CHAPTER 4
4. OPERATIONS IN GENERALIZED FUZZY TOPOLOGICAL SPACES

4.1. Introduction and Preliminaries

Let $X$ be a non-empty set and $\mathcal{F} = \{\lambda \mid \lambda : X \to [0, 1]\}$ be the family of all fuzzy sets defined on $X$. Let $\gamma : \mathcal{F} \to \mathcal{F}$ be a function such that $\lambda \leq \nu$ implies that $\gamma(\lambda) \leq \gamma(\nu)$ for every $\lambda, \nu \in \mathcal{F}$. That is, each $\gamma$ is a monotonic function defined on $\mathcal{F}$. We denote the collection of all monotonic functions defined on $\mathcal{F}$ by $\Gamma(\mathcal{F})$ or simply $\Gamma$. Let us agree in calling operation, any element of $\Gamma$. In section 4.2, we define enlarging and quasi-enlarging operators and discuss their properties. In section 4.3, we define friendly operators and discuss their properties. We also prove that the family of $\gamma-$fuzzy open sets of a friendly operator is closed under finite intersection and we call such spaces as quasi-fuzzy topological spaces. All the results of this section are from [6].

4.2. Enlarging and quasi-Enlarging operations

**Definition 4.2.1.** Let $X$ be a non-empty set and $\gamma \in \Gamma$. An operation $\gamma \in \Gamma$ is said to be enlarging if $\lambda \leq \gamma(\lambda)$ for every $\lambda \in \mathcal{F}$. If $B \subset \mathcal{F}$, then $\gamma \in \Gamma$ is said to be $B-$enlarging if $\lambda \leq \gamma(\lambda)$ for every $\lambda \in B$. We will denote the family of all enlarging operations by $\Gamma_e$ and the family of all $B-$enlarging operations by $\Gamma_B$. 
The easy proof of the following Theorem 4.2.2 is omitted.

**Theorem 4.2.2.** Let $X$ be a non-empty set and $\mathcal{F}$ be the family of all fuzzy sets defined on $X$. If $C \subseteq B \subseteq \mathcal{F}$, then $\Gamma_B \subseteq \Gamma_C$. $\Gamma_C = \Gamma_B$, if $B = \mathcal{F}$.

**Definition 4.2.3.** An operation $\gamma \in \Gamma$, is said to be *quasi-enlarging* (QE) if $\gamma(\lambda) \leq \gamma(\lambda \wedge \gamma(\lambda))$ for every $\lambda \in \mathcal{F}$. An operation $\gamma \in \Gamma$, is said to be *weakly quasi-enlarging* (WQE) if $\lambda \wedge \gamma(\lambda) \leq \gamma(\lambda \wedge \gamma(\lambda))$ for every $\lambda \in \mathcal{F}$. If $\gamma \in \Gamma_e$, then $\lambda \wedge \gamma(\lambda) = \lambda$ for every $\lambda \in \mathcal{F}$ and so $\gamma$ is quasi-enlarging. If $\gamma$ is defined by $\gamma(\lambda) = \beta$ for every $\lambda \in \mathcal{F}$, then also $\gamma$ is quasi-enlarging. If $\gamma \in \Gamma$ is quasi-enlarging, then it is weakly quasi-enlarging, since $\lambda \wedge \gamma(\lambda) \leq \gamma(\lambda) \leq \gamma(\lambda \wedge \gamma(\lambda))$.

The following Example 4.2.4 shows that a weakly quasi-enlarging operation need not be a quasi-enlarging operation.

**Example 4.2.4.** Let $X = \{x, y, z\}$. Define $\gamma : \mathcal{F} \to \mathcal{F}$, by $\gamma(\lambda) = \overline{0}$, if $\lambda = \overline{0}$; $\gamma(\lambda) = \chi_{\{y\}}$, if $\lambda \leq \chi_{\{x\}}$; $\gamma(\lambda) = \chi_{\{z\}}$, if $\lambda \leq \chi_{\{z\}}$ and $\gamma(\lambda) = \overline{1}$ if otherwise.

