CHAPTER 6
6. \( \gamma \)-FUZZY SPACES

In section 6.1, we discuss the properties of \( \gamma \)-fuzzy spaces. In particular, we prove that the interior operator of a GFT is a friendly operator. Also, we characterize \( \gamma \)-fuzzy \( \beta \)-open sets, \( \gamma \)-fuzzy locally closed sets and \( \gamma \)-fuzzy pre-open sets. In section 6.2, we define \( \gamma \)-fuzzy locally closed sets and discuss the properties of these sets. In section 6.3, we define \( \delta \)-fuzzy sets and discuss the properties of these sets. In section 6.4, we define \( \gamma \)-fuzzy generalized closed sets and discuss the properties of these sets.

6.1. More about \( \gamma \)-fuzzy spaces

Let \( X \) be a non-empty set, \( \gamma \in \Gamma_4 \), \( \mu \) be the collection of all \( \gamma \)-fuzzy open sets. A fuzzy set \( \nu \) is said to be \( \gamma \)-dense if \( \overline{1} = c_\gamma(\nu) \). In this section, we establish some of the properties of \( i_\gamma \) and \( c_\gamma \) in a \( \gamma \)-fuzzy space (i.e., \( \gamma \in \Gamma_4 \)) and also prove \( i_\gamma \in \Gamma_4 \). Also, we characterize \( \gamma \)-fuzzy \( \beta \)-open sets, \( \gamma \)-fuzzy locally closed sets and \( \gamma \)-fuzzy pre-open sets.

**Theorem 6.1.1.** Let \( X \) be a non-empty set, \( \gamma \in \Gamma_4 \) and \( \mu \) be the collection of all \( \gamma \)-fuzzy open sets. Then the following hold.

(a) If \( \nu \) is a \( \gamma \)-fuzzy open set and \( \lambda \in \mathcal{F} \), then \( \nu \land i_\gamma(\lambda) = i_\gamma(\nu \land \lambda) \) and so \( i_\gamma \in \Gamma_4 \).

(b) If \( \nu \) is a \( \gamma \)-fuzzy open set and \( \lambda \in \mathcal{F} \), then \( \nu \land c_\gamma(\lambda) \leq c_\gamma(\nu \land \lambda) \).
(c) \( i_\gamma(\lambda \lor \omega) \leq i_\gamma(\lambda) \lor \omega \) where \( \omega \) is a \( \gamma \)-fuzzy closed set and \( \lambda \in \mathcal{F} \).

(d) \( c_\gamma(\lambda \lor \omega) = c_\gamma(\lambda) \lor \omega \) where \( \omega \) is a \( \gamma \)-fuzzy closed set and \( \lambda \in \mathcal{F} \).

(e) If \( \nu \) is a \( \gamma \)-fuzzy open set and \( \psi \) is a \( \gamma \)-fuzzy dense, then \( c_\gamma(\nu \land \psi) = c_\gamma(\nu) \).

Proof (a) Let \( \nu \) be \( \gamma \)-fuzzy open set and \( \lambda \in \mathcal{F} \). Then \( \nu \land i_\gamma(\lambda) \) is a \( \gamma \)-fuzzy open set, by Theorem 4.3.6, such that \( \nu \land i_\gamma(\lambda) \leq \nu \land \lambda \). Therefore, \( \nu \land i_\gamma(\lambda) \leq i_\gamma(\nu \land \lambda) = i_\gamma(\nu) \land i_\gamma(\lambda) = \nu \land i_\gamma(\lambda) \). Therefore, \( \nu \land i_\gamma(\lambda) = i_\gamma(\nu \land \lambda) \). Since set of all \( i_\gamma \)-fuzzy open sets coincides with the set of all \( \gamma \)-fuzzy open sets, it follows that \( i_\gamma \in \Gamma_4 \).

(b) Let \( x_t \in \nu \land c_\gamma(\lambda) \) and \( \theta \) be an arbitrary \( \gamma \)-fuzzy open set containing \( x_t \).

Since \( \theta \land \nu \) is a \( \gamma \)-fuzzy open set containing \( x_t \) and \( x_t \in c_\gamma(\nu \land \lambda) \), \( (\theta \land \nu) \land \lambda \) and so \( \theta q(\nu \land \lambda) \) which implies that \( x_t \in c_\gamma(\nu \land \lambda) \). Therefore, \( \nu \land c_\gamma(\lambda) \leq c_\gamma(\nu \land \lambda) \).

(c) Now \( \bar{I} - i_\gamma(\lambda \lor \omega) = c_\gamma((\bar{I} - \lambda) \land (\bar{I} - \omega)) \geq c_\gamma((\bar{I} - \lambda) \land (\bar{I} - \omega)) \), by (b). Therefore, \( \bar{I} - i_\gamma(\lambda \land \omega) \geq (\bar{I} - i_\gamma(\lambda)) \land (\bar{I} - \omega) = \bar{I} - (i_\gamma(\lambda) \lor \omega) \) and so \( i_\gamma(\lambda \lor \omega) \leq i_\gamma(\lambda) \lor \omega \).

(d) Now \( \bar{I} - c_\gamma(\lambda \lor \omega) = i_\gamma((\bar{I} - \lambda) \land (\bar{I} - \omega)) = i_\gamma((\bar{I} - \lambda) \land (\bar{I} - \omega)) = i_\gamma((\bar{I} - \lambda) \land (\bar{I} - \omega)) = (\bar{I} - c_\gamma(\lambda)) \land (\bar{I} - \omega) = (\bar{I} - (c_\gamma(\lambda) \lor \omega) \) and so \( c_\gamma(\lambda \lor \omega) = c_\gamma(\lambda) \lor \omega \).

(e) Since \( \nu \land \psi \leq \nu \), \( c_\gamma(\nu \land \psi) \leq c_\gamma(\nu) \). By (b), \( c_\gamma(\nu \land \psi) \geq c_\gamma(\psi) \land \nu = \nu \) which implies that \( c_\gamma(\nu \land \psi) \geq c_\gamma(\nu) \) and so \( c_\gamma(\nu \land \psi) = c_\gamma(\nu) \).

