and that the pressure and density 
there and then are \(p, \rho\) respectively; then if the \(p_0, \rho_0\) are the corresponding values 
in the equilibrium state, we may write

\[
\rho = \rho_0 (1 + s), \quad p = p_0 + c^2 \rho_0 s
\]

where \(s\) is called the condensation of the gas and \(c^2\) is given by equation (1.5). If we substitute these expressions in the equation of the motion

\[
\rho \frac{Dv}{Dt} = -\text{grad} \ p
\]

where

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}
\]

and restrict ourselves to small oscillations of the gas, we find that

\[
\rho_0 \frac{\partial v}{\partial t} = -c^2 \rho_0 \text{grad} \ s. \tag{1.4}
\]

Similarly, the continuity equation

\[
\frac{D\rho}{Dt} + \rho \text{ div } v = 0 \tag{1.5}
\]

is equivalent, in this approximation, to the equation

\[
\rho_0 \frac{\partial s}{\partial t} + \rho_0 \text{ div } v = 0. \tag{1.6}
\]

If the motion of the gas is irrotational, then there exists a scalar function \(\phi\) with the property that

\[
v = -\text{grad} \ \phi. \tag{1.7}
\]

Substituting from equation (1.6) into equation (1.4), we find that for small oscillations

\[
\text{grad} \left( \frac{\partial \phi}{\partial t} - c^2 s \right) = 0. \tag{1.8}
\]
Similarly, equation (1.5) is equivalent to
\[ \frac{\partial s}{\partial t} = \nabla^2 \phi. \] (1.9)
Eliminating \( s \) between (1.7) and (1.8), we find that \( \phi \) satisfies the wave equation
\[ \nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}. \]

(e) Electromagnetic Waves

If we write
\[ H = \text{curl} \ A, \quad E = \frac{1}{c} \frac{\partial A}{\partial t} - \text{grad} \ \phi \]
then Maxwell’s equations
\[
\begin{align*}
\text{div} \ E &= 4\pi \rho, \\
\text{div} \ H &= 0, \\
\text{curl} \ E &= \frac{-1}{c} \frac{\partial H}{\partial t}, \\
\text{curl} \ H &= \frac{4\pi i}{c} + \frac{1}{c} \frac{\partial E}{\partial t}
\end{align*}
\]
are satisfied identically provided that \( A \) and \( \phi \) satisfy the equations
\[
\begin{align*}
\nabla^2 A &= \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \frac{4\pi i}{c}, \\
\nabla^2 \phi &= \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - 4\pi \rho.
\end{align*}
\]
Therefore in the absence of charges or currents, \( \phi \) and the components of \( A \) satisfy the wave equation.

(f) Elastic Waves in Solids

If \((u, v, w)\) denotes the components of the displacement vector \( \mathbf{v} \) at the point \((x, y, z)\), then the components of the stress vector are given by the equations
\[
\begin{align*}
(\sigma_x, \sigma_y, \sigma_z) &= \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2\mu \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right) \\
(\tau_{xz}, \tau_{zx}, \tau_{xy}) &= \mu \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right)
\end{align*}
\]
where \( \lambda \) and \( \mu \) are Lame’s constants. The equations of motion are
\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho X = \rho \frac{\partial^2 u}{\partial t^2}, \text{ etc.},
\]
where $F = (X, Y, Z)$ in the body force at $(x, y, z)$. If we write
\[ F = \text{grad } \phi + \text{curl } \psi \]
then it is easily shown that the displacement vector can be taken in the form
\[ \mathbf{v} = \text{grad } \phi + \text{curl } \psi \]
provided that $\phi$ and $\psi$ satisfy the equations. The equations of motion are
\[ \frac{\partial^2 \phi}{\partial t^2} - c_1^2 \nabla^2 \phi = \phi, \quad \frac{\partial^2 \psi}{\partial t^2} - c_2^2 \nabla^2 \psi = \psi \]
where $c_1, c_2$ are given by
\[ c_1^2 = \frac{\lambda + 2\mu}{\rho}, c_2^2 = \frac{\mu}{\rho}. \]
Hence, in the absence of body forces, $\phi$ and the components of $\psi$ each satisfies a wave equation.