Then, $\lambda \wedge \gamma(\lambda) = \overline{0}$, if $\lambda = \overline{0}$; $\lambda \wedge \gamma(\lambda) = \overline{0}$, if $\lambda \leq \chi_{\{z\}}$; $\lambda \wedge \gamma(\lambda) \leq \chi_{\{z\}}$, if $\lambda \leq \chi_{\{z\}}$ and $\lambda \wedge \gamma(\lambda) = \lambda$ if otherwise. Therefore, $\gamma(\lambda \wedge \gamma(\lambda)) = \overline{0}$, if $\lambda = \overline{0}$; $\gamma(\lambda \wedge \gamma(\lambda)) = \overline{0}$, if $\lambda \leq \chi_{\{z\}}$; $\gamma(\lambda \wedge \gamma(\lambda)) = \chi_{\{z\}}$, if $\lambda \leq \chi_{\{z\}}$ and $\lambda \wedge \gamma(\lambda) = \overline{1}$, if otherwise and so it follows that $\gamma$ is a weakly quasi-enlarging operator. If $\lambda = \chi_{\{x\}}$, then $\gamma(\lambda) = \chi_{\{y\}}$ but $\gamma(\lambda \wedge \gamma(\lambda)) = \gamma(\overline{0}) = \overline{0}$ and so $\gamma$ is not a quasi-enlarging operator.

**Remark 4.2.5.** If $\gamma_1, \gamma_2 \in \Gamma$, then the composition $\gamma_1 \circ \gamma_2$ of the two operations $\gamma_1$ and $\gamma_2$ is again an operation and we write $\gamma_1 \gamma_2$ instead of $\gamma_1 \circ \gamma_2$. 
The following Theorem 4.2.6 shows that the composition of enlarging operators is again an enlarging operator and Theorem 4.2.8 below gives a property of quasi-enlarging operators.

**Theorem 4.2.6.** Let $X$ be a non-empty set and $\mathcal{F}$ be the family of all fuzzy sets defined on $X$. If $B \subseteq \mathcal{F}$, and $\gamma_1$ and $\gamma_2$ are $B$-enlarging, then $\gamma_1 \gamma_2$ is also $B$-enlarging.

**Proof.** Suppose $\lambda \in B$. Then $\lambda \leq \gamma_1(\lambda)$ and $\lambda \leq \gamma_2(\lambda)$. Now, $\lambda \leq \gamma_1(\gamma_2(\lambda))$, since $\gamma_1 \in \Gamma$. Therefore, $\gamma_1 \gamma_2$ is $B$-enlarging.

**Corollary 4.2.7.** If $\gamma_1, \gamma_2 \in \Gamma_e$, then $\gamma_1 \gamma_2 \in \Gamma_e$.

**Theorem 4.2.8.** Let $X$ be a non-empty set, $\mathcal{F}$ be the family of all fuzzy sets defined on $X$ and $B \subseteq \mathcal{F}$. If $\gamma \in \Gamma$ is quasi-enlarging, $\{\gamma(\lambda) \mid \lambda \in \mathcal{F}\} \subseteq B$ and $\delta \in \Gamma_B$, then $\delta \gamma$ is quasi-enlarging.

**Proof.** Let $\lambda \in \mathcal{F}$. Since $\gamma$ is quasi-enlarging, $\gamma(\lambda) \leq \gamma(\lambda \wedge \gamma(\lambda))$. Since $\gamma(\lambda) \in B$ and $\delta \in \Gamma_B$, $\gamma(\lambda) \leq \delta(\gamma(\lambda))$ and so $\gamma(\lambda) \leq \gamma(\lambda \wedge \delta \gamma(\lambda))$. Therefore, $\delta \gamma(\lambda) \leq \delta \gamma(\lambda \wedge \delta \gamma(\lambda))$. Hence $\delta \gamma$ is quasi-enlarging.

**Theorem 4.2.9.** Let $X$ be a non-empty set and $\gamma \in \Gamma$. Then $i_{\gamma}$ is quasi-enlarging and $c_{\gamma}$ is enlarging.