The following Theorem 6.1.2 shows that the intersection of two \( \gamma \)-fuzzy \( \alpha \)-open sets is a \( \gamma \)-fuzzy \( \alpha \)-open set and the intersection of a \( \gamma \)-fuzzy semi-open (resp. \( \gamma \)-fuzzy pre-open, \( \gamma \)-fuzzy \( \beta \)-open, \( \gamma \)-fuzzy \( b \)-open) set with a
\( \gamma \)-fuzzy \( \alpha \)-open set is a \( \gamma \)-fuzzy semi-open (resp. \( \gamma \)-fuzzy pre-open, \( \gamma \)-fuzzy \( \beta \)-open, \( \gamma \)-fuzzy \( b \)-open) set.

**Theorem 6.1.2** Let \( X \) be a non-empty set, \( \gamma \in \Gamma_4 \) and \( \mu \) be the collection of all \( \gamma \)-fuzzy open sets. Then the following hold.

(a) \( \nu \wedge \lambda \) is a \( \gamma \)-fuzzy semi-open (resp. \( \gamma \)-fuzzy pre-open, \( \gamma \)-fuzzy \( \beta \)-open, \( \gamma \)-fuzzy \( b \)-open) set whenever \( \nu \) is a \( \gamma \)-fuzzy \( \alpha \)-open set and \( \lambda \) is a \( \gamma \)-fuzzy semi-open (resp. \( \gamma \)-fuzzy pre-open, \( \gamma \)-fuzzy \( \beta \)-open, \( \gamma \)-fuzzy \( b \)-open) set.

(b) \( \nu \wedge \lambda \) is a \( \gamma \)-fuzzy \( \alpha \)-open set whenever \( \nu \) and \( \lambda \) are \( \gamma \)-fuzzy \( \alpha \)-open sets.

**Proof:** (a) Suppose \( \nu \) is a \( \gamma \)-fuzzy \( \alpha \)-open set and \( \lambda \) is a \( \gamma \)-fuzzy semi-open set. Then \( \nu \wedge \lambda \subseteq i_\gamma c_\gamma i_\gamma (\nu) \wedge i_\gamma (\lambda) = c_\gamma (i_\gamma c_\gamma (\nu) \wedge i_\gamma (\lambda)) \subseteq c_\gamma i_\gamma (c_\gamma i_\gamma (\nu) \wedge i_\gamma (\lambda)) = c_\gamma i_\gamma (\nu \wedge \lambda) = c_\gamma i_\gamma (\nu \wedge \lambda). \) Therefore, \( \nu \wedge \lambda \) is a \( \gamma \)-fuzzy semi-open set.

Suppose \( \nu \) is a \( \gamma \)-fuzzy \( \alpha \)-open set and \( \lambda \) is a \( \gamma \)-fuzzy pre-open set. Then \( \nu \wedge \lambda \subseteq i_\gamma c_\gamma i_\gamma (\nu) \wedge i_\gamma (\lambda) = i_\gamma (c_\gamma i_\gamma (\nu) \wedge i_\gamma (\lambda)) \subseteq i_\gamma c_\gamma (i_\gamma (\nu) \wedge i_\gamma (\lambda)) = i_\gamma c_\gamma (\nu \wedge \lambda) \subseteq i_\gamma c_\gamma (\nu \wedge \lambda). \) Therefore, \( \nu \wedge \lambda \) is a \( \gamma \)-fuzzy pre-open set.

Suppose \( \nu \) is a \( \gamma \)-fuzzy \( \alpha \)-open set and \( \lambda \) is a \( \gamma \)-fuzzy \( \beta \)-open set. Then \( \nu \wedge \lambda \subseteq i_\gamma c_\gamma i_\gamma (\nu) \wedge i_\gamma c_\gamma (\lambda) \subseteq c_\gamma (i_\gamma c_\gamma (\nu) \wedge i_\gamma c_\gamma (\lambda)) \subseteq c_\gamma i_\gamma (c_\gamma i_\gamma (\nu) \wedge i_\gamma c_\gamma (\lambda)) \subseteq c_\gamma i_\gamma (\nu \wedge \lambda) = c_\gamma i_\gamma (\nu \wedge \lambda) \) and so \( \nu \wedge \lambda \) is a \( \gamma \)-fuzzy \( \beta \)-open set.
Suppose $\nu$ is a $\gamma-$ fuzzy $\alpha-$ open set and $\lambda$ is a $\gamma-$ fuzzy $b-$ open set. Then
\[ \nu \land \lambda \leq \nu \land (c_\gamma i_\gamma(\nu) \lor i_\gamma c_\gamma(\lambda)) = (\nu \land c_\gamma i_\gamma(\nu)) \lor (\nu \land i_\gamma c_\gamma(\lambda)) \leq c_\gamma i_\gamma(\nu \land \lambda) \lor i_\gamma c_\gamma(\nu \land \lambda) \]
and so $\nu \land \lambda$ is a $\gamma-$ fuzzy $b-$ open set.

(b) Suppose $\nu$ and $\lambda$ are $\gamma-$ fuzzy $\alpha-$ open sets. Then
\[ \nu \land \lambda \leq i_\gamma c_\gamma(\nu) \land i_\gamma c_\gamma(\lambda) \leq i_\gamma (c_\gamma(\nu) \land i_\gamma c_\gamma(\lambda)) = i_\gamma c_\gamma(i_\gamma(\nu) \land i_\gamma c_\gamma(\lambda)) \leq i_\gamma c_\gamma(i_\gamma(\nu) \land c_\gamma i_\gamma(\lambda)) \leq i_\gamma c_\gamma(i_\gamma(\nu) \land \lambda) = i_\gamma c_\gamma(\nu \land \lambda) \]
and so $\nu \land \lambda$ is a $\gamma-$ fuzzy $\alpha-$ open set.

