Contra-$\pi g\alpha$-Continuous
&
Almost Contra-$\pi g\alpha$-Continuous Functions

- Introduction
- Contra-$\pi g\alpha$-Continuous Functions
- Almost Contra-$\pi g\alpha$-Continuous Functions
- Functions With $\pi g\alpha$-Closed Graphs
CHAPTER IV

CONTRA-\(\pi\alpha\)-CONTINUOUS
AND

ALMOST CONTRA-\(\pi\alpha\)-CONTINUOUS FUNCTIONS

4.1 Introduction

Dontchev [36] introduced the notion of contra-continuity and obtained some results concerning compactness, S-closedness and strong-S-closedness. Various topologists introduced new generalizations of contra-continuity in [22, 41, 68, 69] and generalizations of almost contra-continuity in [49, 140, 141]. Long[89], Dube et al.[45] and Jankovic[73] introduced the notions of closed graphs, semi-closed graphs and \(\alpha\)-closed graphs respectively. In this chapter, we introduce the concept of contra-\(\pi\alpha\)-continuous functions, almost contra-\(\pi\alpha\)-continuous functions, \(\pi\alpha\)-closed graphs, contra-\(\pi\alpha\)-closed graphs, \(\pi\alpha\)-regular graphs and study their properties.

4.2 Contra-\(\pi\alpha\)-Continuous Functions

In this section, we introduce and study the concept of contra-\(\pi\alpha\)-continuous functions which is weaker than contra-\(\alpha\)-continuous, contra-\(\pi\gamma\gamma\)-continuous but stronger than contra-\(\pi\gamma\gamma\pi\)-continuous functions.

Definition 4.2.1 : A function \(f : (X,\tau) \rightarrow (Y,\sigma)\) is called contra-\(\pi\alpha\)-continuous if \(f^{-1}(V)\) is \(\pi\alpha\)-closed in \((X,\tau)\) for each open set \(V\) of \((Y,\sigma)\).

Proposition 4.2.2: i) Every contra-continuous function is contra-\(\pi\alpha\)-continuous.

ii) Every contra-\(\alpha\)-continuous function is contra-\(\pi\alpha\)-continuous.

iii) Every contra-\(\pi\gamma\gamma\)-continuous function is contra-\(\pi\alpha\)-continuous.

iv) Every contra-\(\pi\alpha\gamma\gamma\)-continuous function is contra-\(\pi\gamma\gamma\)-continuous.
Remark 4.2.3: Converse of above statements is not true as the following example shows.

Example 4.2.4: a) Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$, $\sigma = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is contra-$\pi g \alpha$-continuous but not contra-continuous.

b) Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{\phi, X, \{b, c\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is contra-$\pi g \alpha$-continuous but not contra-$\alpha$-continuous.

c) Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{d\}, \{a, d\}, \{a, c, d\}\}$, $\sigma = \{\phi, X, \{c\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is contra-$\pi g \alpha$-continuous but not contra-$\pi g$-continuous.

d) Let $X = \{a, b, c, d, e\}$, $\tau = \{\phi, X, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$, $\sigma = \{\phi, X, \{a\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is contra-$\pi g \alpha$-continuous but not contra-$\pi g \alpha$-continuous since $\{a\}$ is $\pi g \alpha$-closed in $(X, \sigma)$ but not $\pi g \alpha$-closed in $(X, \tau)$.

We have the following diagram:

```
   perfectly continuous
      ↓
   contra-continuous
      ↓
contra-\(\alpha\)-continuous    contra-\(\pi g\)-continuous
      ↓
contra-\(\pi g \alpha\)-continuous
      ↓
contra-\(\pi g \alpha\)-continuous
      ↓
contra-\(\pi g p\)-continuous
```
Definition 4.2.5: A space $(X, \tau)$ is called
i) $\pi gamma$-locally indiscrete if every $\pi gamma$-open set is closed.
ii) a $T_{\pi gamma}$ space if every $\pi gamma$-closed set is $\pi gamma$-closed.

