2. Mathematical Analysis

In this chapter the problem is formulated and the numerical method employed is discussed in detail.

2.1 Statement of the Problem

Consider a vertical layer of fluid with thickness $2h$ bounded by two parallel sidewalls located at $x = \pm h$. The temperatures of both walls are kept constant and are equal. In the discussion to follow this constant temperature serves as the reference point. Internal heat sources of uniform volume density $Q$ are distributed uniformly through the volume. As a result a laminar flow is developed. We choose a Cartesian coordinate system, where $x$- and $z$- axes are normal and parallel to the walls with the corresponding unit vectors $\hat{i}$ and $\hat{k}$ respectively. The origin of the coordinate system is located in the midplane of the layer. The governing equations of the resulting flow with the Boussinesq approximation are

\[
\frac{\partial \mathbf{v}^*}{\partial t^*} + (\mathbf{v}^* \cdot \nabla^*) \mathbf{v}^* = -\frac{1}{\rho} \nabla^* p^* + \nu (\nabla^*)^2 \mathbf{v}^* + g\beta T^* \hat{k}
\]  

(2.1.1)

\[
\frac{\partial T^*}{\partial t^*} + (\mathbf{v}^* \cdot \nabla^*) T^* = \kappa (\nabla^*)^2 T^* + \frac{Q}{\rho c_p}
\]  

(2.1.2)

\[
\text{div} \, \mathbf{v}^* = 0
\]  

(2.1.3)

If the nondimensional variables $x = x^*/h$, $z = z^*/h$, $t = t^*/(h^2/\nu)$, $\mathbf{v} = \mathbf{v}^*/(gh^3/2\nu)$, $p = p^*/(\rho gh^3/2)$ & $T = T^*/(qh^2/2)$ are introduced and $q = Q/(\rho c_p \kappa), Gr = g\beta q h^5/(2\nu^2)$ & $Pr = \nu/\chi$, in the dimensionless variables equations (2.1.1)-(2.1.3) become

\[
\frac{\partial \tilde{v}}{\partial t} + Gr(\tilde{v} \cdot \nabla) \tilde{v} = -\nabla p + \nabla^2 \tilde{v} + T \hat{k}
\]  

(2.1.4)

\[
\frac{\partial T}{\partial t} + Gr(\tilde{v} \cdot \nabla) T = \frac{1}{Pr} \nabla^2 T + q
\]  

(2.1.5)
where \( \vec{v}, T \) and \( p \) are respectively, the velocity of the fluid, temperature and pressure. A steady plane parallel solution is sought for the equations (2.1.4)-(2.1.6) of the following type:

\[
\vec{v} = [0, 0, u_0(x)], \quad T = T_0(x), \quad p = p_0(z)
\]  

(2.1.7)

The flow (2.1.7), may be realized in the middle portion of a sufficiently long vertical layer of fluid where the end effects are negligible.

The case of a closed channel is considered. This warrants the fluid flow through any cross section of the channel to be zero and hence

\[
\int_{-1}^{1} v_0(x) \, dx = 0
\]  

(2.1.8)

The method of small perturbations is applied to investigate the stability of the flow (2.1.7). The interest is on the perturbed motion \( \vec{v}_0 + \vec{v}, T_0 + T \), and \( p_0 + p \), where \( \vec{v}, T \) and \( p \) are small unsteady perturbations, \( \vec{v}_0 = v_0 \hat{k} \). Here the attention is restricted to plane perturbations. Then equations (2.1.4)-(2.1.6) for the perturbed state take the form:

\[
\frac{\partial \vec{v}}{\partial t} + Gr[(\vec{v}_0 \cdot \nabla) \vec{v} + (\vec{v} \cdot \nabla) \vec{v}_0] = -\nabla p + \nabla^2 \vec{v} + T \hat{k}
\]  

(2.1.9)

\[
\frac{\partial T}{\partial t} + Gr[(\vec{v}_0 \cdot \nabla) T + (\vec{v} \cdot \nabla) T_0] = \frac{1}{Pr} \nabla^2 T
\]  

(2.1.10)

\[
div \vec{v} = 0
\]  

(2.1.11)