(g) Integrodifferential Equations

Consider the following integrodifferential equation
\[
\frac{\partial^2 z(x, t)}{\partial t^2} + c(t) \frac{\partial^2 z(x, t)}{\partial t} - M \left( \int_{-\infty}^{+\infty} \left| \frac{\partial^2 z(x, s)}{\partial s} \right|^2 ds \right) \frac{\partial^2 z(x, t)}{\partial x^2} + z(t, x)
\]
\[ = h(t, x, u(t, x)), \quad 0 \leq t < \infty, \quad x \in \mathbb{R}, \]
\[ z(0, x) = z_0(x), \quad \frac{\partial z(x, 0)}{\partial t} = z_1(x), \quad x \in \mathbb{R}. \]

Equations of this type have occurred during the study of the nonlinear behaviour of elastic strings [29]. The basic physical assumptions are that the longitudinal strain of the string is very small and that the tension $F$ is uniform along the string but may vary with time to accommodate changes in the arc length of the string. The nonlinearity arises from the assumption that $F$ depends on the arc length $S$ of the string at time $t \geq 0$ by the relation $F = F_0 + C[(S - L)/L]$ where $F_0$ is the minimum tension, $L$ is the minimum length and $C$ is a physical constant.
There are other types of integrodifferential equations similar to the above and, for example, the one occurring in the study of dynamical buckling of the hinged extensible beam which is either stretched or compressed by axial force. A mathematical model for this problem is the hyperbolic equation

\[
\frac{\partial^2 z}{\partial t^2} + \frac{\partial^4 z}{\partial x^4} - \left( \alpha + \beta \int_0^L \left( \frac{\partial z(x, s)}{\partial s} \right)^2 ds \right) \frac{\partial^2 z}{\partial x^2} + g \left( \frac{\partial z}{\partial t} \right) = 0,
\]

in which \( \alpha, \beta, L > 0 \), \( z(x, t) \) is the deflection of the point \( x \) of the beam at the time \( t \), \( g \) is a nondecreasing numerical function, and \( L \) is the length of the beam. The nonlinear friction force \( g(\frac{\partial z}{\partial t}) \) is the dissipative term. When \( g = 0 \), this equation reduces to the equation introduced in [87] as a model for the transverse motion of an extensible beam whose ends are held a fixed distance apart. These equations take the abstract form as

\[
z'' + A^2 z + M(\|A^1 z\|_H)Az + g(z') = 0
\]

where \( A \) is a linear operator in a Hilbert space \( H \) and \( M \) and \( g \) are real functions.

(h) Abstract Equations

In many cases, it is advantageous to treat the second order abstract differential equations directly rather than convert them to first order systems. For instance, reduction to first order is of no particular help in a problem as elementary as the growth of solutions of

\[
x''(t) = (A + cI)x(t)
\]

in terms of the growth of solutions of

\[
x''(t) = Ax(t).
\]

In other problems, such as singular perturbation, direct consideration of second order equation leads to simpler and more inclusive theories. Finally, the integral formulation associated with \( x''(t) = Ax(t), x(0) = x_0, x'(0) = y_0 \) has proven useful in other fields, such as the control theory of hyperbolic equations. These and other reasons give motivation to the study of second order differential equations in Banach spaces. For physical motivation and mathematical formulation one can see the references [8,9,13,14,23,32-35,40,41,43,45,46,65,68,71,79,81,82,85,86].

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A useful tool for the study of abstract second order equations is the theory of strongly continuous cosine families of bounded linear operators which can be briefly described below.

1.3. STRONGLY CONTINUOUS COSINE FAMILIES

The theory of strongly continuous cosine families of bounded linear operators has applications in many branches of analysis and in particular to the solution of initial and boundary value problems for second order partial differential equations. It is closely related to the solution of second order ordinary differential equations in Banach spaces. In recent years, the theory of strongly continuous cosine families of bounded linear operators has been extensively applied to study the existence problems in second order differential equations [11,12,56,57,72]. The most fundamental and extensive work on cosine families is that of Fattorini in [19-22]. Important additions and simplifications to the theory have also been made by Sova [69,70], Goldstein [25-28], Nagy [47-49], Travis and Webb [73-77].

The theory of strongly continuous cosine families of bounded linear operators in Banach space is kindred in spirit to the theory of strongly continuous semigroups of bounded linear operators in Banach space and it is equally appealing because of its great generality and simplicity. Strongly continuous cosine families of bounded linear operators are related to abstract linear second order differential equations in the same manner that strongly continuous semigroups of bounded linear operators are related to abstract linear first order differential equations. Roughly speaking, every second order differential equation of the form \( x'' = Ax \) which is well posed in a certain sense gives rise to a strongly continuous cosine family of bounded linear operators with infinitesimal generator \( A \), and conversely, every strongly continuous cosine family of bounded linear operators with infinitesimal generator \( A \) may be associated with the well-posed second order differential equation \( x'' = Ax \).

