VI. CONTROLLABILITY OF SECOND ORDER NONLINEAR TIME VARYING DELAY INTEGRODIFFERENTIAL SYSTEMS

6.1. INTRODUCTION

Naito and Park [53] established the approximate controllability for delay Volterra systems by using the Leray-Schauder degree theorem. Balachandran and Sakthivel [4] studied the controllability of nonlinear first order integrodifferential systems with time varying delays in Banach spaces by using the Schaefer fixed point theorem. In [58], Ntouyas and Tsamatos proved the global existence theorems for second order semilinear time varying delay integrodifferential equations in Banach spaces by using a fixed point approach. In this chapter, we derive a set of sufficient conditions for the controllability of semilinear second order delay integrodifferential systems in Banach spaces by using the theory of strongly continuous cosine families of bounded linear operators and the Schaefer fixed point theorem.

6.2. PRELIMINARIES

We consider the semilinear second order delay control system of the form

\[ x''(t) = Ax(t) + Bu(t) + f(t, x(\sigma_1(t)), \int_0^t h(t, s)g(s, x(\sigma_2(s)), x'(\sigma_3(s))) ds, x'(\sigma_4(t))) \]

\[ x(0) = x_0, \quad x'(0) = y_0, \quad t \in J = [0, T], \quad (6.1) \]

where the state \( x(\cdot) \) takes values in a Banach space \( X \), \( x_0, y_0 \in X \), \( A \) is a linear infinitesimal generator of the strongly continuous cosine family \( C(t) \), \( t \in \mathbb{R} \), of bounded linear operators in \( X \), \( g : J \times X \times X \to X \), \( h : J \times J \to \mathbb{R} \), \( f : J \times X \times X \times X \to X \) are given functions, \( B \) is a bounded linear operator from \( U \) to \( X \) and the control function \( u(\cdot) \) is given in \( L^2(J, U) \), a Banach space of admissible control functions, with \( U \) being a Banach space. Moreover \( \sigma_i : I \to I, \ i = 1, 2, 3, 4 \) are continuous functions such that \( \sigma_i(t) \leq t, \ i = 1, 2, 3, 4 \).

Assume the following conditions on \( A \).

\( (H_1) \quad A \) is the infinitesimal generator of a strongly continuous cosine family \( C(t) \), \( t \in \mathbb{R} \), of bounded linear operators from \( X \) into itself.
The infinitesimal generator of a strongly continuous cosine family $C(t), t \in \mathbb{R}$, is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2}C(t)x \bigg|_{t = 0}, \quad x \in D(A),$$

where $D(A) = \{ x \in X : C(t)x \text{ is twice continuously differentiable in } t \}.$

Define $E = \{ x \in X : C(t)x \text{ is once continuously differentiable in } t \}.$

To establish our main theorem we need the following assumptions:

- $(H_2)$ $C(t), t > 0$ is compact.
- $(H_3)$ $Bu(t)$ is continuous in $t$ and $\|B\| \leq M_1$ for some constant $M_1 > 0$.
- $(H_4)$ The linear operator $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^T S(t, s)Bu(s)ds$$

induces a bounded invertible operator $\tilde{W} : L^2(J, U)/\ker W \to X$ such that $\|\tilde{W}^{-1}\| \leq M_2$ for some constant $M_2 > 0$.

- $(H_5)$ $g : J \times X \times X \to X$ is continuous in $t$ and the function $h : J \times J \to \mathbb{R}$ is measurable.

- $(H_6)$ There exists a continuous function $n : J \to [0, \infty)$ such that

$$\|g(t, x, y)\| \leq n(t)\Omega(\|x\| + \|y\|), \quad t \in J, \quad x, y \in X,$$

where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

- $(H_7)$ There exists a constant $L$ such that $\|h(t, s)\| \leq L$, for $t, s \in J$.

- $(H_8)$ $f(t, \cdot, \cdot, \cdot) : X \times X \times X \to X$ is continuous for each $t \in J$, and the function $f(\cdot, x, y, z) : J \to X$ is strongly measurable for each $x, y, z \in X$.