**Proof.** If $\lambda \in \mathcal{F}$, then $i_{\gamma}(\lambda) = i_{\gamma} i_{\gamma}(\lambda) = i_{\gamma}(\lambda \wedge i_{\gamma}(\lambda))$, since $i_{\gamma}(\lambda) \leq \lambda$. So $i_{\gamma}$ is quasi-enlarging. Again, $i_{\gamma}(\overline{1} - \lambda) \leq \overline{1} - \lambda$ and so $\lambda = \overline{1} - (\overline{1} - \lambda) \leq \overline{1} - i_{\gamma}(\overline{1} - \lambda) = c_{\gamma}(\lambda)$. Therefore, $c_{\gamma}$ is enlarging.
**Theorem 4.2.10.** Let $X$ be a non-empty set, $\gamma \in \Gamma$ and $\mu$ be the family of all $\gamma-$fuzzy open sets. If $\delta \in \Gamma$, such that $i_{\gamma} \delta$ is quasi-enlarging and $\kappa \in \Gamma_{\mu}$, then $\kappa i_{\gamma} \delta$ is quasi-enlarging.

**Proof.** If $\lambda \in \mathcal{F}$, then $i_{\gamma} \delta(\lambda) \in \mu$. By Theorem 4.2.8, it follows that $\kappa i_{\gamma} \delta$ is quasi-enlarging.

**Corollary 4.2.11.** Let $X$ be a non-empty set, $\gamma \in \Gamma$ and $\mu$ be the family of all $\gamma-$fuzzy open sets. If $\kappa \in \Gamma_{\mu}$, then $\kappa i_{\gamma}$ is quasi-enlarging.

**Proof.** If $\delta : \mathcal{F} \to \mathcal{F}$ is the identity operator, then $i_{\gamma} \delta = i_{\gamma}$ is quasi-enlarging and so the proof follows from Theorem 4.2.10.

Let $\{\gamma_i \in \Gamma \mid i \in \Delta \neq \emptyset\}$ be a family of operations. Define $\varphi : \mathcal{F} \to \mathcal{F}$ by $\varphi(\lambda) = \vee \{\gamma_i(\lambda) \mid i \in \Delta\}$ for every $\lambda \in \mathcal{F}$.

The following Theorem 4.2.12 gives some properties of $\varphi$.

**Theorem 4.2.12.** Let $X$ be a non-empty set. Let $\{\gamma_i \in \Gamma \mid i \in \Delta \neq \emptyset\}$ be a family of operations. Define $\varphi : \mathcal{F} \to \mathcal{F}$ by $\varphi(\lambda) = \vee \{\gamma_i(\lambda) \mid i \in \Delta\}$ for every $\lambda \in \mathcal{F}$. Then the following hold.

(a) $\varphi \in \Gamma$.

(b) If each $\gamma_i$ is $B-$enlarging, then so is $\varphi$.

(c) If each $\gamma_i$ is quasi-enlarging, then so is $\varphi$.

(d) If each $\gamma_i$ is weakly quasi-enlarging, then so is $\varphi$.

**Proof.** (a) If $\lambda \leq \nu$, then $\gamma_i(\lambda) \leq \gamma_i(\nu)$ and so $\varphi(\lambda) = \vee \{\gamma_i(\lambda) \mid i \in \Delta\} \leq \vee \{\gamma_i(\nu) \mid i \in \Delta\} = \varphi(\nu)$. Therefore, $\varphi \in \Gamma$.

(b) If each $\gamma_i$ is $B-$enlarging, then $\varphi(\lambda) = \vee \{\gamma_i(\lambda) \mid i \in \Delta\} \leq \vee \{\gamma_i(\nu) \mid i \in \Delta\} = \varphi(\nu)$. Therefore, $\varphi$ is $B-$enlarging.