It is convenient to introduce the stream function \( \Psi(x, z) \) as
We set
\[ \Psi(x, z, t) = \phi(x) e^{-\lambda t + ikz} \]
\[ T(x, z, t) = \theta(x) e^{-\lambda t + ikz} \]
\[ \text{(2.1.13)} \]

where \( \phi \) and \( \theta \) are the amplitudes of the normal perturbations, \( k \) the wavenumber and \( \lambda \) a complex eigenvalue. Substituting (2.1.13) in (2.1.9) and (2.1.10) we obtain the amplitude equations

\[ \phi^{(4)} - 2k^2\phi^{(2)} + k^4\phi + ikGr[v_0^{(2)}\phi - v_0(\phi^{(2)} - k^2\phi)] + \theta^{(1)} = -\lambda(\phi^{(2)} - k^2\phi) \]
\[ \text{(2.1.14)} \]
\[ \frac{1}{Pr}(\theta^{(2)} - k^2\theta) + ikGr(T_0^{(1)}\phi - v_0\theta) = -\lambda\theta \]
\[ \text{(2.1.15)} \]

The velocity and temperature perturbations vanish at the sidewalls and hence the boundary conditions are

\[ \phi(\pm 1) = 0, \quad \phi^{(1)}(\pm 1) = 0, \quad \theta(\pm 1) = 0 \]
\[ \text{(2.1.16)} \]

For fixed values of \( k, Gr \) and \( Pr \), the boundary value problem (2.1.14)-(2.1.16) determines the corresponding \( \lambda = Re(\lambda) + iIm(\lambda) \).

### 2.2 Spectral Collocation Method

Since instability, as a rule occurs for large values of the parameters concerned, the numerical computations of the spectrum in hydrodynamic stability problems is a difficult one. From the mathematical point of view this means that it is necessary to solve an eigensystem for a set of differential equations with a small parameter as the coefficient of highest ordered derivative. Therefore it becomes necessary to use an efficient numerical method.
The widespread application of spectral methods to stability studies is based on several reasons. First, they guarantee high accuracy in the final results. Second, they can take into account any symmetry of the problem. Finally, they can provide a dependable control of different kinds of errors [Canuto et al. (1988)]. Spectral methods make use of the excellent approximation properties of orthogonal polynomials to discretize the boundary value problem (2.1.12)-(2.1.14). The thesis adopts a spectral method which uses zeroes of a Chebyshev polynomial as collocation points. This method, suggested initially by Babenko and Vasilev (1983) and developed later by Kolyshkin and Vaillancourt (1989), has proven superiority in convergence over the usual Galerkin method. The steps involved in this method are provided for the sake of completeness.

The following fundamental interpolation polynomials are introduced

\[ P_n(x, \phi) = \sum_{j=1}^{n} p_{nj}(x)\phi_j, \quad Q_n(x, \phi) = \sum_{j=1}^{n} q_{nj}(x)\phi_j, \quad S_n(x, \phi) = \sum_{j=1}^{n} s_{nj}(x)\phi_j, \]

\[ U_n(x, \Theta) = \sum_{j=1}^{n} p_{nj}(x)\theta_j \quad \& \quad V_n(x, \Theta) = \sum_{j=1}^{n} q_{nj}(x)\theta_j \]

(2.2.1)

where \( \Phi = (\phi_1, ..., \phi_n)^T \in \mathbb{R}^n, \Theta = (\theta_1, ..., \theta_n)^T \in \mathbb{R}^n, \phi_j = \phi(x_j) \) and \( \theta_j = \theta(x_j) \) and

\[ p_{nj}(x) = \frac{T_n(x)}{(x - x_j)T_n'(x_j)}, \quad q_{nj}(x) = \frac{p_{nj}(x)(1 - x^2)}{(1 - x_j^2)}, \quad s_{nj}(x) = \frac{q_{nj}(x)(1 - x^2)}{(1 - x_j^2)} \]

(2.2.2)

Here \( T_n(x) \) denotes the Chebyshev polynomial of the first kind of degree \( n \) whose zeroes are

\[ x_j = \cos\left(\frac{(2j - 1)\pi}{2n}\right), \ j = 1,...,n. \]

(2.2.3)