- $(H_9)$ For every positive constant $k$ there exists $\alpha_k \in L^1(J)$ such that

$$\sup_{\|x\|,\|y\|,\|z\| \leq k} \|f(t, x, y, z)\| \leq \alpha_k(t) \quad t \in J \text{ a.e.}$$
There exists a continuous function \( m : J \to [0, \infty) \) such that
\[
\|f(t, x, y, z)\| \leq m(t)\Omega_0(\|x\| + \|y\| + \|z\|), \quad t \in J, \quad x, y, z \in X,
\]
where \( \Omega_0 : [0, \infty) \to (0, \infty) \) is a continuous nondecreasing function and
\[
\int_0^T \hat{m}(s) ds < \int_0^\infty \frac{ds}{\Omega_0(s) + \Omega(s)}
\]
where \( \hat{m}(t) = \max\{M(T + 1)m(t), L\eta(t)\} \)
\( M = \sup\{\|C(t)\| : t \in J\} \), and \( M* = \sup\{\|AS(t)\| : t \in J\} \), and
\[
c = (M + M*)\|x_0\| + (1 + T)M\|y_0\| + (1 + T)MTM_1M_2
\]
\[
\left[\|x_1\| + M\|x_0\| + MT\|y_0\| + MT \int_0^T m(s)\Omega_0 \left(\|x(s)\| + \|x'(s)\|\right) + L \int_0^T \eta(s)\Omega(\|x(s)\| + \|x'(s)\|) ds \right].
\]
Then the system (6.1) has a mild solution of the form
\[
x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t - s)Bu(s)ds + \int_0^t S(t - s)
\]
\[
f(s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_{\alpha}(\tau)), x'(\sigma_4(\tau)))d\tau, x'(\sigma_4(\sigma))ds).
\]

**Definition 6.1.** The system (6.1) is said to be controllable on \( J \) if for every \( x_0 \in D(A), y_0 \in E \) and \( x_1 \in X \) there exists a control \( u \in L^2(J, U) \) such that the solution \( x(\cdot) \) of (6.1) satisfies \( x(T) = x_1 \).

### 6.3. CONTROLLABILITY RESULT

**Theorem 6.1.** Suppose \((H_1)-(H_{10})\) hold. Then the system (6.1) is controllable on \( J \).

**Proof.** Using the assumption \((H_4)\), for an arbitrary function \( x(\cdot) \), we define the control
\[
u(t) = \tilde{W}^{-1} \left[ x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T - s)
\]
\[
f(s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_{\alpha}(\tau)), x'(\sigma_4(\sigma)))d\tau, x'(\sigma_4(\sigma))ds) \right](t).
\]
Using this control, we will show that the operator defined by

\[(Fx)(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s) \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\tau)) ds + \int_0^t S(t-s)BW^{-1} \left[ x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T-\theta) \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) d\theta \right] (s)ds \]

has a fixed point. This fixed point is then a solution of the equation (6.2).

Clearly, \((Fx)(T) = x_1\), which means that the control \(u\) steers the system from the initial state \(x_0\) to \(x_1\) in time \(T\), provided we obtain a fixed point of the nonlinear operator \(F\).

Consider the space \(Z = C^1(J, X)\) with norm

\[ \|x\|^* = \max\{\|x\|, \|x'\|\}. \]

In order to study the controllability problem for the system (6.1), we apply the Schaefer fixed point theorem to the following system

\[ x''(t) = Ax(t) + Bu(t) + \lambda f \left( t, x(\sigma_1(t)), \int_0^t h(t, s)g(s, x(\sigma_2(s)), x'(\sigma_3(s))) ds, x'(\sigma_4(t)) \right), \]

\[ x(0) = \lambda x_0, \quad x'(0) = \lambda y_0, \quad t \in J, \quad \lambda \in (0, 1). \] (6.3)

Let \(x\) be a mild solution of the system (6.3). Then from

\[ x(t) = \lambda(C(t)x_0 + S(t)y_0) + \lambda \int_0^t S(t-s) \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\tau)) ds + \lambda \int_0^t S(t-s)BW^{-1} \left[ x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T-\theta) \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) d\theta \right] (s)ds \]

we have

\[ \|x(t)\| \leq M\|x_0\| + MT\|y_0\| + MT \int_0^t m(s)\Omega_0 \left( \|x(\sigma_1(s))\| + \|x'(\sigma_4(s))\| \right) ds \]
\[
+ L \int_0^T n(\tau)\Omega(\|x(\sigma_2(\tau))\| + \|x'(\sigma_3(\tau))\|)d\tau \, ds \\
+ MT^2 M_1 M_2 [\|x_1\| + M \|x_0\| + MT \|y_0\| \\
+ MT \int_0^T m(s)\Omega_0(\|x(\sigma_1(s))\| + \|x'(\sigma_4(s))\| \\
+ L \int_0^T n(\tau)\Omega(\|x(\sigma_2(\tau))\| + \|x'(\sigma_3(\tau))\|)d\tau \, ds)]
\]