The following basic results concerning strongly continuous cosine families are established in [22,31,36,38,42,64,84].

**Definition 1.1.** A one parameter family \( C(t), t \in \mathbb{R} \), of bounded linear operators mapping the Banach space \( X \) into itself is called a strongly continuous cosine family if and only if

\[(i) \ C(s + t) + C(s - t) = 2C(s)C(t) \quad \text{for all } s, t \in \mathbb{R};\]
(ii) \( C(0) = I \);

(iii) \( C(t)x \) is continuous in \( t \) on \( \mathbb{R} \) for each fixed \( x \in X \).

If \( C(t), t \in \mathbb{R}, \) is a strongly continuous cosine family in \( X \), then \( S(t), t \in \mathbb{R}, \) is the associated sine family of operators in \( X \) defined by

\[
S(t)x = \int_0^t C(s)x \, ds, \quad x \in X, \quad t \in \mathbb{R}.
\]

**Proposition 1.1.** Let \( C(t), t \in \mathbb{R}, \) be a strongly continuous cosine family in \( X \). Then the following are true:

(i) \( C(t) = C(-t) \) for all \( t \in \mathbb{R} \);

(ii) \( C(s), S(s)C(t) \) and \( S(t) \) commute for all \( s, t \in \mathbb{R} \);

(iii) \( S(t)x \) is continuous in \( t \) on \( \mathbb{R} \) for each fixed \( x \in X \);

(iv) \( S(s + t) + S(s - t) = 2S(s)C(t) \) for all \( s, t \in \mathbb{R} \);

(v) \( S(s + t) = S(s)C(t) + S(t)C(s) \) for all \( s, t \in \mathbb{R} \);

(vi) \( S(t) = -S(-t) \) for all \( t \in \mathbb{R} \);

(vii) there exist constants \( K \geq 1 \) and \( \omega \geq 0 \) such that \( |C(t)| \leq Ke^{\omega|t|} \) for all \( t \in \mathbb{R} \);

(viii) \( |S(t) - S(t')| \leq K \int_t^{t'} e^{\omega|s|} ds \), for all \( t, t' \in \mathbb{R} \) for all \( t \in \mathbb{R} \);

**Proposition 1.2.** Let \( C(t), t \in \mathbb{R}, \) be a strongly continuous cosine family in \( X \) with infinitesimal generator \( A \). The following are true:

(i) \( D(A) \) is dense in \( X \) and \( A \) is closed operator in \( X \);

(ii) if \( x \in X \) and \( r, s \in \mathbb{R} \), then \( z \overset{\text{def}}{=} \int_r^s S(u)x \, du \in D(A) \) and \( Az = C(s)x - C(r)x \);

(iii) if \( x \in X \) and \( r, s \in \mathbb{R} \), then \( z \overset{\text{def}}{=} \int_0^s \int_0^r C(u)C(v)x \, dv \, du \in D(A) \) and \( Az = 2^{-1}(C(s + r)x - C(s - r)x) \);

(iv) if \( x \in X \), then \( S(t)x \in E \);
(v) if $x \in E$, then $S(t)x \in D(A)$ and $\frac{d}{dt} C(t)x = AS(t)x$;

(vi) if $x \in D(A)$, then $C(t)x \in D(A)$ and $\frac{d^2}{dt^2} C(t)x = AC(t)x = C(t)Ax$;

(vii) if $x \in E$, then $\lim_{t \to 0} AS(t)x = 0$;

(viii) if $x \in E$, then $S(t)x \in D(A)$ and $\frac{d^2}{dt^2} S(t)x = AS(t)x$;

(ix) if $x \in D(A)$, then $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$;

(x) $C(t+s) - C(t-s) = 2AS(t)S(s)$ for all $s, t \in R$.

**Proposition 1.3.** Let $C(t), t \in R$, be a strongly continuous cosine family in $X$. The operator $A : X \to X$ defined by

$$Ax = \lim_{t \to 0} \frac{C(2t)x - x}{2t^2},$$

with domain those $x \in X$ for which this limit exists, is the infinitesimal generator of the cosine family $C(t), t \in R$.