Theorem 4.2.6: i) If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\pi gamma$-continuous and $(X, \tau)$ is $\pi gamma$-locally indiscrete, then $f$ is contra-continuous.
ii) If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra-$\pi gamma$-continuous and $(X, \tau)$ is a $\pi gamma$-$T_{\pi gamma}$ space, then $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra-$\alpha$-continuous.
iii) If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra-$\pi gamma$-continuous, pre-$\alpha$-closed surjection and if $X$ is a $\pi gamma$-$T_{\pi gamma}$ space, then $Y$ is locally indiscrete.
iv) If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra-$\pi gamma$-continuous and $X$ is a $\pi gamma$-space, then $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra-continuous.
v) If $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra-$\pi gamma$-continuous and if $X$ is a $T_{\pi gamma}$ space, then $f$ is contra-$\pi gamma$-continuous.

Proof: i) Let $V$ be open in $(Y, \sigma)$. By assumption, $f^{-1}(V)$ is $\pi gamma$-open in $X$. Since $X$ is $\pi gamma$-locally indiscrete, $f^{-1}(V)$ is closed in $X$. Hence $f$ is contra-continuous.
ii) Let $V$ be open set in $Y$. By assumption, $f^{-1}(V)$ is $\pi gamma$-closed in $X$. Since $X$ is a $\pi gamma$-$T_{\pi gamma}$ space, $f^{-1}(V)$ is $\alpha$-closed in $X$. Hence $f$ is contra-$\alpha$-continuous.
iii) Let $V$ be open in $(Y, \sigma)$. By assumption, $f^{-1}(V)$ is $\pi gamma$-closed in $(X, \tau)$ and hence $\alpha$-closed in $X$. Since $f$ is a pre-$\alpha$-closed surjection, $f(f^{-1}(V)) = V$ is $\alpha$-closed in $Y$. Now $\text{cl}(V) = \text{cl}(\text{int}(V)) \subseteq \text{cl}(\text{int}(\text{cl}(V))) \subseteq V$ shows that $V$ is closed in $Y$. Therefore $Y$ is locally indiscrete.
iv) Let $V$ be open set in $Y$. By assumption, $f^{-1}(V)$ is $\pi gamma$-closed in $X$. Since $X$ is a $\pi gamma$-space, $f^{-1}(V)$ is closed in $X$. Hence $f$ is contra-continuous.
v) Let $V$ be open set in $Y$. By assumption, $f^{-1}(V)$ is $\pi gamma$-closed in $X$. Since $X$ is a $T_{\pi gamma}$ space, $f^{-1}(V)$ is $\pi gamma$-closed in $X$. Hence $f$ is contra-$\pi gamma$-continuous.

Remark 4.2.7: $\pi gamma O(X, x)$ and $\pi gamma C(X, x)$ represent $\pi gamma$-open and $\pi gamma$-closed sets in $X$ containing $x$. 

53
**Theorem 4.2.8**: Suppose $\pi\text{G}\alpha\text{O}(X,\tau)$ is closed under arbitrary unions. Then the following are equivalent for a function $f : (X,\tau) \rightarrow (Y,\sigma)$:

1. $f$ is contra-$\pi\text{G}\alpha$-continuous.

2. For every closed subset $F$ of $Y$, $f^{-1}(F) \in \pi\text{G}\alpha\text{O}(X)$.

3. For each $x \in X$ and each $F \in C(Y,f(x))$, there exists $U \in \pi\text{G}\alpha\text{O}(X, x)$ such that $f(U) \subset F$.

**Proof**: $1 \iff 2$ and $2 \Rightarrow 3$ is obvious.

3$\Rightarrow 2$. Let $F$ be any closed set of $Y$ and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in \pi\text{G}\alpha\text{O}(X, x)$ such that $f(U_x) \subset F$. Therefore we obtain

$f^{-1}(F) = \bigcup \{U_x : x \in f^{-1}(F)\}$ and $f^{-1}(F)$ is $\pi\text{G}\alpha$-open.

**Theorem 4.2.9**: If $f : (X,\tau) \rightarrow (Y,\sigma)$ is contra-$\pi\text{G}\alpha$-continuous and $U$ is open in $X$, then $f/U : (U,\tau) \rightarrow (Y,\sigma)$ is contra-$\pi\text{G}\alpha$-continuous.

**Proof**: Let $V$ be closed in $Y$. Since $f : (X,\tau) \rightarrow (Y,\sigma)$ is contra-$\pi\text{G}\alpha$-continuous, $f^{-1}(V)$ is $\pi\text{G}\alpha$-open in $X$. $(f/U)^{-1}(V) = f^{-1}(V) \cap U$ is $\pi\text{G}\alpha$-open in $X$. By Proposition 2.2.19(i), $(f/U)^{-1}(V)$ is $\pi\text{G}\alpha$-open in $U$.