The mapping of \([-1,1]\) onto itself and denser distribution of zeroes near the boundaries (Fig.2.1) improve the spectral accuracy of Chebyshev expansion. One can easily verify that \( S_n(x, \Phi) \) and \( V_n(x, \Theta) \) satisfy the boundary conditions (2.1.16) for \( \phi \) and \( \theta \) respectively. The Chebyshev collocation method consists of replacing the functions \( \phi^{(4)}, \phi^{(2)}, \phi, \theta^{(2)}, \theta^{(1)} \) and \( \theta \)
in (2.1.14) and (2.1.15) with the interpolation polynomials $S^{(i)}_n(x, \Phi)$, $Q^{(2)}_n(x, \Phi)$, $P_n(x, \Phi)$, $V^{(2)}_n(x, \Theta)$, $U^{(1)}_n(x, \Theta)$ and $U_n(x, \Theta)$ respectively and setting $x = x_i (i = 1, ..., n)$ in the corresponding equations. Then the invertible matrices $C = \left(q^{(4)}_{n_i}(x_i)\right)_{i=1}^n$ and $D = \left(q^{(2)}_{n_i}(x_i)\right)_{i=1}^n$ are constructed. Finally to obtain the finite dimensional generalized eigenvalue problem

\[(A - \lambda B)\bar{u} = 0 \tag{2.2.4}\]

where $\bar{u} = (\phi_1, ..., \phi_n, \theta_1, ..., \theta_n)^T$, the unknown functions in (2.1.14) and (2.1.15) are replaced with the corresponding interpolation polynomials and set $x = x_i$. Then (2.1.14) and (2.1.15) are multiplied by $C^{-1}$ and $D^{-1}$. Solutions of the form (2.2.1) are more convenient [Canuto et al. (1988)] than those obtained by traditional collocation methods. The advantages are the resulting matrix $B$ is nonsingular and the base functions considerably reduces the condition number of matrices in this spectral method.

2.3 Solution to Generalized Eigenvalue Problem

In this section the iterative procedure called QZ method developed by Moler and Stewart (1973) to determine the nontrivial solutions of $A\bar{u} - \lambda B\bar{u}$ where $A$ and $B$ are real matrices of order $m$ is outlined.

When $B$ is invertible this problem is equivalent to the eigenvalue problem $B^{-1}A\bar{u} = \lambda \bar{u}$. When $B$ is singular such a reduction is impossible and the problem turns to be ill-posed with the characteristic polynomial $\text{det}(A - \lambda B)$ of degree less than $m$. There is no sharp distinction between singular and invertible matrices in a numerical method and the pathological features associated with singular $B$ get transferred to the nearly singular $B$. QZ method computes the eigensystem that is unaffected by nearly singular $B$. It provides an iterative procedure for computing the following decomposition:

Generalized Schur Decomposition: If $A$ and $B$ are in $C^{m\times m}$ then there exist unitary $Q$ and $Z$ such that $Q^*AZ = T$ and $Q^*BZ = S$ are upper triangular with * representing the
transjugate. If for some \( l, t_{ll} \) and \( s_{ll} \) are both zero, then \( \lambda(A, B) = C \). Otherwise,

\[
\lambda(A, B) = \frac{t_{ll}}{s_{ll}} : s_{ll} \neq 0
\]  

(2.3.1)

This method is based on the elementary Hermitian matrices. Accordingly, the following matrices denoted by \( H_r(l) \) are introduced. \( H_r(l) \) represents the class of symmetric, orthogonal matrices of the form

\[
I + \bar{w}_2 \bar{w}_1^T
\]  

(2.3.2)

where \( \bar{w}_1^T \bar{w}_1 = -2 \), \( \bar{w}_2 \) a scalar multiple of \( w_1 \), only components \( l, l+1, \ldots, l+r-1 \) of \( \bar{w}_1 \) nonzero and \( l^{th} \) component of \( \bar{w}_1 \) one. This definition enables the following:

(i) Given any vector \( \bar{u} \) it is easy to choose a \( Q \) of \( H_r(l) \) such that \( Q\bar{u} = \bar{u} + (\bar{w}_1^T \bar{u})\bar{w}_2 \) has its \( l+1, \ldots, l+r-1 \) components equal to zero, its \( l^{th} \) component changed and all other components unchanged. Since \( l^{th} \) component of \( \bar{w}_1 \) is one, the computation of \( Q\bar{y} \) for any \( \bar{y} \) requires only \( 2r-1 \) multiplications and \( 2r-1 \) additions which is less than those using standard Hermitians.