(c) If each $\gamma_i$ is quasi-enlarging, then $\varphi(\lambda) = \vee \{\gamma_i(\lambda) \mid i \in \Delta\} \leq \vee \{\gamma_i(\nu) \mid i \in \Delta\} = \varphi(\nu)$. Therefore, $\varphi$ is quasi-enlarging.

(d) If each $\gamma_i$ is weakly quasi-enlarging, then $\varphi(\lambda) = \vee \{\gamma_i(\lambda) \mid i \in \Delta\} \leq \vee \{\gamma_i(\nu) \mid i \in \Delta\} = \varphi(\nu)$. Therefore, $\varphi$ is weakly quasi-enlarging.
(b) Let \( \lambda \in B \). Then, by hypothesis, \( \lambda \leq \gamma_i(\lambda) \) for every \( i \in \Delta \neq \emptyset \). Therefore, 
\[ \lambda \leq \bigvee \gamma_i(\lambda) = \varphi(\lambda) \] and so \( \varphi \) is \( B \)-enlarging.

(c) Suppose each \( \gamma_i \) is quasi-enlarging. Then for \( \lambda \in \mathcal{F} \), 
\[ \varphi(\lambda) = \bigvee \gamma_i(\lambda) \leq \bigvee \gamma_i(\lambda \wedge \gamma_i(\lambda)) = \varphi(\lambda \wedge \varphi(\lambda)) \] and so \( \varphi \) is quasi-enlarging.

(d) For \( \lambda \in \mathcal{F} \), 
\[ \lambda \wedge \varphi(\lambda) = \lambda \wedge (\bigvee \gamma_i(\lambda)) = \bigvee (\lambda \wedge \gamma_i(\lambda)) \leq \bigvee \gamma_i(\lambda \wedge \gamma_i(\lambda)) \leq \bigvee \gamma_i(\lambda \wedge \varphi(\lambda)) = \varphi(\lambda \wedge \varphi(\lambda)). \] Therefore, \( \varphi \) is weakly quasi-enlarging.

### 4.3 Friendly functions

**Definition 4.3.1.** Let \( X \) be a non-empty set and \( A \subset \mathcal{F} \). We say that an operation \( \gamma \in \Gamma \) is \( A \)-friendly, if \( \nu \wedge \gamma(\lambda) \leq \gamma(\nu \wedge \lambda) \) for every \( \lambda \in \mathcal{F} \) and \( \nu \in A \).

The following Example 4.3.2 gives examples of \( A \)-friendly operators. It is clear that if \( \gamma \) is \( A \)-friendly and \( B \subset A \), then \( \gamma \) is a \( B \)-friendly operator. Theorem 4.3.3 below shows that the composition of friendly operators is again a friendly operator. Theorem 4.3.4 shows that arbitrary union of friendly operators is again a friendly operator.

**Example 4.3.2.** (a) If \( \gamma : \mathcal{F} \to \mathcal{F} \) is defined by \( \gamma(\lambda) = \theta \) for every \( \lambda \in \mathcal{F} \) for some \( \theta \in \mathcal{F} \), then \( \gamma \) is \( A \)-friendly for every \( A \subset \mathcal{F} \).

(b) In any fuzzy topological space \( (X, \tau) \), the fuzzy interior and closure operators
\( i_\tau \) and \( c_\tau \) are \( \tau \)-friendly. That is, the following hold.

(i) \( i_\tau (\lambda) \land \nu \leq i_\tau (\lambda \land \nu) \) for every \( \lambda \in \mathcal{F} \) and \( \nu \in \tau \).

(ii) \( c_\tau (\lambda) \land \nu \leq c_\tau (\lambda \land \nu) \) for every \( \lambda \in \mathcal{F} \) and \( \nu \in \tau \).

**Theorem 4.3.3.** Let \( X \) be a non-empty set, \( \gamma, \gamma_1 \in \Gamma \) and \( \mathcal{A} \subset \mathcal{F} \). If \( \gamma \) and \( \gamma_1 \) are \( \mathcal{A} \)-friendly operators, then so is \( \gamma_1 \gamma \).