**Theorem 4.2.10**: Suppose $\pi\text{G}\alpha\text{O}(X,\tau)$ is closed under arbitrary unions. Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be a function and $\{U_i : i \in I\}$ be a cover of $X$ such that $U_i \in \pi\text{G}\alpha\text{C}(X)$ and regular open for each $i \in I$. If $f/U_i : (U_i,\tau_i) \rightarrow (Y,\sigma)$ is contra-$\pi\text{G}\alpha$-continuous for each $i \in I$, then $f$ is contra-$\pi\text{G}\alpha$-continuous.

**Proof**: Suppose that $F$ is any closed set of $Y$. We have

$f^{-1}(F) = \bigcup \{f^{-1}(F) \cap U_i : i \in I\} = \bigcup \{(f/U_i)^{-1}(F) : i \in I\}$. Since $f/U_i$ is contra-$\pi\text{G}\alpha$-continuous for each $i \in I$, it follows $(f/U_i)^{-1}(F) \in \pi\text{G}\alpha\text{O}(U_i)$. By Proposition 2.2.19(ii), we have $f^{-1}(F) \in \pi\text{G}\alpha\text{O}(X)$, which implies $f$ is contra-$\pi\text{G}\alpha$-continuous.

**Theorem 4.2.11**: For a function $f : (X,\tau) \rightarrow (Y,\sigma)$ the following are equivalent:

1) $f$ is perfectly continuous.

2) $f$ is contra-$\pi\text{G}\alpha$-continuous and regular-continuous.
3) $f$ is contra-$\pi\alpha$-continuous and $\pi$-continuous.

**Proof**: Follows from Lemma 2.2.9.

**Theorem 4.2.12**: Suppose $\pi G_{\alpha}O(X,\tau)$ is closed under arbitrary unions. If $f : (X,\tau) \rightarrow (Y,\sigma)$ is contra-$\pi\alpha$-continuous and $Y$ is regular, then $f$ is $\pi\alpha$-continuous.

**Proof**: Let $x$ be an arbitrary point of $X$ and $V$ an open set of $Y$ containing $f(x)$. Then $Y$ is regular implies that there exists an open set $W$ in $Y$ containing $f(x)$ such that $\text{cl}(W) \subseteq V$. Since $f$ is contra-$\pi\alpha$-continuous by Theorem 4.2.8 there exists $U \in \pi G_{\alpha}O(X,x)$ such that $f(U) \subseteq \text{cl}(W)$. Then $f(U) \subseteq \text{cl}(W) \subseteq V$. Hence $f$ is $\pi\alpha$-continuous.

**Definition 4.2.13**: A function $f : (X,\tau) \rightarrow (Y,\sigma)$ is said to be

i) I.C. $\pi\alpha$-continuous if for each $x \in X$ and each closed set $F$ of $Y$ containing $f(x)$, there exists an $\pi\alpha$-open set $U$ in $X$ containing $x$, such that $\text{int}(f(U)) \subseteq F$.

ii) $(\pi\alpha, S)$-open if $f(U) \in \text{SO}(Y)$ for every $U \in \pi G_{\alpha}O(X)$.

**Theorem 4.2.14**: If a function $f : (X,\tau) \rightarrow (Y,\sigma)$ is I.C. $\pi\alpha$-continuous and $(\pi\alpha, S)$-open, then $f$ is contra-$\pi\alpha$-continuous.

**Proof**: Let $x \in X$ be arbitrary and $V \in C(Y, f(x))$. By hypothesis, there exists a $U \in \pi G_{\alpha}O(X, x)$ such that $\text{int}(f(U)) \subseteq V$. Since $f$ is $(\pi\alpha, S)$-open, $f(U) \in \text{SO}(Y)$. It follows that $f(U) \subseteq \text{cl}(\text{int}(f(U))) \subseteq \text{cl}(V) \subseteq V$. By Theorem 4.2.8, $f$ is contra-$\pi\alpha$-continuous.

**Definition 4.2.15**: A space $X$ is said to be

i) strongly-$S$-closed[36] if every closed cover of $X$ has a finite sub cover.

ii) mildly compact[169] if every clopen cover of $X$ has a finite subcover.

iii) strongly-$S$-Lindelof if every closed cover of $X$ has a countable subcover.

iv) $\pi\alpha$-Lindelof if every cover of $X$ by $\pi\alpha$-open sets has a countable subcover.