### 1.4. THE CAUCHY PROBLEM

Let $X$ be a general Banach space and let $A$ be a linear operator with domain $D(A)$ in $X$ and range in $X$. Let us take $D(A)$ to be dense in $X$. A solution of the abstract differential equation

$$x''(t) = Ax(t) \quad (1.10)$$

in $[0, \infty)$ is a twice continuously differentiable $X$-valued function $x(t)$ such that $x(t) \in D(A)$ and (1.10) is satisfied for $t \geq 0$.

The Cauchy problem or initial value problem for (1.10) in $t \geq 0$ is that of finding solutions satisfying the initial conditions

$$x(0) = x_0, \quad x'(0) = y_0. \quad (1.11)$$

The Cauchy problem for (1.10) is well posed in $t \geq 0$ if and only if

(a) *(Existence).* There exists a dense subspace $D$ of $E$ such that for any $x_0, y_0 \in D$, there exists a solution $x(t)$ of (1.10) in $t \geq 0$ satisfying (1.11).
(b) (Continuous dependence). There exists a function $m : [0, \infty) \to (0, \infty)$ such that

$$\|x(t)\| \leq m(t)(\|x(0) + x'(0)\|), \quad t \geq 0.$$ 

The Cauchy problem for (1.10) is uniformly posed in $t \geq 0$ if (a) and (b) hold and the function $m(t)$ is nondecreasing in $t \geq 0$.

Consider the equation

$$x''(t) = Ax(t) + f(t) \quad (1.12)$$

The solutions of (1.12) are defined in the same way as solutions of the corresponding homogeneous part; if the Cauchy problem for (1.10) is well posed in $t \geq 0$, uniqueness of solutions of the initial value problem for the homogeneous equation implies that there is at most one solution of the inhomogeneous equation satisfying the initial conditions

$$x(0) = x_0, \quad x'(0) = y_0 \quad (1.13)$$

Let $f(t)$ be a continuous function defined in $0 \leq t \leq T$, and let $x(t)$ be a solution of (1.12) satisfying (1.13). Assuming that $t > 0$ is fixed and that $C(t), S(t)$ are the solution operators of (1.10) we obtain using the equation that the function

$$C(t - s)x(s) + S(t - s)x'S(t - s)f(s))$$

is continuously differentiable in $0 < s < t$ with derivative $S(t - s)f(s)$. Integrating in $0 < t < T$ we obtain

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t - s)f(s)ds. \quad (1.14)$$

It is quite natural to ask whether the Cauchy problem for the equation

$$x''(t) = Ax(t)$$

in the Banach space $X$ can be reduced to the Cauchy problem for a first order system. It is obvious that it can be reduced to the system

$$x'_1(t) = x_2(t), \quad x'_2(t) = Ax_1(t), \quad (1.15)$$

where $x_1(t) = x(t)$. However, if (1.15) is considered in the space $X \times X$, it may give rise to an improperly posed Cauchy problem due to the fact that $x'(t)$, unlike $x(t)$, may fail to depend continuously on $x(0), x'(0)$.

However, the reduction (1.15) will always succeed if $x_1(t)$ is measured in a different norm.
Example 1.1. We can write the one-dimensional wave equation $y_{tt} = y_{xx}$ in the form (1.10) in the following way: Let $X = L^2(-\infty, \infty)$ and $x''(t) = Ax(t)$ where $D(A)$ is the set of all $x \in X$ such that $x''$ belongs to $L^2$. Then the Cauchy problem for (1.10) is well posed with the strongly continuous cosine family $C(t)$ and the associated sine family $S(t)$. Obviously, $C''(t)x$ is given by

$$C'(t)x(\cdot) = \frac{1}{2}(u'(\cdot + t) - u'(\cdot - t));$$

thus it is not a bounded operator in $X$.

Proposition 1.4. Let $C(t), t \in \mathbb{R}$ be a strongly continuous cosine family in $X$ with infinitesimal generator $A$. If $g : \mathbb{R} \to X$ is continuously differentiable, $x \in D(A)$, $y \in E$, and

$$w(t) \equiv C(t)x + S(t)y + \int_0^t S(t - s)g(s)ds, \quad t \in \mathbb{R},$$

then $w(t) \in D(A)$ for $t \in \mathbb{R}$, $w$ is twice continuously differentiable, and $w$ satisfies

$$w''(t) = Aw(t) + g(t), \quad t \in \mathbb{R}, \quad w(0) = x, \quad w'(0) = y. \quad (1.16)$$