Denoting by \( v(t) \) the right-hand side of the above inequality we have

\[
\|x(t)\| \leq v(t)
\]

and

\[
v'(t) = MTm(t)\Omega_0(\|x(\sigma_1(t))\| + \|x'(\sigma_4(t))\| \\
+ L \int_0^t n(s)\Omega(\|x(\sigma_2(s))\| + \|x'(\sigma_3(s))\|)ds).
\]

But

\[
x'(t) = \lambda(AS(t)x_0 + C(t)y_0) + \lambda \int_0^t C(t - s) \\
\quad \cdot f \left( s, x(\sigma_1(s)), \int_0^t h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds \\
+ \lambda \int_0^t C(t - s)BW^{-1} \left[ x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T - \theta) \\
\quad \cdot f \left( \theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) \right) d\theta \right] (s) ds
\]

Thus we have

\[
\|x'(t)\| \leq M^*\|x_0\| + M\|y_0\| + M \int_0^t m(s)\Omega_0(\|x(\sigma_1(s))\| + \|x'(\sigma_4(s))\| \\
+ L \int_0^t n(\tau)\Omega(\|x(\sigma_2(\tau))\| + \|x'(\sigma_3(\tau))\|)d\tau \, ds \\
+ MT M_1 M_2 [\|x_1\| + M \|x_0\| + MT \|y_0\| \\
+ MT \int_0^T m(s)\Omega_0(\|x(\sigma_1(s))\| + \|x'(\sigma_4(s))\| \\
+ L \int_0^T n(\tau)\Omega(\|x(\sigma_2(\tau))\| + \|x'(\sigma_3(\tau))\|)d\tau \, ds)]
\]

74
Denoting by $r(t)$ the right-hand side of the above inequality we have

$$\|x'(t)\| \leq r(t),$$

$$r'(t) = Mm(t)\Omega_0 \left( \|x(t)\| + \|x'(t)\| + L \int_0^t n(s)\Omega(\|x(s)\| + \|x'(s)\|)ds \right), \quad t \in J.$$

Let

$$w(t) = v(t) + r(t) + L \int_0^t n(s)\Omega(v(s) + r(s))ds, \quad t \in J.$$

Then

$$w(0) = v(0) + r(0) = c, \quad v(t) + r(t) \leq w(t), \quad t \in J,$$

and

$$w'(t) = v'(t) + r'(t) + Ln(t)\Omega(v(t) + r(t))$$

$$\leq MTm(t)\Omega_0(w(t)) + Mm(t)\Omega_0(w(t)) + Ln(t)\Omega(w(t))$$

$$= \tilde{m}(t) (\Omega_0(w(t)) + \Omega(w(t))), \quad t \in J.$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega_0(s) + \Omega(s)} \leq \int_0^t \tilde{m}(s)ds < \int_0^\infty \frac{ds}{\Omega_0(s) + \Omega(s)}, \quad t \in J.$$

This inequality implies that there is a constant $K$ such that

$$v(t) + r(t) \leq w(t) \leq K, \quad t \in J.$$

Then

$$\|x\|^* = \max\{\|x\|, \|x'\|\} \leq K,$$

where $K$ depends only on $T$ and on the functions $m, n, \Omega_0$ and $\Omega$.

We shall now prove that the operator $F : Z \to Z$ defined by

$$(Fx)(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)$$

$$f \left( s, x(\sigma_1(s)) \right) \left[ \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right] ds$$

$$+ \int_0^T S(t-s)B\tilde{W}^{-1} \left[ x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T-\theta)$$

$$f \left( \theta, x(\sigma_1(\theta)) \right) \left[ \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) \right] d\theta \right] (s) ds$$

is a completely continuous operator.
Let $B_k = \{ x \in \mathbb{Z} : \|x\|^* \leq k \}$ for some $k \geq 1$. We first show that $F$ maps $B_k$ into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leq T$,

$\|(Fx)(t_1) - (Fx)(t_2)\| \leq \|C(t_1) - C(t_2)\| \|x_0\| + \|S(t_1) - S(t_2)\| \|y_0\|$