**Proof.** Suppose \( \mathcal{A} \subset \mathcal{F} \) such that \( \gamma \) and \( \gamma_1 \) are \( \mathcal{A} \)-friendly. Then, \( \gamma (\lambda) \land \nu \leq \gamma (\lambda \land \nu) \) for every \( \lambda \in \mathcal{F} \) and \( \nu \in \mathcal{A} \), and \( \gamma_1 (\lambda) \land \nu \leq \gamma_1 (\lambda \land \nu) \) for every \( \lambda \in \mathcal{F} \) and \( \nu \in \mathcal{A} \). Replacing \( \lambda \) by \( \gamma (\lambda) \) in the second inequality, we get \( \gamma_1 \gamma (\lambda) \land \nu \leq \gamma_1 (\gamma (\lambda) \land \nu) \leq \gamma_1 \gamma (\lambda \land \nu) \). Therefore, \( \gamma_1 \gamma \) is an \( \mathcal{A} \)-friendly operator.

**Theorem 4.3.4.** Let \( X \) be a non-empty set, \( \mathcal{A} \subset \mathcal{F} \) and \( \gamma_i \) is \( \mathcal{A} \)-friendly for every \( i \in \Delta \). Then \( \varphi = \bigvee \gamma_i \) is \( \mathcal{A} \)-friendly.

**Proof.** If \( \lambda \in \mathcal{F} \), then for \( \nu \in \mathcal{A} \), \( \varphi (\lambda) \land \nu = (\bigvee \gamma_i) (\lambda) \land \nu = (\bigvee (\gamma_i (\lambda))) \land \nu = \bigvee (\gamma_i (\lambda) \land \nu) \leq \bigvee \gamma_i (\lambda \land \nu) = \varphi (\lambda \land \nu) \). Therefore, \( \varphi \) is an \( \mathcal{A} \)-friendly operator.

Using friendly operators, next we construct quasi-enlarging operators using a generalized fuzzy topology (GFT). Let \( \mu \subset \mathcal{F} \) be arbitrary. For \( \lambda \in \mathcal{F} \), define \( i_\mu (\lambda) = \bigvee \{ \beta \in \mu \mid \beta \leq \lambda \} \) and \( i_\mu (\lambda) = 0 \), if no such \( \beta \in \mu \) exists. Let \( \mu' = \{ \bar{\lambda} - \lambda \mid \lambda \in \mu \} \). Define \( c_\mu (\lambda) = \land \{ \beta \in \mu \mid \lambda \leq \beta \} \) and \( c_\mu (\lambda) = \bar{1} \), if no such \( \beta \in \mu' \) exists. If \( \mu \) is the family of all \( \gamma \)-fuzzy open sets, then \( c_\gamma = c_\mu \) and \( i_\gamma = i_\mu \).

**Theorem 4.3.5** Let \( \mu \subset \mathcal{F} \) be a GFT. If \( \gamma \in \Gamma \), is \( \mu \)-friendly, then \( i_\mu \gamma \) is
quasi-enlarging.

**Proof.** If $\xi \in \mathcal{F}$, then $i_\mu \gamma(\xi) = \gamma(\xi) \land i_\mu \gamma(\xi)$ by Lemma 2.9. Since $\gamma$ is $\mu-$friendly, $\gamma(\xi) \land i_\mu \gamma(\xi) \leq \gamma(\xi \land i_\mu \gamma(\xi))$. Therefore, $i_\mu \gamma(\xi) = i_\mu i_\mu \gamma(\xi) \leq i_\mu \gamma(\xi \land i_\mu \gamma(\xi))$ and so $i_\mu \gamma$ is quasi-enlarging.

**Theorem 4.3.6.** Let $\mu \subset \mathcal{F}$ and $\gamma \in \Gamma$ be $\mu-$friendly. If $\nu \in \mu$ and $\xi$ is a $\gamma-$fuzzy open set, then $\xi \land \nu$ is again a $\gamma-$fuzzy open set.