**Theorem 6.1.3.** Let $X$ be a non-empty set, $\gamma \in \Gamma_4$ and $\mu$ be the collection of all $\gamma-$ fuzzy open sets. If $\nu$ is a $\gamma-$ fuzzy open set and $\lambda \in \mathcal{F}$, then the following hold.

(a) $\nu \land i_\sigma(\lambda) \leq i_\sigma(\nu \land \lambda)$.
(b) $\nu \land i_\alpha(\lambda) \leq i_\alpha(\nu \land \lambda)$.
(c) $\nu \land i_\pi(\lambda) \leq i_\pi(\nu \land \lambda)$.
(d) $\nu \land i_\beta(\lambda) \leq i_\beta(\nu \land \lambda)$.
(e) $\nu \land i_\beta(\lambda) \leq i_\beta(\nu \land \lambda)$.
(f) $\nu \land c_\sigma(\lambda) \leq c_\sigma(\nu \land \lambda)$.
(g) $\nu \land c_\alpha(\lambda) \leq c_\alpha(\nu \land \lambda)$.
(h) $\nu \land c_\pi(\lambda) \leq c_\pi(\nu \land \lambda)$.
(i) $\nu \land c_\beta(\lambda) \leq c_\beta(\nu \land \lambda)$.
(j) $\nu \land c_\beta(\lambda) \leq c_\beta(\nu \land \lambda)$.

**Proof:** (a) Let $\nu$ be a $\gamma-$ fuzzy open set and $\lambda \in \mathcal{F}$. Then $\nu \land i_\sigma(\lambda)$ is a
γ—fuzzy semi-open set by Theorem 6.1.2(a), such that \( \nu \wedge i_\sigma(\lambda) \leq \nu \wedge \lambda \). Therefore, \( \nu \wedge i_\sigma(\lambda) \leq i_\sigma(\nu \wedge \lambda) \).

Similarly, we can prove (b), (c), (d) and (e).

(f) Let \( x_t \in \nu \wedge c_\sigma(\lambda) \) and \( \psi \) be an arbitrary γ—fuzzy \( \sigma \)—open set containing \( x_t \).
Since \( x_t \in c_\sigma(\lambda) \), \( (\psi \wedge \nu)q_\lambda \) and so \( \psi(\nu \wedge \lambda) \) which implies that \( x_t \in c_\sigma(\nu \wedge \lambda) \).
Therefore, \( \nu \wedge c_\sigma(\lambda) \leq c_\sigma(\nu \wedge \lambda) \).
Similarly, we can prove (g), (h), (i) and (j).

The following Corollary 6.1.4 shows that if \( \gamma \in \Gamma_4 \), then \( i_\alpha \in \Gamma_4 \) and Theorem 6.1.3 (b) above is also true for γ—fuzzy \( \alpha \)—open sets. The proof follows from Theorem 5.1.2(b) and the fact that the set of all γ—fuzzy \( \alpha \)—open sets coincides with the set of all \( i_\alpha \)—fuzzy open sets.

**Corollary 6.1.4.** Let \( X \) be a non-empty set, \( \gamma \in \Gamma_4 \) and \( \mu \) be the collection of all γ—fuzzy open sets. If \( \nu, \lambda \in \mathcal{F} \), then the following hold.

(a) \( i_\alpha(\nu \wedge \lambda) = i_\alpha(\nu) \wedge i_\alpha(\lambda) \).

(b) If \( \nu \) is a γ—fuzzy \( \alpha \)—open set, then \( \nu \wedge i_\alpha(\lambda) = i_\alpha(\nu \wedge \lambda) \).

(c) \( i_\alpha \in \Gamma_4 \).

The following Corollary 6.1.5 follows from Theorem 6.1.3.

**Corollary 6.1.5.** Let \( X \) be a non-empty set, \( \gamma \in \Gamma_4 \) and \( \mu \) be the collection of all γ—fuzzy open sets. If \( \nu \) is a γ—fuzzy open set and \( \lambda \in \mathcal{F} \), then the following hold.
(a) \(c_t(\nu \wedge c_t(\lambda)) = c_t(\nu \wedge \lambda)\).
(b) \(c_a(\nu \wedge c_a(\lambda)) = c_a(\nu \wedge \lambda)\).
(c) \(c_p(\nu \wedge c_p(\lambda)) = c_p(\nu \wedge \lambda)\).
(d) \(c_\beta(\nu \wedge c_\beta(\lambda)) = c_\beta(\nu \wedge \lambda)\).
(e) \(c_b(\nu \wedge c_b(\lambda)) = c_b(\nu \wedge \lambda)\).

Let \(X\) be a non-empty set and \(\gamma \in \Gamma\). A fuzzy set \(\lambda\) of \(\mathcal{F}\) is said to be \(\gamma\)-fuzzy regular set if \(\lambda = \gamma(\lambda)\).

The following Theorem 6.1.6 shows that the intersection of two \(i_\gamma c_\gamma\)-fuzzy regular sets is again a \(i_\gamma c_\gamma\)-fuzzy regular set and Theorem 6.1.7 below gives characterizations of \(\gamma\)-fuzzy \(\beta\)-open sets in \(\gamma\)-fuzzy spaces.

**Theorem 6.1.6.** Let \(X\) be a non-empty set, \(\gamma \in \Gamma_4\) and \(\lambda\) and \(\nu\) are \(i_\gamma c_\gamma\)-fuzzy regular sets. Then \(\lambda \wedge \nu\) is a \(i_\gamma c_\gamma\)-fuzzy regular set.

