Chapter 5

Numerical Solutions of Nonlinear Black-Scholes Equation

5.1 Introduction

In this chapter, we give numerical solutions for a nonlinear Black-Scholes equation for pricing derivative securities in the presence of transaction costs as well as a volatile risk. For this, we apply the semidiscretization numerical scheme to a model based on Black-Scholes partial differential equation with a nonlinear volatile term that consists of transaction costs too. Transaction costs as well as the risk from a volatile portfolio are described by the variance of the synthesized portfolio. Generally the time lag between two consecutive transaction costs acts as the deciding factor in both transaction costs as well as volatile portfolio. This theory is capable of valuing options as well as other derivative securities over moderate intervals in which transaction costs and volatile risk portfolio are negligible. But the bid-ask spreads, if exist, prevent the application of classical Black-Scholes equation. There is a need for frequent portfolio adjustments to maintain the delta hedge, but the adjustments tend to increase the risk from a volatile portfolio.

Consider the following Black-Scholes equation discussed by Sevcovic [118]

\[ V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_s - rV = 0 \]  

(5.1.1)

where the \( \sigma^2 \) is given by

\[ \sigma^2 = \sigma^2 \left( 1 + 3\left( \frac{C^2 M}{2\pi} S V_{SS} \right)^{\frac{1}{3}} \right) \]  

(5.1.2)

with \( C \) representing the round trip transaction cost per unit dollar of transaction and \( M \), a constant, called risk premium coefficient that represents the marginal
value of investor's exposure to a risk. In addition to this, the usual terminal conditions are
\[ V(S, T) = \max(S - E, 0), \quad V(0, t) = 0, \quad (5.1.3) \]
\[ \frac{V(S, T)}{S} \to 1 \quad \text{as} \quad S \to \infty \quad (5.1.4) \]
where \( E \) is the exercise price.

Already Barles and Soner [17] derived the more complex model that includes transaction costs with the modified volatility
\[ \sigma^2 = \sigma^2 \left( 1 + \psi \left[ \exp(r(T - t)\sigma^2 S^2 V_{SS}) \right] \right) \]
where \( \psi \) is the solution of the nonlinear initial value problem
\[ \psi'(A) = \frac{\psi(A) + 1}{2\sqrt{A\psi(A) - A}}. \]

Ankudinova and Ehrhardt derived the numerical solutions for (5.1.1) and (5.1.3) using Crank-Nicolson method and Rigal compact schemes in [39]. Further Company et al. [43] discussed the model (5.1.3) using semi-discretization process which we use in this chapter.

Regarding the literature available for numerical solutions of Black-Scholes equation, Han and Wu [64] introduced a fast numerical method for computing American option pricing problems governed by the Black-Scholes equation with an artificial boundary condition. Compact finite difference method is used for evaluating the equation. Hong and Wee [68] discussed the problem of convergence of option prices jointly with the costs from the locally risk-minimizing strategies when the jump-diffusion models converge to the Black-Scholes model. Then McCartin and Labadie [104] applied the Crandall-Douglas scheme for the heat equation to compute the option price. In this work, the authors studied the numerical approximation using the above scheme so as to determine suitable discretization parameters and permit reasonable placement of the far-field boundary. Discretization of the associated free boundary problem, when expressed as a variational inequality, leads to a linear complementarity problem at each time step. This methodology produces an accurate and efficient result. The inverse problem of estimating volatility function from a Black-Scholes model with a known option price is derived using implicit-finite difference method by Cho et al. [39]. Zhao et al. [119] designed a compact finite difference method to obtain quick solutions for valuing the American options.
with optimal exercise boundary conditions. This method converts Black-Scholes partial differential equation into an ordinary differential equation and use a values of function at three points to approximate a linear combination of the values of derivatives of the same function at the same three points. This method is more efficient than other methods for solving American option problems.

Matache et al. [103] solved a parablic equation of the form $u_t + Au = 0$ in $(0, T) \times \Omega$, $\Omega \subset \mathbb{R}$ where $A$ is a strongly elliptic integro-differential operator and $\mathbb{R}$ is any bounded interval. Here a discontinuous Galerkin (dG) discretization in time and a wavelet discretization in space are used. The complexity of the above algorithm is linear and is applicable to purely discontinuous Levy processes arising in finance. Cont and Voltchkova [45] studied the link between option prices in exponential Levy models and the related partial integro-differential equations (PIDEs) in the case of European options and options with a single or double barriers. Here the Black-Scholes partial differential equation is transformed into the following partial differential equation:

$$
C_t(t, S) + rsC_S(t, S) + \frac{\sigma^2 S^2}{2} C_{SS}(t, S) - rC(t, S) + \int \nu(dx) \left[ C(t, Se^x) - C(t, S) - S(e^x - 1)C_S(t, S) \right] = 0.
$$

(5.1.5)

Here the main difference between the general option pricing model and the integro-differential model is the presence of jumps following Markov process. The Viscosity solution is alone derived as the classical solution is not possible for pure jump models following the method adopted in [46]. Later the same authors in [44] explained the characterization of prices of European and barrier options in exponential Levy models in terms of viscosity solutions of partial integrodifferential equations. Hence solving such equations like (5.1.5) using finite difference methods involves several approximations: localization of the equation to a bounded domain, treatment of the singularity due to small jumps, discretization of the equation in space and iteration in time.