$+ \| \int_0^{t_1} [S(t_1 - s) - S(t_2 - s)]$

$f \left( s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds \|

+ \| \int_0^{t_2} S(t_2 - s)$

$f \left( s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds \|

+ \| \int_0^{t_1} [S(t_1 - s) - S(t_2 - s)]B\bar{W}^{-1} \left[ x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T - \theta)$

$f \left( \theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) \right) d\theta \right] (s)ds \|

+ \| \int_0^{t_2} S(t_2 - s)B\bar{W}^{-1} \left[ x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T - \theta)$

$f \left( \theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) \right) d\theta \right] (s)ds \|

\leq \|C(t_1) - C(t_2)\| \|x_0\| + \|S(t_1) - S(t_2)\| \|y_0\|

+ \int_0^{t_1} \|S(t_1 - s) - S(t_2 - s)\| \alpha_k(s)ds + \int_0^{t_2} \|S(t_2 - s)\| \alpha_k(s)ds

+ \int_0^{t_1} \|S(t_1 - s) - S(t_2 - s)\| M_1 M_2 \left[ \|x_1\| + M \|x_0\| + MT \|y_0\| \right]$

+ $MT \int_0^T \alpha_k(\theta)d\theta \right] ds$

+ $\int_0^{t_2} \|S(t_2 - s)\| M_1 M_2 \left[ \|x_1\| + M \|x_0\| + MT \|y_0\| + MT \int_0^T \alpha_k(\theta)d\theta \right] ds,$

and similarly

$\|(Fx)'(t_1) - (Fx)'(t_2)\| \leq \|C'(t_1) - C'(t_2)\| \|x_0\| + \|S'(t_1) - S'(t_2)\| \|y_0\|

+ \| \int_0^{t_1} [C(t_1 - s) - C(t_2 - s)]$

$f \left( s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds \|

+ \| \int_0^{t_2} C(t_2 - s)$
\[ f \left( s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds \]

\[ + \left\| \int_0^{t_1} [C(t_1 - s) - C(t_2 - s)] BW^{-1} \left[ x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T - \theta) f \left( \theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) \right) d\theta (s)ds \right\| \]

\[ + \left\| \int_{t_1}^{t_2} C(t_2 - s) BW^{-1} \left[ x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T - \theta) f \left( \theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) \right) d\theta (s)ds \right\| \]

\[ \leq \| A(S(t_1) - S(t_2)) \| \| x_0 \| + \| C(t_1) - C(t_2) \| \| y_0 \| \]

\[ + \int_0^{t_1} \| C(t_1 - s) - C(t_2 - s) \| \alpha_k(s) ds + \int_{t_1}^{t_2} \| C(t_2 - s) \| \alpha_k(s) ds \]

\[ + \int_0^{t_1} \| C(t_1 - s) - C(t_2 - s) \| M_1 M_2 \left[ \| x_1 \| + M \| x_0 \| + MT \| y_0 \| \right] \]

\[ + MT \int_0^T \alpha_k(\theta) d\theta \] \[ds \]

\[ + \int_{t_1}^{t_2} \| C(t_2 - s) \| M_1 M_2 \left[ \| x_1 \| + M \| x_0 \| + MT \| y_0 \| + MT \int_0^T \alpha_k(\theta) d\theta \right] ds \]

The right-hand sides are independent of \( y \in B_k \) and tends to zero as \( t_1 \to t_2 \), since \( C(t), S(t) \) are uniformly continuous for \( t \in J \) and the compactness of \( C(t), S(t) \) for \( t > 0 \) imply the continuity in the uniform operator topology. The compactness of \( S(t) \) follows from that of \( C(t) \).

Thus \( F \) maps \( B_k \) into an equicontinuous family of functions. It is easy to see that the family \( FB_k \) is uniformly bounded.

Next we show \( FB_k \) is compact. Since we have shown \( FB_k \) is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that \( F \) maps \( B_k \) into a precompact set in \( X \).

Let \( 0 < t \leq T \) be fixed and \( \epsilon \) a real number satisfying \( 0 < \epsilon < t \). For \( x \in B_k \) we define

\[ (F_x)(t) = C(t)x_0 + S(t)y_0 + \int_0^{t-\epsilon} S(t - s) f \left( s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds \]
+ \int_{t-\epsilon}^{t} S(t-s)B\bar{W}^{-1}\left[ x_1 - C(T)x_0 - S(T)y_0 - \int_{0}^{T} S(T-\theta) \\
\quad f\left(\theta, x(\sigma_1(\theta)), \int_{0}^{\theta} h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta))\right) d\theta\right](s)ds.