**Theorem 4.2.16**: If $f : (X,\tau) \rightarrow (Y,\sigma)$ is contra-$\pi\alpha$-continuous and $K$ is $\pi G_{\alpha}O$-compact relative to $X$, then $f(K)$ is strongly-$S$-closed in $Y$. 

55
Proof: Let \{H_\alpha : \alpha \in \mathbb{V}\} be any cover of \(f(K)\) by closed sets of the subspace \(f(K)\). For each \(\alpha \in \mathbb{V}\), there exists a closed set \(K_\alpha\) of \(Y\) such that \(H_\alpha = K_\alpha \cap f(K)\). For each \(x \in K\), there exists \(\alpha(x) \in \mathbb{V}\) such that \(f(x) \in K_{\alpha(x)}\) and by Theorem 4.2.8 there exists \(U_x \in \pi GaO(X,x)\) such that \(f(U_x) \subseteq K_{\alpha(x)}\). Since the family \(\{U_x : x \in K\}\) is a cover of \(K\) by \(\pi Ga\)-open sets of \(X\), there exists a finite subset \(K_0\) of \(K\) such that \(K \subseteq \bigcup\{U_x : x \in K_0\}\). Therefore we obtain \(f(K) \subseteq \bigcup\{ f(U_x) : x \in K_0\} \subseteq \bigcup\{ K_{\alpha(x)} : x \in K_0\}\). Thus \(f(K) = \bigcup\{ H_{\alpha(x)} : x \in K_0\}\) and hence \(f(K)\) is strongly-\(S\)-closed.

Corollary 4.2.17: If \(f : (X,\tau) \rightarrow (Y,\sigma)\) is a contra-\(\pi Ga\)-continuous surjection and \(X\) is \(\pi GaO\)-compact, then \(Y\) is strongly-\(S\)-closed.

Theorems 4.2.18: a) If \(f : (X,\tau) \rightarrow (Y,\sigma)\) is a contra-\(\pi Ga\)-continuous, \(\pi\)-continuous surjection and \(X\) is mildly compact, then \(Y\) is compact.

b) If \(f : (X,\tau) \rightarrow (Y,\sigma)\) is a contra-\(\pi Ga\)-continuous surjection and \(X\) is \(\pi Ga\)-Lindelöf, then \(Y\) is strongly-\(S\)-Lindelöf.

Proof: a) Let \(\{V_\alpha : \alpha \in \mathbb{V}\}\) be an open cover of \(Y\). Since \(f\) is contra-\(\pi Ga\)-continuous and \(\pi\)-continuous, by Theorem 4.2.11 \(\{f^{-1}(V_\alpha) : \alpha \in \mathbb{V}\}\) is a clopen cover of \(X\) and there exists a finite subset \(\mathbb{V}_0\) of \(\mathbb{V}\) such that \(X = \bigcup\{f^{-1}(V_\alpha) : \alpha \in \mathbb{V}_0\}\). Since \(f\) is a surjection, \(Y = \bigcup\{V_\alpha : \alpha \in \mathbb{V}_0\}\) and hence \(Y\) is compact.

b) Let \(\{V_\alpha : \alpha \in \mathbb{V}\}\) be a closed cover of \(Y\). Since \(f\) is contra-\(\pi Ga\)-continuous, \(\{f^{-1}(V_\alpha) : \alpha \in \mathbb{V}\}\) is a \(\pi Ga\)-open cover of \(X\) and hence there exists a countable cover \(\mathbb{V}_0\) of \(\mathbb{V}\), such that \(X = \bigcup\{f^{-1}(V_\alpha) : \alpha \in \mathbb{V}_0\}\). Since \(f\) is a surjection, \(Y = \bigcup\{V_\alpha : \alpha \in \mathbb{V}_0\}\) and hence \(Y\) is strongly-\(S\)-Lindelöf.

Theorem 4.2.19: Let \((X,\tau)\) be \(\pi Ga\)-connected and \((Y,\sigma)\) be \(T_1\). If \(f : (X,\tau) \rightarrow (Y,\sigma)\) is contra-\(\pi Ga\)-continuous, then \(f\) is constant.