**Proof.** Since $\xi$ is a $\gamma-$fuzzy open set, $\xi \leq \gamma(\xi)$. Then for $\nu \in \mu$, $\nu \land \xi \leq \nu \land \gamma(\xi) \leq \gamma(\nu \land \xi)$ and so $\nu \land \xi$ is a $\gamma-$fuzzy open set.

**Corollary 4.3.7.** Let $\gamma \in \Gamma$, $\mu$ be the family of all $\gamma-$fuzzy open sets and $\gamma$ be $\mu-$friendly. Then $\lambda \land \nu \in \mu$ whenever $\lambda \in \mu$ and $\nu \in \mu$.

**Definition 4.3.8.** A new subfamily of $\Gamma$, namely $\Gamma_4$ is defined as $\Gamma_4 = \{\gamma \in \Gamma \mid \gamma$ is $\mu_\gamma-$friendly$\}$ where $\mu_\gamma$ is the family of all $\gamma-$fuzzy open sets. Hence, if $\gamma \in \Gamma_4$, then the GFTS $(X, \gamma)$ is closed under finite intersection, by Corollary 4.3.7. If $\gamma \in \Gamma_4$, we call such spaces as *Quasi-fuzzy topological spaces* or a $\gamma-$fuzzy space. Clearly, if $\gamma \in \Gamma_1$, then $\mu_\gamma$ is a fuzzy topological space.

The following Example 4.3.9 shows that $\gamma \in \Gamma_4$ does not imply that $\gamma \in \Gamma_1$.

**Example 4.3.9.** Let $X = \mathbb{R}$, the set of all real numbers and $\mathcal{F}$ be the family of all fuzzy sets defined on $X$. Define $\gamma : \mathcal{F} \to \mathcal{F}$ by $\gamma(\lambda) = \bar{\alpha}$ if $\bar{\alpha} \leq \lambda$, and $\gamma(\lambda) = \bar{0}$ if otherwise, where $0 < \alpha < 1$. Clearly, $\gamma \not\in \Gamma_1$. Since $\{\bar{0}, \bar{\alpha}\}$ is the family of all $\gamma-$fuzzy open sets, it follows that $\gamma \in \Gamma_4$. 
Theorem 4.3.10. If $X$ is a non-empty set, $\mathcal{F}$ is the family of all fuzzy sets defined on $X$ and $\gamma \in \Gamma_4$, then the following hold.

(a) $i_\gamma(\lambda \land \nu) = i_\gamma(\lambda) \land i_\gamma(\nu)$ for every fuzzy sets $\lambda, \nu \in \mathcal{F}$.

(b) $c_\gamma(\lambda \lor \nu) = c_\gamma(\lambda) \lor c_\gamma(\nu)$ for every fuzzy sets $\lambda, \nu \in \mathcal{F}$.

Proof. (a) Since $i_\gamma(\lambda) \leq \lambda$ and $i_\gamma(\nu) \leq \nu$, by Corollary 4.3.7, $i_\gamma(\lambda) \land i_\gamma(\nu)$ is a $\gamma-$fuzzy open set contained in $\lambda \land \nu$ and so $i_\gamma(\lambda) \land i_\gamma(\nu) \leq i_\gamma(\lambda \land \nu)$. Clearly, $i_\gamma(\lambda \land \nu) \leq i_\gamma(\lambda) \land i_\gamma(\nu)$.

(b) Since $\lambda \lor \nu \leq c_\gamma(\lambda) \lor c_\gamma(\nu) \leq c_\gamma(\lambda \lor \nu)$, it follows that $c_\gamma(\lambda \lor \nu) = c_\gamma(\lambda) \lor c_\gamma(\nu)$ for every fuzzy sets $\lambda, \nu \in \mathcal{F}$.

Lemma 4.3.11. Let $\lambda \in \mathcal{F}$, $\gamma \in \Gamma$ and $\mu$ be the family of all $\gamma-$fuzzy open sets. Then a fuzzy point $x_t \in c_\gamma(\lambda)$ if and only if for every $\gamma-$fuzzy open set $\nu$ of $x_t$, $\nu \bar{q} \lambda$.