**Proof:** Suppose \(\lambda\) and \(\nu\) are \(i_\gamma c_\gamma\)-fuzzy regular sets. Now \(\lambda \wedge \nu = i_\gamma c_\gamma(\lambda) \wedge i_\gamma c_\gamma(\beta) = i_\gamma(c_\gamma(\lambda) \wedge c_\gamma(\nu))\) by Theorem 4.3.10 and so \(i_\gamma c_\gamma(\lambda) \wedge i_\gamma c_\gamma(\nu) \geq i_\gamma c_\gamma(\lambda \wedge \nu)\). Since the intersection of two \(\gamma\)-fuzzy open sets is a \(\gamma\)-fuzzy open set, \(\lambda \wedge \nu = i_\gamma(\lambda \wedge \nu) \leq i_\gamma c_\gamma(\lambda \wedge \nu)\). Therefore, \(\lambda \wedge \nu = i_\gamma c_\gamma(\lambda \wedge \nu)\) which implies that \(\lambda \wedge \nu\) is a \(i_\gamma c_\gamma\)-fuzzy regular set.

**Theorem 6.1.7.** Let \(X\) be a non-empty set, \(\gamma \in \Gamma_4\) and \(\lambda\) be a fuzzy set in \(\mathcal{F}\). Then the following statements are equivalent:

(a) \(\lambda\) is a \(\gamma\)-fuzzy \(\beta\)-open set.
(b) \( c_\gamma(\lambda) = c_\gamma i_\gamma c_\gamma(\lambda) \).

(c) \( c_\gamma(\lambda) \) is a \( c_\gamma i_\gamma \)— fuzzy regular set.

(d) There is a \( \gamma \)— fuzzy pre-open set \( \nu \) such that \( \nu \leq \lambda \leq c_\gamma(\nu) \).

(e) \( c_\gamma(\lambda) \) is a \( \gamma \)— fuzzy semi-open set.

(f) \( c_\sigma(\lambda) \) is a \( \gamma \)— fuzzy semi-open set.

(g) \( c_\pi(\lambda) \) is a \( \gamma \)— fuzzy \( \beta \)— open set.

**Proof:** The equivalence of (a) and (b) is clear.

(a) \( \Rightarrow \) (c). If \( \lambda \) is a \( \gamma \)— fuzzy \( \beta \)— open set, then \( c_\gamma(\lambda) = c_\gamma i_\gamma c_\gamma(\lambda) \) and so \( c_\gamma(\lambda) \) is a \( c_\gamma i_\gamma \)— fuzzy regular set.

(c) \( \Rightarrow \) (d). Let \( \nu = i_\pi(\lambda) \). Then \( \nu \) is a \( \gamma \)— fuzzy pre-open set such that \( \nu \leq \lambda \).

Now \( c_\gamma(\nu) = c_\gamma(i_\pi(\lambda)) = c_\gamma i_\gamma c_\gamma(\lambda) \). Therefore, \( c_\gamma(\nu) = c_\gamma(\lambda) \) and so \( \nu \leq \lambda \leq c_\gamma(\nu) \).

(d) \( \Rightarrow \) (a). Suppose \( \nu \) is a \( \gamma \)— fuzzy pre-open set such that \( \nu \leq \lambda \leq c_\gamma(\nu) \).

Then \( c_\gamma(\nu) = c_\gamma(\lambda) \). Since \( \nu \) is a \( \gamma \)— fuzzy pre-open set, \( \nu \leq i_\gamma c_\gamma(\nu) \) and so \( \lambda \leq c_\gamma(\lambda) = c_\gamma(\nu) \leq c_\gamma i_\gamma c_\gamma(\nu) \leq c_\gamma i_\gamma c_\gamma(\lambda) \) and so \( \lambda \) is a \( \gamma \)— fuzzy \( \beta \)— open set.

(e) \( \Rightarrow \) (f) is clear.

(e) \( \Rightarrow \) (f). Suppose \( c_\gamma(\lambda) \) is a \( \gamma \)— fuzzy semi-open set. Now \( i_\gamma c_\gamma(\lambda) = i_\gamma c_\sigma(\lambda) \), and so \( i_\gamma c_\gamma(\lambda) \leq c_\sigma(\lambda) \leq c_\gamma(c_\sigma(\lambda)) = c_\gamma(\lambda) \). Therefore, \( i_\gamma c_\gamma(\lambda) \leq c_\sigma(\lambda) \leq c_\gamma(\lambda) \leq c_\gamma i_\gamma c_\gamma(\lambda) \). Since \( i_\gamma c_\gamma(\lambda) \) is a \( \gamma \)— fuzzy open set, \( c_\sigma(\lambda) \) is a \( \gamma \)— fuzzy semi-open set.

(f) \( \Rightarrow \) (a). Suppose \( c_\sigma(\lambda) \) is a \( \gamma \)— fuzzy semi-open set. Then \( \lambda \leq c_\sigma(\lambda) \leq
\( c_\gamma i_\gamma (c_\sigma (\lambda)) = c_\gamma i_\gamma c_\gamma (\lambda) \), and so \( \lambda \) is a \( \gamma \)-fuzzy \( \beta \)-open set.

\((a) \Rightarrow (g)\). Suppose \( \lambda \) is a \( \gamma \)-fuzzy \( \beta \)-open set. Since every \( \gamma \)-fuzzy open set is a \( \gamma \)-fuzzy pre-open set, \( c_\pi (\lambda) \leq c_\gamma (\lambda) \leq c_\gamma i_\gamma c_\gamma (\lambda) = c_\gamma i_\gamma c_\gamma (c_\pi (\lambda)) \), and so \((g)\) follows.

\((g) \Rightarrow (a)\). Suppose \( c_\pi (\lambda) \) is a \( \gamma \)-fuzzy \( \beta \)-open set. Then \( \lambda \leq c_\pi (\lambda) \leq c_\gamma i_\gamma c_\gamma (c_\pi (\lambda)) = c_\gamma i_\gamma c_\gamma (\lambda) \). Therefore \( \lambda \) is a \( \gamma \)-fuzzy \( \beta \)-open set.