In the following section (5.2), we will give some preliminaries about the method, the results for the linear and nonlinear Black-Scholes model in section (5.3) in the light of [43]. Throughout this chapter we state the results given in [43] along with the same notations since the problem here the same except the volatility term.
5.2 Semidiscretization Technique

Change the variable \( \tau = T - t \) in (5.1.1) and \( V(S,t) = U(S,\tau) \). This results in the following equation

\[
U_\tau - \frac{1}{2} \sigma^2 S^2 U_{SS} - r S U_S + r U = 0
\]

(5.2.1)

with the terminal and boundary conditions

\[
U(S,0) = \max(S - E, 0), \quad U(0, \tau) = 0, \quad (5.2.2)
\]

\[
\frac{U(S,\tau)}{S} \to 1 \quad \text{as} \quad S \to \infty.
\]

(5.2.3)

Then the finite difference approximations is given by (See [1, 135])

\[
U_S(S,\tau) = \frac{U(S - 2h, \tau) - 8U(S - h, \tau) + 8U(S + h, \tau) - U(S + 2h, \tau)}{12h} + O(h^4),
\]

(5.2.4)

\[
U_{SS}(S,\tau) = \frac{-U(S - 2h, \tau) - 16U(S - h, \tau) - 30U(S, \tau) + 16U(S + h, \tau) - U(S + 2h, \tau)}{12h^2} + O(h^4).
\]

(5.2.5)

For the sake of convenience, subdivide the interval such that \( S_i = E - L + ih, \quad 0 \leq i \leq N \) where \( Nh = 2L \) as explained in [43]. Let \( u_i(\tau) \) correspond to \( U(S_i,\tau) \). This satisfies the ordinary differential equation

\[
\frac{du_i(\tau)}{d\tau} = Bu_i(\tau) + w
\]

(5.2.6)

where \( u(\tau) = [u_1, u_2, \ldots, u_{N-1}]^T \) and

\[
B = \begin{bmatrix}
\gamma_1 & \delta_1 & \xi_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\beta_2 & \gamma_2 & \delta_2 & \xi_2 & 0 & 0 & 0 & \cdots & 0 \\
\alpha_3 & \beta_3 & \gamma_3 & \delta_3 & \xi_3 & 0 & 0 & \cdots & 0 \\
0 & \alpha_4 & \beta_4 & \gamma_4 & \delta_4 & \xi_4 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha_{N-1} & \beta_{N-1} & \gamma_{N-1}
\end{bmatrix}
\]

(5.2.7)
and

\[
\begin{bmatrix}
\alpha_1 u_{-1} + \beta_1 u_0 \\
\alpha_2 u_0 \\
0 \\
\vdots \\
\vdots \\
\vdots \\
0 \\
\xi_{N-2} u_N \\
\delta_{N-1} u_N + \xi_{N-1} u_{N+1}
\end{bmatrix}
\]

\[
w = 
\begin{bmatrix}
\alpha_1 u_{-1} + \beta_1 u_0 \\
\alpha_2 u_0 \\
0 \\
\vdots \\
\vdots \\
\vdots \\
0 \\
\xi_{N-2} u_N \\
\delta_{N-1} u_N + \xi_{N-1} u_{N+1}
\end{bmatrix}
\]

\[
(5.2.8)
\]

\[
\alpha_i = \alpha_i(\tau) = -\frac{\sigma_i^2 S^2}{24 h^2} + \frac{r S_i}{12 h}
\]

\[
\beta_i = \beta_i(\tau) = \frac{2\sigma_i^2 S^2}{3 h^2} - \frac{2r S_i}{3 h}
\]

\[
\gamma_i = \gamma_i(\tau) = -\frac{15\sigma_i^2 S^2}{12 h^2} - r
\]

\[
\delta_i = \delta_i(\tau) = \frac{2\sigma_i^2 S^2}{3 h^2} + \frac{2r S_i}{3 h}
\]

\[
\xi_i = \xi_i(\tau) = -\frac{\sigma_i^2 S^2}{24 h^2} - \frac{r S_i}{12 h}
\]

(5.2.9)

Here the value of \(\sigma_i = \sigma(\varphi_S(S_i, \tau))\) using (5.1.2).

5.3 Semidiscretization method for Linear and Nonlinear Black-Scholes equation

If we assume that there is no transaction cost then \(\delta = \sigma\) which leads to linear classical Black-Scholes equation. Hence using semidiscretization technique this equation is solved in works like [43, 57]. Instead of choosing the method of applying boundary values [57] let us apply the Lagangian interpolation method as in [43].