Proof: Assume \(Y\) is non-empty. Since \(Y\) is a \(T_1\)-space and \(f\) is contra-\(\pi Ga\)-continuous, \(\Omega = \{f^{-1}(y) : y \in Y\}\) is a disjoint \(\pi Ga\)-open partition of \(X\). If \(|\Omega| \geq 2\), then \(X\) can be written as the disjoint union of \(\pi Ga\)-open sets which is a contradiction. Therefore \(|\Omega| = 1\) and hence \(f\) is a constant.
Theorem 4.2.20: a) If \( f : (X, \tau) \to (Y, \sigma) \) is a contra-\( \pi \gamma \alpha \)-continuous, \( \pi \)-continuous surjection and \( X \) is connected, then \( Y \) has an indiscrete topology.

b) If \( f : (X, \tau) \to (Y, \sigma) \) is a contra-\( \pi \gamma \alpha \)-continuous, \( \pi \)-continuous surjection and \( X \) is connected, then \( Y \) is connected.

c) If \( f : (X, \tau) \to (Y, \sigma) \) is a contra-\( \pi \gamma \alpha \)-continuous surjection and \( X \) is \( \pi \Gamma \alpha \)-connected, then \( Y \) is connected.

Proof: a) Suppose that there exists a proper open set \( V \) of \( Y \). Since \( f \) is contra-\( \pi \gamma \alpha \)-continuous and \( \pi \)-continuous, \( f^{-1}(V) \) is \( \pi \gamma \alpha \)-closed and \( \pi \)-open in \( X \). By Lemma 2.2.9, \( f^{-1}(V) \) is a proper clopen set in \( X \) which is a contradiction to the fact that \( X \) is connected. Therefore \( Y \) has an indiscrete topology.

b) Suppose \( Y \) is not connected. There exist non empty disjoint open sets \( V_1 \) and \( V_2 \) such that \( Y = V_1 \cup V_2 \). Since \( f \) is contra-\( \pi \gamma \alpha \)-continuous and \( \pi \)-continuous, \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are \( \pi \gamma \alpha \)-closed and \( \pi \)-open in \( X \) and hence clopen by Lemma 2.2.9.

Also \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are non empty disjoint sets in \( X \) such that \( X = f^{-1}(V_1) \cup f^{-1}(V_2) \) which shows that \( X \) is not connected. Hence \( Y \) is connected.

c) Suppose \( Y \) is not connected. There exist non empty disjoint open sets \( V_1 \) and \( V_2 \) such that \( Y = V_1 \cup V_2 \). Since \( f \) is contra-\( \pi \gamma \alpha \)-continuous, \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are \( \pi \gamma \alpha \)-closed sets in \( X \) such that \( X = f^{-1}(V_1) \cup f^{-1}(V_2) \) and \( f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset \) which shows that \( X \) is not \( \pi \Gamma \alpha \)-connected. Hence \( Y \) is connected.

Definition 4.2.21: A subset \( A \) of a space \( X \) is said to be

i) \( \pi \gamma \alpha \)-dense if \( \pi \gamma \alpha \text{-cl}(A) = X \).

ii) \( \pi \gamma \alpha \)-\( T_1 \) if for each pair of distinct points \( x \) and \( y \) in \( X \), there exist \( \pi \gamma \alpha \)-open sets \( U \) and \( V \) containing \( x \) and \( y \) respectively such that \( y \notin U \) and \( x \notin V \).

iii) \( \pi \gamma \alpha \)-Hausdorff (or \( \pi \gamma \alpha \)-\( T_2 \)) if for each pair of distinct points \( x \) and \( y \) in \( X \), there exist \( U \in \pi \Gamma \alpha O(X, x) \) and \( V \in \pi \Gamma \alpha O(Y, y) \) such that \( U \cap V = \emptyset \).

iv) clopen \( T_2 \) [169] (clopen Hausdorff or ultra Hausdorff) if for each pair of distinct points \( x \) and \( y \) in \( X \), there exist disjoint clopen sets \( U \) and \( V \) in \( X \) such that \( x \in U \) and \( y \in V \).

v) ultra normal [169] if every two disjoint closed sets of \( X \) can be separated by clopen sets.
vi) weakly Hausdorff[168] if each element is an intersection of regular closed sets.

**Theorem 4.2.22:** Suppose \( \pi G\alpha C(X,\tau) \) is closed under arbitrary intersections. If \( f : (X,\tau) \rightarrow (Y,\sigma) \) and \( g : (X,\tau) \rightarrow (Y,\sigma) \) are contra-\( \pi G\alpha \)-continuous functions and \( Y \) is Urysohn, then \( E = \{ x \in X : f(x) = g(x) \} \) is \( \pi G\alpha \)-closed in \( X \).