Proof. Suppose $x_t \in c_\gamma(\lambda)$. Let $\nu$ be a $\gamma-$fuzzy open set of $x_t$. If $\nu \bar{q} \lambda$, then $\lambda \leq (\bar{1} - \nu)$. Since $(\bar{1} - \nu)$ is a $\gamma-$fuzzy closed set, $c_\gamma(\lambda) \leq (\bar{1} - \nu)$. Since $x_t \not\in (\bar{1} - \nu)$, $x_t \not\in c_\gamma(\lambda)$, a contradiction. Therefore, $\nu \bar{q} \lambda$.

Conversely, suppose $x_t \not\in c_\gamma(\lambda)$. Since $c_\gamma(\lambda) = \land\{\xi \mid \lambda \leq \xi \text{ and } \xi \text{ is a } \gamma-\text{fuzzy closed set}\}$, there is a $\gamma-$fuzzy closed set $\xi \geq \lambda$ such that $x_t \not\in \xi$. Then $\bar{1} - \xi$ is a $\gamma-$fuzzy open sets such that $x_t \in (\bar{1} - \xi)$. By hypothesis, $(\bar{1} - \xi) \bar{q} \lambda$. Since $\xi \geq \lambda$, $(\bar{1} - \xi) \bar{q} \lambda$, a contradiction to the hypothesis. Hence $x_t \in c_\gamma(\lambda)$.

Theorem 4.3.12. Let $\mathcal{A} \subset \mathcal{F}$, $\gamma \in \Gamma$ be $\mathcal{A}-$ friendly and $\mu$ be the family of all $\gamma-$fuzzy open sets. Then $c_\gamma$ is $\mathcal{A}-$ friendly.
Proof. Let $\nu \in A, \xi \in \mathcal{F}$ and $x_t \in \nu \land c_\mu(\xi)$. If $x_t \in \omega \in \mu$, then by Theorem 3.2.5, $\nu \land \omega$ is a $\gamma-$fuzzy open set containing $x_t$. By Lemma 4.3.11, $(\omega \land \nu)q\xi$. Then clearly, $\omega q(\nu \land \xi)$ and so $x_t \in c_\mu(\nu \land \xi)$. Hence $\nu \land c_\mu(\xi) \leq c_\mu(\nu \land \xi)$ which implies that $c_\gamma$ is $\mathcal{A}-$ friendly.

Corollary 4.3.13. If $X$ is a non-empty set, $\mathcal{F}$ is the family of all fuzzy sets on $X$, $\gamma \in \Gamma_4$ and $\mu$ is the family of all $\gamma-$fuzzy open sets, then the following hold.

(a) $c_\gamma(\nu) \land \xi \leq c_\gamma(\nu \land \xi)$ for every fuzzy sets $\nu, \xi \in \mu$.

(b) $c_\gamma(c_\gamma(\nu) \land \xi) = c_\gamma(\nu \land \xi)$ for every fuzzy sets $\nu, \xi \in \mu$.

(c) $i_\gamma(\nu \lor \xi) \leq i_\gamma(\nu) \lor \xi$ for every fuzzy set $\nu$ and $\mu-$fuzzy closed set $\xi$.

(d) $i_\gamma(\nu \lor \xi) = i_\gamma(i_\gamma(\nu) \lor \xi)$ for every fuzzy set $\nu$ and $\mu-$fuzzy closed set $\xi$.

Proof. (a) The proof follows from Theorem 4.3.12.

(b). Since $\nu \land \xi \leq c_\gamma(\nu) \land \xi$, the proof follows from (a).