### 6.2. \( \gamma \)-fuzzy locally closed sets

Let \( X \) be a non-empty set and \( \gamma \in \Gamma \). A fuzzy set \( \lambda \) of \( \mathcal{F} \) is said to be a \( \gamma \)-fuzzy locally closed set if \( \lambda = \nu \wedge \psi \) where \( \nu \) is a \( \gamma \)-fuzzy open set and \( \psi \) is a \( \gamma \)-fuzzy closed set. Since \( \bar{I} \) is \( \gamma \)-fuzzy closed set, every \( \gamma \)-fuzzy open set is a \( \gamma \)-fuzzy locally closed set.

The following Theorem 6.2.1 gives a characterization of \( \gamma \)-fuzzy locally closed sets. Theorem 6.2.2 shows that for \( \gamma \)-fuzzy dense sets (i.e., \( \lambda \) is \( \gamma \)-fuzzy dense set if and only if \( c_\gamma (\lambda) = \bar{I} \)), the notions of \( \gamma \)-fuzzy open set and \( \gamma \)-fuzzy locally closed set are equivalent:

**Theorem 6.2.1.** Let \( X \) be a non-empty set, \( \gamma \in \Gamma_4 \) and \( \lambda \) be a fuzzy set of \( \mathcal{F} \). Then the following statements are equivalent:

(a) \( \lambda \) is a \( \gamma \)-fuzzy locally closed set.

(b) \( \lambda = \nu \wedge c_\gamma (\lambda) \) for some \( \gamma \)-fuzzy open set \( \nu \).
Proof. (a) \(\Rightarrow\) (b). Suppose \(\lambda = \nu \land \psi\) where \(\nu\) is a \(\gamma\)-fuzzy open set and \(\psi\) is a \(\gamma\)-fuzzy closed set. Then \(\lambda \leq \psi\) implies that \(c_\gamma(\lambda) \leq \psi\) and also, \(\lambda = \lambda \land c_\gamma(\lambda) = (\nu \land \psi) \land c_\gamma(\lambda) = \nu \land (\psi \land c_\gamma(\lambda)) = \nu \land c_\gamma(\lambda)\).

(b) \(\Rightarrow\) (a) is clear.

**Theorem 6.2.2.** Let \(X\) be a non-empty set, \(\gamma \in \Gamma\) and \(\lambda\) be a \(\gamma\)-fuzzy dense set of \(\mathcal{F}\). Then the following statements are equivalent:

(a) \(\lambda\) is a \(\gamma\)-fuzzy open set.

(b) \(\lambda\) is a \(\gamma\)-fuzzy locally closed set.

**Proof.** It is enough to prove (b) implies (a). Suppose \(\lambda\) is a \(\gamma\)-fuzzy dense and \(\gamma\)-fuzzy locally closed set. Then by Theorem 6.2.1, \(\lambda = \nu \land c_\gamma(\lambda)\) for some \(\gamma\)-fuzzy open set \(\nu\). Therefore, \(\lambda = \nu \land \mathbb{1} = \nu\) and so \(\lambda\) is a \(\gamma\)-fuzzy open set.

The following Theorem 6.2.3 gives decompositions of \(\gamma\)-fuzzy open sets in \(\gamma\)-fuzzy spaces.

**Theorem 6.2.3.** Let \(X\) be a non-empty set, \(\gamma \in \Gamma_4\) and \(\lambda \in \mathcal{F}\). Then the following statements are equivalent:

(a) \(\lambda\) is a \(\gamma\)-fuzzy open set.

(b) \(\lambda\) is a \(\gamma\)-fuzzy \(\alpha\)-open set and a \(\gamma\)-fuzzy locally closed set.

(c) \(\lambda\) is a \(\gamma\)-fuzzy pre-open set and a \(\gamma\)-fuzzy locally closed set.

**Proof:** It is enough to prove that (c) implies (a).

(c) \(\Rightarrow\) (a). Suppose \(\lambda\) is a \(\gamma\)-fuzzy pre-open set and \(\gamma\)-fuzzy locally closed set.
Since \( \lambda \) is a \( \gamma \)-fuzzy pre-open set, \( \lambda \leq i_\gamma c_\gamma(\lambda) \). Since \( \lambda \) is a \( \gamma \)-fuzzy locally closed set, \( \lambda = \nu \land c_\gamma(\lambda) \) for some \( \gamma \)-fuzzy open set \( \nu \). Now \( \lambda = \lambda \land i_\gamma c_\gamma(\lambda) = (\nu \land c_\gamma(\lambda)) \land i_\gamma c_\gamma(\lambda) = \nu \land i_\gamma c_\gamma(\lambda) = i_\gamma(\nu \land c_\gamma(\lambda)) \), by Theorem 6.1.1(a). Therefore, \( \lambda = i_\gamma(\lambda) \) which implies that \( \lambda \) is a \( \gamma \)-fuzzy open set.

### 6.3. \( \delta \)-fuzzy sets

Let \( X \) be a non-empty set, \( \gamma \in \Gamma \) and \( \lambda \in \mathcal{F} \). Then \( \lambda \) is said to be \( \delta \)-fuzzy set if \( i_\gamma c_\gamma(\lambda) \leq c_\gamma i_\gamma(\lambda) \). We denote the family of all \( \delta \)-fuzzy sets by \( \Delta \). A fuzzy set \( \lambda \) is said to be \( \gamma \)-fuzzy rare set if \( i_\gamma c_\gamma(\lambda) = \bar{0} \). The \( \gamma \)-boundary of a fuzzy set \( \lambda \) of \( \mathcal{F} \), denoted by \( bd(\lambda) \), is given by \( bd_\gamma(\lambda) = c_\gamma(\lambda) \land c_\gamma(\bar{1} - \lambda) \). Clearly, every \( \gamma \)-fuzzy rare set is a \( \delta \)-fuzzy set, since \( i_\gamma c_\gamma(\lambda) = \bar{0} \leq c_\gamma i_\gamma(\lambda) \) and every \( \gamma \)-fuzzy open set, as well as every \( \gamma \)-fuzzy closed set, is a \( \delta \)-fuzzy set.