Since $C(t), S(t)$ are compact operators, the set $Y_{\epsilon}(t) = \{(Fx)(t) : x \in B_k\}$ is precompact in $X$ for every $\epsilon, 0 < \epsilon < t$. Moreover for every $x \in B_k$ we have

$$
\| (Fx)(t) - (F_{\epsilon}x)(t) \| \leq \int_{t-\epsilon}^{t} \| S(t-s) \|
$$

$$
f \left( s, x(\sigma_1(s)), \int_{0}^{s} h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds
$$

$$
+ \int_{t-\epsilon}^{t} \| S(t-s)B\bar{W}^{-1}\left[ x_1 - C(T)x_0 - S(T)y_0 - \int_{0}^{T} S(T-\theta) \\
\quad f\left(\theta, x(\sigma_1(\theta)), \int_{0}^{\theta} h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta))\right) d\theta\right](s)ds.
$$

and

$$
\| (Fx)'(t) - (F_{\epsilon}x)'(t) \| \leq \int_{t-\epsilon}^{t} \| C(t-s) \|
$$

$$
f \left( s, x(\sigma_1(s)), \int_{0}^{s} h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds
$$

$$
+ \int_{t-\epsilon}^{t} \| C(t-s)B\bar{W}^{-1}\left[ x_1 - C(T)x_0 - S(T)y_0 - \int_{0}^{T} S(T-\theta) \\
\quad f\left(\theta, x(\sigma_1(\theta)), \int_{0}^{\theta} h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta))\right) d\theta\right](s)ds.
$$

Therefore there are precompact sets arbitrarily close to the set $\{(Fx)(t) : x \in B_k\}$. Hence the set $\{(Fx)(t) : x \in B_k\}$ is precompact in $X$.

It remains to show that $F : Z \rightarrow Z$ is continuous. Let $\{x_n\}_{n=0}^{\infty} \subseteq Z$ with $x_n \rightarrow x$ in $Z$. Then there is an integer $q$ such that $\|x_n(t)\| \leq q, \|x'_n(t)\| \leq q$ for
all $n$ and $t \in J$, so $\|x(t)\| \leq q, \|x'(t)\| \leq q$ and $x, x' \in Z$. By $(H_2)$

$$f \left( t, x_n(\sigma_1(t)), \int_0^t h(t, s)g(s, x_n(\sigma_2(s)), x_n(\sigma_3(s)))ds, x'_n(\sigma_4(t)) \right)$$

$$\longrightarrow f \left( t, x(\sigma_1(t)), \int_0^t h(t, s)g(s, x(\sigma_2(s)), x'(\sigma_3(s)))ds, x'(\sigma_4(t)) \right)$$

for each $t \in J$ and since

$$\left\| f \left( t, x_n(\sigma_1(t)), \int_0^t h(t, s)g(s, x_n(\sigma_2(s)), x_n(\sigma_3(s)))ds, x'_n(\sigma_4(t)) \right) \right\| \leq 2\alpha_q(t),$$

we have by dominated convergence theorem

$$\|F x_n - F x\| = \sup_{t \in J} \left\| \int_0^t S(t - s) \left[ f \left( s, x_n(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x_n(\sigma_2(\tau)), x_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(s)) \right) \right. \right.$$

$$- f \left( s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) \right] ds$$

$$- \int_0^t S(t - s)B\hat{W}^{-1}$$

$$\int_0^T \left[ f \left( \theta, x_n(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x_n(\sigma_2(\tau)), x_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(\theta)) \right) \right.$$

$$- f \left( \theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) \right) \right] d\theta ds$$

$$\leq \int_0^T \left\| S(t - s) \right.$$

$$\left[ f \left( s, x_n(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x_n(\sigma_2(\tau)), x_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(s)) \right) \right.$$

$$- f \left( s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) \right\| ds$$

$$+ \int_0^T \left\| S(t - s)B\hat{W}^{-1} \right.$$
Thus $F$ is continuous. This completes the proof that $F$ is completely continuous.

Finally the set $\zeta(F) = \{x \in Z : x = \lambda Fx, \lambda \in (0, 1)\}$ is bounded, as we proved in the first step. Consequently by Schaefer's theorem the operator $F$ has a fixed point in $Z$. This means that any fixed point of $F$ is a mild solution of (6.1) on $J$ satisfying $(Fx)(t) = x(t)$. Thus the system (6.1) is controllable on $J$. 

\[ \star \star \star \star \star \]