**Proof:** Let \( x \in X - E \). Then \( f(x) \neq g(x) \). Since \( Y \) is Urysohn, there exist open sets \( V \) and \( W \) such that \( f(x) \in V \), \( g(x) \in W \) and \( \text{cl}(V) \cap \text{cl}(W) = \phi \). Since \( f \) and \( g \) are contra-\( \pi G\alpha \)-continuous, \( f^{-1}(\text{cl}(V)) \) and \( g^{-1}(\text{cl}(W)) \) are \( \pi G\alpha \)-open sets in \( X \). Let \( U = f^{-1}(\text{cl}(V)) \) and \( V = g^{-1}(\text{cl}(W)) \). Then \( U \) and \( V \) are \( \pi G\alpha \)-open sets in \( X \) containing \( x \). Let \( A = U \cap V \). Then \( A \) is \( \pi G\alpha \)-open in \( X \) containing \( x \). Hence, \( f(A) \cap g(A) = f(U \cap V) \cap g(U \cap V) \subseteq f(U) \cap g(V) \subseteq \text{cl}(V) \cap \text{cl}(W) = \phi \). Therefore \( A \cap E = \phi \) and \( x \notin \pi G\alpha \text{-cl}(E) \). Hence \( E \) is \( \pi G\alpha \)-closed in \( X \).

**Theorem 4.2.23:** Suppose \( \pi G\alpha C(X,\tau) \) is closed under arbitrary intersections. If \( f : (X,\tau) \rightarrow (Y,\sigma) \) and \( g : (X,\tau) \rightarrow (Y,\sigma) \) are contra-\( \pi G\alpha \)-continuous functions and \( Y \) is Urysohn and \( f = g \) on a \( \pi G\alpha \)-dense set \( A \subseteq X \), then \( f = g \) on \( X \).

**Proof:** Since \( f \) and \( g \) are contra-\( \pi G\alpha \)-continuous functions and \( Y \) is Urysohn, by Theorem 4.2.22 \( E \) is \( \pi G\alpha \)-closed in \( X \). By assumption, we have \( f = g \) on a \( \pi G\alpha \)-dense set \( A \subseteq X \). Since \( A \subseteq E \) and \( A \) is \( \pi G\alpha \)-dense, \( X = \pi G\alpha \text{-cl}(A) \subseteq \pi G\alpha \text{-cl}(E) = E \). Hence \( f = g \) on \( X \).

**Theorem 4.2.24:** Let \( X \) and \( Y \) be topological spaces. If for each pair of distinct points \( x \) and \( y \) in \( X \), there exists a function \( f \) from \( X \) into \( Y \) such that \( f(x) \neq f(y) \), \( Y \) is an Urysohn space and \( f \) is a contra-\( \pi G\alpha \)-continuous function at \( x \) and \( y \), then \( X \) is \( \pi G\alpha \text{-T}_2 \).

**Proof:** Let \( x \) and \( y \) be any two distinct points in \( X \). Then since \( Y \) is Urysohn, there exists a function \( f \) from \( X \) into \( Y \) such that \( f(x) \neq f(y) \). Let \( a = f(x) \) and \( b = f(y) \). Then \( a \neq b \). Since \( Y \) is Urysohn space, there exist open sets \( V \) and \( W \) containing \( a \) and \( b \) respectively such that \( \text{cl}(V) \cap \text{cl}(W) = \phi \). Since \( f \) is a contra-\( \pi G\alpha \)-continuous function at \( x \) and \( y \) there exist \( \pi G\alpha \)-open sets \( A = f^{-1}(\text{cl}(V)) \) and \( B = f^{-1}(\text{cl}(W)) \) containing \( x \) and \( y \) respectively such that \( f(A) \subseteq \text{cl}(V) \) and \( f(B) \subseteq \text{cl}(W) \). Then \( f(A) \cap f(B) = \phi \). So \( A \cap B = \phi \). Hence \( X \) is \( \pi G\alpha \text{-T}_2 \).
**Theorem 4.2.25:**

a) If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is contra-\( \pi \alpha \)-continuous injection and \( Y \) is weakly Hausdorff, then \( X \) is \( \pi \alpha - \mathcal{T}_1 \).

b) If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is contra-\( \pi \alpha \)-continuous injection and \( Y \) is ultra Hausdorff, then \( X \) is \( \pi \alpha - \mathcal{T}_2 \).

c) If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is a contra-\( \pi \alpha \)-continuous closed injection and \( Y \) is ultra normal, then \( X \) is \( \pi \alpha \)-normal.