The following Theorem 6.3.1 gives some properties of \( \gamma \)-fuzzy rare sets.

**Theorem 6.3.1** Let \( X \) be a non-empty set, \( \gamma \in \Gamma \) and \( \mu \) be the family of all \( \gamma \)-fuzzy open sets. Then the following hold.

(a) \( \bar{0} \) is a \( \gamma \)-fuzzy rare set.

(b) If \( \nu \leq \lambda \) and \( \lambda \) is a \( \gamma \)-fuzzy rare set, then \( \nu \) is a \( \gamma \)-fuzzy rare set.

(c) If \( \lambda \) is a \( \gamma \)-fuzzy rare set, then \( bd_\gamma(\lambda) \) is a \( \gamma \)-fuzzy rare set.

**Proof.** (a) If \( \Phi = \vee \{\lambda \mid \lambda \in \mu\} \), then \( c_\gamma i_\gamma(\bar{1}) = c_\gamma(\Phi) = \bar{1} \) and so \( \bar{1} - c_\gamma i_\gamma(\bar{1}) = \bar{0} \)
which implies that $i_\gamma c_\gamma(\bar{0}) = \bar{0}$.

(b) The proof is clear.

(c) Since $\lambda$ is $\gamma-$fuzzy rare set, $i_\gamma c_\gamma(\lambda) = \bar{0}$. Now $i_\gamma c_\gamma(bd(\lambda)) = i_\gamma c_\gamma(c_\gamma(\lambda) \land c_\gamma(\bar{1} - \lambda)) \leq i_\gamma c_\gamma c_\gamma(\lambda) = i_\gamma c_\gamma(\lambda) = \bar{0}$. Therefore, $bd(\lambda)$ is a $\gamma-$fuzzy rare set.

The following Theorem 6.3.2 shows that in a $\gamma-$fuzzy space, the union of two $\gamma-$fuzzy rare sets is again a $\gamma-$fuzzy rare set.

**Theorem 6.3.2.** Let $X$ be a non-empty set, $\gamma \in \Gamma_4$, and $\lambda$, $\nu$ be $\gamma-$fuzzy rare sets of $\mathcal{F}$. Then $\lambda \lor \nu$ is also a $\gamma-$fuzzy rare set.

**Proof.** Now $i_\gamma c_\gamma(\lambda \lor \nu) = i_\gamma(c_\gamma(\lambda) \lor c_\gamma(\nu))$, by Theorem 4.3.10(b) and so $i_\gamma c_\gamma(\lambda \lor \nu) \leq i_\gamma c_\gamma(\lambda) \lor c_\gamma(\nu)$ by Theorem 6.1.1(c) and so $i_\gamma c_\gamma(\lambda \lor \nu) \leq \bar{0} \lor c_\gamma(\nu) = c_\gamma(\nu)$. Therefore, $i_\gamma c_\gamma(\lambda \lor \nu) \leq i_\gamma c_\gamma(\nu) = \bar{0}$ and so $\lambda \lor \nu$ is a $\gamma-$fuzzy rare set.

The following Theorems 6.3.3 and 6.3.4 give some properties of $\delta-$fuzzy sets.

**Theorem 6.3.3.** Let $X$ be a non-empty set and $\gamma \in \Gamma$. Then the following hold.

(a) If $\lambda$ is a $\gamma-$fuzzy semi-open, then $\lambda \in \Delta$.
(b) If $\lambda \in \Delta$, then $\bar{1} - \lambda \in \Delta$.

**Proof:** (a) If $\lambda$ is a $\gamma-$fuzzy semi-open set, then $\lambda \leq c_\gamma i_\gamma(\lambda)$. Now, $i_\gamma c_\gamma(\lambda) \leq i_\gamma c_\gamma c_\gamma i_\gamma(\lambda) \leq i_\gamma c_\gamma i_\gamma(\lambda) \leq c_\gamma i_\gamma(\lambda)$ and so $\lambda \in \Delta$.

(b) $\lambda \in \Delta$ implies that $i_\gamma c_\gamma(\lambda) \leq c_\gamma i_\gamma(\lambda)$ and so $\bar{1} - c_\gamma i_\gamma(\lambda) \leq \bar{1} - i_\gamma c_\gamma(\lambda)$ which in turn implies that $i_\gamma(\bar{1} - i_\gamma(\lambda)) \leq c_\gamma(\bar{1} - c_\gamma(\lambda))$ and so $i_\gamma c_\gamma(\bar{1} - \lambda) \leq c_\gamma i_\gamma(\bar{1} - \lambda)$. 
Hence $I - \lambda \in \Delta$.

**Theorem 6.3.4.** Let $X$ be a non-empty set and $\gamma \in \Gamma_4$. If $\lambda \in \Delta$ and $\nu \in \Delta$, then $\lambda \land \nu \in \Delta$.

**Proof:** $\lambda, \nu \in \Delta$ implies that $i_\gamma c_\gamma(\lambda) \leq c_\gamma i_\gamma(\lambda)$ and $i_\gamma c_\gamma(\nu) \leq c_\gamma i_\gamma(\nu)$. Now $i_\gamma c_\gamma(\lambda \land \nu) \leq i_\gamma (c_\gamma(\lambda) \land c_\gamma(\nu)) = i_\gamma c_\gamma(\lambda) \land i_\gamma c_\gamma(\nu)$. Since $\lambda \in \Delta$, it follows that $i_\gamma c_\gamma(\lambda \land \nu) \leq c_\gamma i_\gamma(\lambda) \land i_\gamma c_\gamma(\nu) \leq c_\gamma(i_\gamma(\lambda) \land i_\gamma(\nu))$. Since $\nu \in \Delta$, it follows that $i_\gamma c_\gamma(\lambda \land \nu) \leq c_\gamma(i_\gamma(\lambda) \land i_\gamma(\nu)) = c_\gamma(i_\gamma(\lambda) \land i_\gamma(\nu)) = c_\gamma(i_\gamma(\lambda) \land \nu)$. Hence $i_\gamma c_\gamma(\lambda \land \nu) \leq c_\gamma(i_\gamma(\lambda \land \nu)$ and so $\lambda \land \nu \in \Delta$.