(ii) When a matrix \( Q \) in \( H_3(l) \) premultiplies a matrix \( A \), only rows \( l, l+1 \) and \( l+2 \) in \( QA \) are changed. If the elements \( l, l+1 \) and \( l+2 \) in a column of \( A \) are zero, they remain zero in \( QA \) also. Similarly if \( Z \in H_3(l) \), only columns \( l, l+1 \) and \( l+2 \) are changed in \( AZ \). If some row has elements \( l, l+1 \) and \( l+2 \) zero, then they remain zero in \( AZ \).

The \( QZ \) method proceeds in four stages. In the first \( A \) is reduced to upper Hessenberg form and at the same time \( B \) is reduced to upper triangular form. In the second step, \( A \) is reduced to quasi-triangular form by Francis implicit double shift while triangular form of \( B \) is maintained. In the third stage the quasi-triangular matrix is effectively reduced to triangular form and the eigenvalues extracted. In the final stage the eigenvectors, if necessary are obtained from the triangular matrices and then transformed back into the
original coordinate system. These are done by a sequence of pre- and post-multiplications by matrices mostly from \( H_2 \) and \( H_3 \). Further details can be seen in Golub and VanLoan (1986).

If an eigenvalue and eigenvector found with some accuracy are not too ill-disposed then they produce a small relative residual. But this can be made sufficiently small by choosing the accuracy in a proper way. The advantages of the method are the numerical results provide a global instability and no initial guesses are required in contrast to methods using orthogonalization at each step and variants of Runge-Kutta methods [see Meyer (1986) and Canuto et al. (1988)].

2.4 Method of Solution

A computer code was written to determine the eigenvalues of (2.2.4) implementing the procedures explained in Sections 2.2 and 2.3. This method is very stable and required computer time is moderate. The real parts of the complex damping rates \( Re(\lambda_i) \), determine the stability of the flow. If \( Re(\lambda_i) \) are all positive, then it is evident that from (2.1.13) that all the small perturbations decay exponentially. This means that the flow (2.1.7) will be stable. It becomes unstable if at least one \( Re(\lambda_i) \) is negative. The marginal curve corresponds to the case when at least for one of the eigenvalues \( Re(\lambda_i)=0 \). This curve divides the region of stability and instability in the \((k, Gr)\) plane for a fixed value of \( Pr \).

The numerical procedure to find the critical values are as follows. For fixed values of \( k \) and \( Pr \) the corresponding \( Gr \) on the marginal (stability) curve was obtained. It is found by iteratively adjusting \( Gr \), called vertical iteration until two successive values of \( Gr \) differ by less than 0.01%. This can be done using bisection or secant rule to drive the growth rate to zero. Then another \( k \) was taken and the corresponding \( Gr \) was calculated for the same \( Pr \). After several such computations the critical value of \( Gr \) was found for the given \( Pr \) as
\[ Gr_c = \min_k Gr(Pr) \]  

(2.4.1)

Then we choose another value of \( Pr \) and continue the computation.

For each value of \( Pr \), \( n \) was increased by one until \( Gr_c \) differs less than 0.5% in the case of channels bounded by two planes and 2% in the case of channels bounded by two cylinders, for two successive values of \( n \). As the curvature effect required more number of terms in the approximation functions (2.2.1), the 0.5% condition was relaxed to 2% in the case of cylindrical annular configurations. A convergence criterion more stringent than this increased the cost significantly. However it was felt that the present choices are good.