(c) If $\xi$ is a $\mu-$fuzzy closed set, then $\omega = \bar{\nu} - \xi \in \mu$ and so by (a), for $\nu \in \mathcal{F}$, $c_\gamma(\nu) \land \omega \leq c_\gamma(\nu \land \omega)$ and so $\bar{\nu} - c_\gamma(\nu \land \omega) \leq \bar{\nu} - (c_\gamma(\nu) \land \omega)$. Therefore, $i_\gamma((\bar{\nu} - \nu) \lor (\bar{\nu} - \omega)) \leq (\bar{\nu} - c_\gamma(\nu)) \lor (\bar{\nu} - \omega)$ and so $i_\gamma((\bar{\nu} - \nu) \lor \xi) \leq i_\gamma(\bar{\nu} - \nu) \lor \xi$.

If $\psi = \bar{\nu} - \nu$, we have $i_\gamma(\psi \lor \xi) \leq i_\gamma(\psi) \lor \xi$.

(d) The proof follows from (c).

Corollary 4.3.14. Let $A \subset \mathcal{F}$ be a GFT, $\gamma \in \Gamma$ be $\mathcal{A}-$ friendly and $\mu$ be the family of all $\gamma-$fuzzy open sets. Then $i_\mu c_\mu$ is quasi-enlarging.

Proof. The proof follows from Theorem 4.3.5 and Theorem 4.3.6.

In the rest of the section, we consider a special type of enlargement whose
domain is a subfamily of \( \mathcal{F} \). A function \( \kappa : \mu \to \mathcal{F} \) is an enlargement if \( \lambda \leq \kappa(\lambda) \) for every \( \lambda \in \mu \).

The following are some examples of enlargements.

**Example 4.3.15.** Let \( X \) be a non-empty set, \( \mathcal{F} \) be the family of all fuzzy sets defined on \( X \) and \( \mu \subset \mathcal{F} \). Define \( \kappa : \mu \to \mathcal{F} \) by

(a) \( \kappa(\lambda) = \lambda \) for every \( \lambda \in \mu \).

(b) \( \kappa(\lambda) = c_\mu(\lambda) \) for every \( \lambda \in \mu \).

(c) \( \kappa(\lambda) = i_\mu c_\mu(\lambda) \) for every \( \lambda \in \mu \).

Then \( \kappa \) is an enlargement in each case.

Let \( \kappa : \mu \to \mathcal{F} \) is an enlargement. Define \( \kappa_\mu = \{ \lambda \in \mathcal{F} \mid \text{For each } x_t \in \lambda, \text{there exists } \nu \in \mu \text{ such that } x_t \in \nu \leq \kappa(\nu) \leq \lambda \} \).

The following Theorem 4.3.16 gives some properties of \( \kappa_\mu \).

**Theorem 4.3.16.** Let \( X \) be a non-empty set, \( \mathcal{F} \) be the family of all fuzzy sets defined on \( X \), \( \mu \subset \mathcal{F} \) and \( \kappa : \mu \to \mathcal{F} \) be an enlargement. Then the following hold.

(a) \( \kappa_\mu \) is a GFT.

(b) If \( \mu \) is a GFT, then \( \kappa_\mu \subset \mu \).

**Proof.** (a) Clearly, \( 0 \in \kappa_\mu \). Let \( \nu_\alpha \in \kappa_\mu \) for every \( \alpha \in \Delta \) and \( \nu = \lor \{ \nu_\alpha \mid \alpha \in \Delta \} \).

If \( x_t \in \nu \), where \( t \in (0, 1] \), then \( x_t \in \nu_\alpha \) for some \( \alpha \in \Delta \). By hypothesis, there is a \( \xi \in \mu \) such that \( x_t \in \xi \leq \kappa(\xi) \leq \nu_\alpha \leq \nu \). Hence \( \nu \in \kappa_\mu \) which implies that \( \kappa_\mu \) is a GFT.
(b) Let $\nu \in \kappa_\mu$. Then for each $x_t \in \nu$ where $t \in (0,1]$, there exists $\xi_x \in \mu$ such that $\kappa(\xi_x) \leq \nu$ and so $x_t \in \xi_x \leq \kappa(\xi_x) \leq \nu$. Hence $\nu = \vee \{\xi_x \mid x \in \nu\}$. Since $\mu$ is a GFT, $\nu \in \mu$ and so $\kappa_\mu \subseteq \mu$.

***