**Theorem 6.3.5.** Let $X$ be a non-empty set, $\gamma \in \Gamma_4$ and $\lambda$ and $\nu$ be fuzzy sets of $\mathcal{F}$ such that $\lambda \in \Delta$. Then $i_\gamma c_\gamma(\lambda \land \nu) = i_\gamma c_\gamma(\lambda) \land i_\gamma c_\gamma(\nu)$.

**Proof:** Since $i_\gamma c_\gamma(\lambda)$ and $i_\gamma c_\gamma(\nu)$ are $\gamma$-fuzzy open sets, $i_\gamma c_\gamma(\lambda) \land i_\gamma c_\gamma(\nu)$ is also $\gamma$-fuzzy open set and so $i_\gamma c_\gamma(\lambda) \land i_\gamma c_\gamma(\nu) = i_\gamma(i_\gamma c_\gamma(\lambda) \land i_\gamma c_\gamma(\nu)) \leq i_\gamma(c_\gamma i_\gamma(\lambda) \land i_\gamma c_\gamma(\nu))$, since $\lambda \in \Delta$. Therefore, $i_\gamma c_\gamma(\lambda) \land i_\gamma c_\gamma(\nu) \leq i_\gamma c_\gamma(i_\gamma(\lambda) \land i_\gamma c_\gamma(\nu)) \leq i_\gamma c_\gamma(i_\gamma(\lambda) \land c_\gamma(\nu)) \leq i_\gamma c_\gamma(i_\gamma(\lambda) \land c_\gamma(\nu)) \leq i_\gamma c_\gamma(i_\gamma(\lambda) \land \nu) \leq i_\gamma c_\gamma(\lambda \land \nu)$. Also $i_\gamma c_\gamma(\lambda \land \nu) \leq i_\gamma c_\gamma(\lambda) \land \nu \leq i_\gamma c_\gamma(\lambda) \land i_\gamma c_\gamma(\nu)$. Hence $i_\gamma c_\gamma(\lambda \land \nu) = i_\gamma c_\gamma(\lambda) \land i_\gamma c_\gamma(\nu)$.

### 6.4. $\gamma$-fuzzy generalized closed Sets

**Definition 6.4.1** Let $X$ be a non-empty set, $\mathcal{F}$ be the family of all fuzzy sets defined on $X$ and $\gamma \in \Gamma$. Let $\kappa \in \{\mu, \alpha, \sigma, \pi, b, \beta\}$. A fuzzy set $\lambda \in \mathcal{F}$ is called a
γ- fuzzy generalized κ- closed set (γ- fuzzy $g_\kappa-$ closed set) if $c_\kappa(\lambda) \leq \nu$ whenever $\lambda \leq \nu$ and $\nu$ is a κ- fuzzy open set. We denote the family of all γ- fuzzy $g_\kappa-$ closed sets by $G_\kappa$. Clearly, every κ- fuzzy closed set is a γ- fuzzy generalized κ- closed set.

**Theorem 6.4.2.** Let $X$ be a non-empty set, $\mathcal{F}$ be the family of all fuzzy sets defined on $X$ and $\gamma \in \Gamma_4$. If $\lambda$ and $\nu$ are γ- fuzzy $g_\mu-$ closed sets in a GFTS $(\mathcal{F}, \mu)$, then $\lambda \lor \nu$ is a γ- fuzzy $g_\mu-$ closed set.

**Proof.** Let $\lambda \lor \nu \leq \omega$ where $\omega \in \mu$. Since $\lambda \leq \omega$, $c_\gamma(\lambda) \leq \omega$ and $\nu \leq \omega$, $c_\gamma(\nu) \leq \omega$. Therefore, $c_\gamma(\lambda) \lor c_\gamma(\nu) \leq \omega$. Since by Theorem 4.3.10(b), $c_\gamma(\lambda \lor \nu) = c_\gamma(\lambda) \lor c_\gamma(\nu)$, $c_\gamma(\lambda \lor \nu) \leq \omega$ and so $\lambda \lor \nu$ is a γ- fuzzy $g_\mu-$ closed set.

**Theorem 6.4.3.** Let $X$ be a non-empty set, $\mathcal{F}$ be the family of all fuzzy sets defined on $X$ and $\gamma \in \Gamma$. Let $\kappa \in \{\mu, \alpha, \sigma, \pi, b, \beta\}$. If $\lambda$ is a γ- fuzzy $g_\kappa-$ closed set and $\lambda \leq \nu \leq c_\kappa(\lambda)$, then $\nu$ is a γ- fuzzy $g_\kappa-$ closed set. In general, the κ- closure of every γ- fuzzy $g_\kappa-$ closed set is a γ- fuzzy $g_\kappa-$ closed set.

**Proof.** Let $\omega$ be a κ- fuzzy open set such that $\omega \geq \nu$. Since $\nu \geq \lambda, \omega \geq \lambda$, and $\lambda$ is a γ- fuzzy $g_\kappa-$ closed set, $\omega \geq c_\kappa(\lambda)$. But $c_\kappa(\lambda) \geq c_\kappa(\nu)$, since $c_\kappa(\lambda) \geq \nu$ and so $\omega \geq c_\kappa(\nu)$. Hence, $\nu$ is a γ- fuzzy $g_\kappa-$ closed set.