In addition to ensure that the errors in eigenvalue computations are minimal for the cylindrical configurations, a performance index \( P \) was defined [see Thangam and Chen (1986)] as

\[ P = \max_{1 \leq i \leq n} \| \lambda_{Bi} \bar{A}_i \bar{u}^i - \lambda_{Ai} \bar{B}_i \bar{u}^i \| \times \left[ (|\lambda_{Bi}| \|A\| \|\bar{u}^i\| + |\lambda_{Ai}| \|B\| \|\bar{u}^i\|) \varepsilon \right]^{-1}. \]  

(2.4.2)

Here \( \lambda_i = \lambda_{Ai}/\lambda_{Bi} \) and \( \bar{u}^i \) are the associated eigensystem. The quantity \( \varepsilon \) specifies the relative precision of the real variable, 10^{-8} in the present case. The notations |.| and ||.|| are used to represent respectively, the norms of vectors and matrices. When \( P \) is less than unity, the performance of the eigensystem code is considered to be excellent in the sense that the residuals \((\lambda_{Bi} \bar{A}_i \bar{u}^i - \lambda_{Ai} \bar{B}_i \bar{u}^i)\) can be made as small as required. The value of \( P \) was monitored and kept below 0.8.

2.5 Nondimensional Quantities

The dimensional characteristics of physical variables render a physical equation to have an arbitrary mathematical form. By dimensional analysis the variables in a physical equation are reorganized in power-product groups that have no dimensions called nondimensional quantities. These quantities play a decisive role in predicting the nature of flow. The great advantage of dimensional analysis is in producing scaling laws and in reducing the number of variables. The following nondimensional quantities appear in this thesis.
(i) **Grashof number** \((Gr)\) is an important quantity appearing in natural convective flows. As differential heating and heat generation are the two sources to produce buoyancy force, \(Gr\) can be constructed in two different ways. Here it is defined as as

\[
Gr = \frac{g \beta q h^5}{2 \nu^2} \quad (2.5.1)
\]

It is proportional to the ratio of buoyancy force to viscous force.

(ii) **Hartmann number** \((Ha)\) arises in magnetohydrodynamic flows and is defined as

\[
Ha = B_0 h \sqrt{\frac{\sigma_e \mu}{\rho \nu}} \quad (2.5.2)
\]

is the ratio of magnetic force to viscous force.

(iii) **Prandtl number** \((Pr)\), a material property of a fluid defined as

\[
Pr = \frac{\nu}{\chi} \quad (2.5.3)
\]

is the ratio of momentum and thermal diffusions.

(iv) **Overall Prandtl number** \((Pr_m)\) is nothing but \(Pr\) of a porous medium composed of solid and fluid phases. It is defined as

\[
Pr_m = \frac{k_m'}{(\rho c_p)_f \nu} \quad (2.5.4)
\]

where \(k_m' = (1 - \phi)k'_s + \phi k'_f\), \(\phi\) being porosity of the medium.

(v) **Radius ratio** \((R)\) serving as a measure of curvature of an annular cylindrical surface is defined as
(vi) **Reynolds number** \((Re)\) plays the same role in forced convection that \(Gr\) plays in free convection. It is the ratio of inertial force to viscous force and is defined as

\[
Re = \left[ \frac{\rho (\overline{v} \cdot \nabla) \overline{v}}{\mu \nabla^2 \overline{v}} \right] = \frac{u_0 h}{\nu} \quad (2.5.6)
\]

(vii) **Source distribution parameter** \((\delta)\) defined as

\[
\delta = \left( \frac{d - r_1}{r_2 - r_1} \right) 100 \quad (2.5.7)
\]
gives the ratio of the relative position (with respect to inner cylinder) of the source distribution maximum to the annulus width.

(viii) **Viscosity ratio** \((\tilde{\mu})\) is defined as

\[
\tilde{\mu} = \frac{\mu_e}{\mu} \quad (2.5.8)
\]

(ix) **Porosity parameter** \((\sigma)\) measures the ratio of Darcy friction in the porous medium to viscous resistance and is defined by

\[
\sigma^2 = \left[ \frac{\mu \overline{v}}{K \nabla^2 \overline{v}} \right] = \frac{h^2}{K} \quad (2.5.9)
\]

(x) **Heat capacity ratio** \((\Omega)\) of a porous medium is defined as

\[
\Omega = \frac{(\rho c)_m}{(\rho c_p)_f} \quad (2.5.10)
\]
Fig. 2.1 Chebyshev polynomial $T_n(x)$ of first kind of degree $n=15$