Conversely, if $g : \mathbb{R} \to X$ is continuous, $w : \mathbb{R} \to X$ is twice continuously differentiable, $w(t) \in D(A)$ for $t \in \mathbb{R}$, and $w$ satisfies (1.10), then

$$w(t) = C(t)x + S(t)y + \int_0^t S(t - s)g(s)ds, \quad t \in \mathbb{R}.$$

Proof. By virtue of (vi) and (vii) of Proposition 1.2, it suffices to show (1.16) for $x = y = 0$. First,

$$\int_0^t S(t - s)g(s)ds = \int_0^t S(t - s) \left( g(0) + \int_0^s g'(u)du \right) ds$$

$$= \int_0^t S(t - s)g(0)ds + \int_0^t \int_u^t S(t - s)g'(u)dsdu$$

$$= \int_0^t S(t - s)g(0)ds + \int_0^t \int_0^t S(s)g'(u)dsdu.$$
Then, (1.16) follows from

\[ w'(t) = \int_0^t C(t - s)g(s)ds + S(0)g(t) = \int_0^t C(s)g(t - s)ds, \]

\[ w''(t) = \int_0^t C(s)g'(t - s)ds + C(t)g(0) = Aw(t) + g(t). \]

To prove the converse statement, observe that

\[ \frac{d}{ds}S(t - s)w'(s) = -C(t - s)w'(s) + S(t - s)w''(s), \]

\[ \frac{d}{ds}C(t - s)w(s) = -S(t - s)Aw(s) + C(t - s)w'(s), \]

implies

\[ -S(t)w'(0) = -\int_0^t C(t - s)w'(s)ds + \int_0^t S(t - s)w''(s)ds, \]

\[ w(t) - C(t)w(0) = -\int_0^t S(t - s)Aw(s)ds + \int_0^t C(t - s)w'(s)ds. \]

The conclusion follows by adding the above two formulas.

1.5. CONTROLLABILITY PROBLEM AND METHODS

Controllability is one of the most important properties of dynamical systems. The problem of controllability is to show the existence of a control function which steers the solution of the system from its initial state to a final state, where the initial and final states may vary over the entire space. There are various approaches to the study of controllability of nonlinear systems which may be classified as follows:

(i) methods based on the stability theory of Lyapunov

(ii) methods for systems defined on a manifold

(iii) methods which are geometrical in nature

(iv) fixed point methods.

Of all the methods, the fixed point method can be effectively used to study the controllability of nonlinear systems. In this method, the controllability problem is transformed to a fixed point problem for an appropriate nonlinear operator in
a function space. An essential part of this approach is to guarantee the existence of an invariant subset for this operator. The controllability problem has been studied in general Banach spaces by using fixed point theorems. Several existence and uniqueness results can be obtained by applying fixed point theorems. Fixed point method is the most powerful method in proving the existence theorems for integral and integrodifferential equations. Due to their importance, several researchers have used different kinds of fixed point theorems.

1.6. CONTROLLABILITY OF NONLINEAR SYSTEMS

Controllability of linear and nonlinear first order systems represented by ordinary differential equations in finite and infinite dimensional spaces has been extensively studied by many authors [1,16,17,24,51,52,59,66,78,88,90]. Several authors extended the concept to infinite dimensional systems in Banach Spaces with bounded operators. Lasiecka and Triggiani [44] studied the exact controllability of semilinear first order systems. Chukwu and Lenhart [15] discussed the controllability of nonlinear first order systems in abstract spaces. Balachandran and Dauer [2] established sufficient conditions for the relative controllability of nonlinear first order neutral Volterra integrodifferential systems whereas Bian [10] considered the nonlinear first order evolution systems with time varying delay. Recently, Park and Han [60] discussed the controllability of second order nonlinear systems in Banach spaces with the help of the Schauder fixed point theorem. Very few papers only have appeared on the controllability of second order systems.

In this work, we have made an attempt to study the controllability of variety of second order nonlinear systems and nonlinear integrodifferential systems in Banach spaces with the help of the theory of strongly continuous cosine families of operators and the fixed point theorems due to Schauder and Schaefer.

1.7. CONTRIBUTIONS OF THE AUTHOR

In the light of the above, the author has obtained some significant results on the following topics.

1. Controllability of second order nonlinear differential systems.
2. Controllability of second order nonlinear Volterra integrodifferential systems.
3. Controllability of second order nonlinear integrodifferential systems.
4. Controllability of second order nonlinear delay integrodifferential systems.
5. Controllability of second order nonlinear time varying delay integrodifferential systems.

The rest of the thesis contains a detailed account of the above topics.