First fix \(S_0 = E - L, S_N = E + L\). Applying \(u_i(\tau) = U(\varphi_S, \tau)\) in (5.2.4) and (5.2.5), we can write
\begin{align*}
    u_{-1} &= 10u_1 - 20u_2 + 15u_3 - 4u_4, \\
    u_0 &= 4u_1 - 6u_2 + 4u_3 - u_4, \\
    u_N &= 4u_{N-1} - 6u_{N-2} + 4u_{N-3} - u_{N-4}, \quad (5.3.1) \\
    u_{N+1} &= 10u_{N-1} - 20u_{N-2} + 15u_{N-3} - 4u_{N-4}.
\end{align*}

Combining (5.2.6)-(5.2.9) along with (5.3.1), we get the solution of the linear Black-Scholes equation as follows:

\[ u(\tau) = e^{At}u(0) \quad (5.3.2) \]

where

\[
    A = \begin{bmatrix}
    m_{11} & m_{12} & m_{13} & m_{14} & 0 & 0 & \cdots & 0 \\
    m_{21} & m_{22} & m_{23} & m_{24} & 0 & 0 & \cdots & \cdots \\
    \alpha_3 & \beta_3 & \gamma_3 & \delta_3 & \xi_3 & 0 & \cdots & \cdots \\
    0 & \alpha_4 & \beta_4 & \gamma_4 & \delta_4 & \xi_4 & \cdots & \cdots \\
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    m_{N-2N-4} & m_{N-2N-3} & m_{N-2N-2} & m_{N-2N-1} & m_{N-2N-4} & m_{N-2N-3} & m_{N-2N-2} & m_{N-2N-1} \\
    m_{N-1N-4} & m_{N-1N-3} & m_{N-1N-2} & m_{N-1N-1} & m_{N-1N-4} & m_{N-1N-3} & m_{N-1N-2} & m_{N-1N-1}
\end{bmatrix}
\quad (5.3.3)
\]

with

\[
    \begin{align*}
    m_{11} &= \gamma_1 + 10\alpha_1 + 4\beta_1, & m_{12} &= \delta_1 - 20\alpha_1 - 6\beta_1, \\
    m_{13} &= \xi_1 + 15\alpha_1 + 4\beta_1, & m_{14} &= -4\alpha_1 - \beta_1, \\
    m_{21} &= \beta_2 + 4\alpha_2, & m_{22} &= \gamma_2 + -6\alpha_2, \\
    m_{23} &= \delta_2 + 4\alpha_2, & m_{24} &= \xi_2 + -\alpha_2, \quad (5.3.4) \\
    m_{N-2N-4} &= \alpha_{N-2} - \xi_{N-2}, & m_{N-2N-3} &= \beta_{N-2} - 4\xi_{N-2}, \\
    m_{N-2N-2} &= \gamma_{N-2} - 6\xi_{N-2}, & m_{N-2N-1} &= \delta_{N-2} - 4\xi_{N-2}, \\
    m_{N-1N-4} &= -\delta_{N-1} - 4\xi_{N-1}, & m_{N-1N-3} &= \alpha_{N-1} + 4\delta_{N-1} + 15\xi_{N-1}, \\
    m_{N-1N-2} &= \beta_{N-1} - 6\delta_{N-1} - 20\xi_{N-1}, & m_{N-1N-1} &= \gamma_{N-1} + 4\delta_{N-1} + 10\xi_{N-1}
\end{align*}
\]

and \( u_i(0) = \max(S_i - E, 0), \quad 1 \leq i \leq N - 1. \)

Now we derive the numerical solution for the nonlinear Black-Scholes model with the volatility given by (5.1.2). Here we use the same method as before instead of linearising of the nonlinear term. Obviously, in the problem under consideration with the aid of (5.2.1) and (5.2.4)-(5.2.6), we get the ordinary differential equation as

\[
    \frac{du}{d\tau} = A(\tau)u(\tau), \quad 0 < \tau \leq T \quad (5.3.5)
\]
where \( A(t) \) is given by (5.3.3).

Applying (5.3.1)-(5.3.4), the solution of (5.3.5) can be written as follows:

\[
u(t) = \left[ \Pi_{a=0}^{n-1} (l + kA(ak)) \right] u(0)
\]

(5.3.6)

where \( \tau = lk \). That is, we have discretized the time \( \tau \) with \( \Delta \tau = \Delta t \). The volatility is given by

\[
\sigma^2 = \sigma^2_0 \left( 1 + 3\left( \frac{C^2 M}{2\pi} S_i \frac{-u_{i-2} - 16u_{i-1} - 30u_i + 16u_{i+1} - u_{i+2}}{12h^2} \right)^{\frac{1}{2}} \right).
\]

(5.3.7)

Notice that the method as well as the solution of the above problem is same as in [43]. But the volatility differentiates the option price as it depends clearly on the transaction cost \( C \) and the risk premium coefficient \( M \). Further, if we increase the time lag \( \Delta t \) between the portfolio adjustments, the transaction costs can be reduced. But this larger value of \( \Delta t \) leads to a higher investor's exposure to the risk from an unprotected portfolio. Hence care should be taken in fixing the intervals so that the risk is under control.