**Theorem 6.4.4.** Let $X$ be a non-empty set, $\mathcal{F}$ be the family of all fuzzy sets defined on $X$ and $\gamma \in \Gamma$. Let $\kappa \in \{\mu, \alpha, \sigma, \pi, b, \beta\}$. Then $\kappa = \{1 - \lambda \mid \lambda \in \kappa\}$ if and only if every fuzzy set of $\mathcal{F}$ is a γ- fuzzy $g_\kappa-$ closed set.
Proof: Suppose that every fuzzy set of $F$ is a $\gamma$-fuzzy $g_{\kappa}$-closed set. Let $\lambda \in \kappa$. Since $\lambda \leq \lambda$ and $\lambda$ is a $\gamma$-fuzzy $g_{\kappa}$-closed set, we have $c_{\kappa}(\lambda) \leq \lambda$, but $\lambda \leq c_{\kappa}(\lambda)$ and therefore, $\lambda = c_{\kappa}(\lambda)$. Hence $\kappa \subseteq \{\overline{1} - \lambda \mid \lambda \in \kappa\}$. Also, if $\nu \in \{\overline{1} - \lambda \mid \lambda \in \kappa\}$, then $\overline{1} - \nu \in \kappa \subseteq \{\overline{1} - \lambda \mid \lambda \in \kappa\}$ and hence $\nu \in \mu$. Therefore, $\kappa = \{\overline{1} - \lambda \mid \lambda \in \kappa\}$.

The converse is easy.

Definition 6.4.5. Let $X$ be a non-empty set, $F$ be the family of all fuzzy sets defined on $X$ and $\gamma \in \Gamma$. Let $\kappa \in \{\mu, \alpha, \sigma, \pi, b, \beta\}$. A fuzzy set $\lambda$ is called a $\gamma$-fuzzy $g_{\kappa}$-open set if and only if $\overline{1} - \lambda$ is a $\gamma$-fuzzy $g_{\kappa}$-closed set.

We shall now prove some properties of $\gamma$-fuzzy $g_{\kappa}$-open sets.

Theorem 6.4.6. Let $X$ be a non-empty set, $F$ be the family of all fuzzy sets defined on $X$ and $\gamma \in \Gamma$. Let $\kappa \in \{\mu, \alpha, \sigma, \pi, b, \beta\}$. A fuzzy set $\lambda$ is a $\gamma$-fuzzy $g_{\kappa}$-open set if and only if $\nu \leq i_{\kappa}(\lambda)$ whenever $\nu$ is $\kappa$-fuzzy closed and $\nu \leq \lambda$.

Proof: Let $\lambda$ be a $\gamma$-fuzzy $g_{\kappa}$-open set and $\nu$ be a $\gamma$-fuzzy $\kappa$-closed set such that $\nu \leq \lambda$. Now $\nu \leq \lambda$ implies that $\overline{1} - \nu \geq \overline{1} - \lambda$ and $\overline{1} - \lambda$ is a $\gamma$-fuzzy $g_{\kappa}$-closed set. Therefore, $c_{\kappa}(\overline{1} - \lambda) \leq \overline{1} - \nu$ and so $\overline{1} - c_{\kappa}(\overline{1} - \lambda) \geq \overline{1} - (\overline{1} - \nu) = \nu$. Hence, $\nu \leq i_{\kappa}(\lambda)$.

Conversely, suppose that $\lambda$ is a fuzzy set such that $\nu \leq i_{\kappa}(\lambda)$ whenever $\nu$ is $\kappa$-fuzzy closed and $\nu \leq \lambda$. We claim that $\overline{1} - \lambda$ is a $\gamma$-fuzzy $g_{\kappa}$-closed set. Let $\overline{1} - \lambda \leq \nu$ where $\nu$ is a $\kappa$-fuzzy open set. Now $\overline{1} - \lambda \leq \nu$ implies that
I - ν ≤ λ. So by hypothesis, we must have I - ν ≤ iκ(λ) and so I - iκ(λ) ≤ ν. But
I - iγ(ν) = cκ(1 - λ) and so cκ(1 - λ) ≤ ν. This shows that I - λ is a γ-fuzzy
gκ-closed set.

**Theorem 6.4.7.** Let X be a non-empty set, F be the family of all fuzzy sets
defined on X and γ ∈ Γ. Let κ ∈ {μ, α, σ, π, b, β}. If iκ(λ) ≤ ν ≤ λ and if λ is
a γ-fuzzy gκ-open set, then ν is a γ-fuzzy gκ-open set.

**Proof.** Given that iγ(λ) ≤ ν ≤ λ. So, we have I - λ ≤ I - ν ≤ I - iκ(λ) = cκ(I - λ).
As, λ is a γ-fuzzy gκ-open set, I - λ is γ-fuzzy gκ-closed set and so by
Theorem 6.4.4, it follows that ν is a γ-fuzzy gκ-open set.

The intersection of two γ-fuzzy gκ-closed sets need not be a γ-fuzzy
gκ-closed set.

The following Theorem 6.4.9 shows that if one of the set is γ-fuzzy κ-closed,
then the intersection is a γ-fuzzy gκ-closed set.

**Theorem 6.4.8.** Let X be a non-empty set, F be the family of all fuzzy sets
defined on X and γ ∈ Γ. If λ and ν are γ-fuzzy gμ-closed sets in a GFTS
(F, μ), then λ ∨ ν is a γ-fuzzy gμ-closed set. If λ is a γ-fuzzy gμ-closed
and ω is a γ-fuzzy μ-closed, then λ ∧ ω is a γ-fuzzy gκ-closed set.
Proof: Suppose $\lambda \land \omega \leq \nu$ where $\nu$ is $\gamma$–fuzzy $\kappa$–open. Then $\lambda \leq (\nu \lor (\bar{1} - \omega))$. Since $\lambda$ is $\gamma$–fuzzy $g_\kappa$–closed, $c_\kappa(\lambda) \leq (\nu \lor (\bar{1} - \omega))$ and so $c_\kappa(\lambda) \land \omega \leq \nu$ and so $c_\kappa(\lambda \land \omega) \leq \nu$ which implies that $\lambda \land \omega$ is a $\gamma$–fuzzy $g_\kappa$–closed set.

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