**Proof:**

a) Since \( Y \) is weakly Hausdorff for any two distinct points \( x \) and \( y \) in \( X \), there exist regular closed sets \( A \) and \( B \) in \( Y \) such that \( f(x) \in A \), \( f(y) \notin A \), \( f(y) \in B \), \( f(y) \notin B \). Since \( f \) is contra-\( \pi \alpha \)-continuous, \( f^{-1}(A) \) and \( f^{-1}(B) \) are \( \pi \alpha \)-open subsets of \( X \) such that \( x \in f^{-1}(A) \), \( y \notin f^{-1}(A) \), \( y \in f^{-1}(B) \), \( x \notin f^{-1}(B) \). Hence \( X \) is \( \pi \alpha - \mathcal{T}_1 \).

b) Let \( x \) and \( y \) be distinct points of \( X \). Then \( f(x) \neq f(y) \) and since \( Y \) is ultra Hausdorff, there exist clopen sets \( A \) and \( B \) containing \( f(x) \) and \( f(y) \) respectively such that \( A \cap B = \emptyset \). By hypothesis, \( f^{-1}(A) \) and \( f^{-1}(B) \) are \( \pi \alpha \)-open sets in \( X \) containing \( x \) and \( y \) respectively such that \( f^{-1}(A) \cap f^{-1}(B) = \emptyset \). Hence \( X \) is \( \pi \alpha - \mathcal{T}_2 \).

c) Let \( A \) and \( B \) be disjoint closed subsets of \( X \). Then \( f \) is a closed injection implies \( f(A) \) and \( f(B) \) are disjoint and closed in \( Y \). Since \( Y \) is ultra normal, \( f(A) \) and \( f(B) \) are separated by disjoint clopen sets \( C \) and \( D \) respectively. Thus \( A \subset f^{-1}(C) \in \pi \alpha C(X, \tau) \), \( B \subset f^{-1}(D) \in \pi \alpha O(X) \) and \( f^{-1}(C) \cap f^{-1}(D) = \emptyset \). Hence \( X \) is \( \pi \alpha \)-normal.

### 4.3 Almost Contra-\( \pi \alpha \)-Continuous Functions

In this section, we introduce the concept of almost contra-\( \pi \alpha \)-continuous functions which is a weaker form of contra-\( \pi \alpha \)-continuous functions. Moreover, we investigate the relationships among almost contra-\( \pi \alpha \)-continuous functions, separation axioms, connectedness and compactness.

**Definition 4.3.1:** A function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is said to be almost contra-\( \pi \alpha \)-continuous if \( f^{-1}(V) \in \pi \alpha C(X, \tau) \) for each \( V \in RO(Y, \sigma) \).
Theorem 4.3.2: Suppose $\pi G\alpha O(X,\tau)$ is closed under arbitrary unions. The following statements are equivalent for a function $f: (X,\tau) \rightarrow (Y,\sigma)$:

1. $f$ is almost contra-$\pi G\alpha$-continuous.
2. $f^{-1}(F) \in \pi G\alpha O(X,\tau)$, for every $F \in RC(Y,\sigma)$.
3. For each $x \in X$ and each regular closed set $F$ in $Y$ containing $f(x)$, there exists a $\pi G\alpha$-open set $U$ in $X$ containing $x$ such that $f(U) \subset F$.
4. For each $x \in X$ and each regular open set $V$ in $Y$ not containing $f(x)$, there exists a $\pi G\alpha$-closed set $K$ in $X$ not containing $x$ such that $f^{-1}(V) \subset K$.
5. $f^{-1}(\text{int}(\text{cl}(G))) \in \pi G\alpha C(X,\tau)$ for every open subset $G$ of $Y$.
6. $f^{-1}(\text{cl}(\text{int}(F))) \in \pi G\alpha O(X,\tau)$ for every closed subset $F$ of $Y$.

**Proof:**

1 $\Rightarrow$ 2: Let $F \in RC(Y,\sigma)$. Then $Y-F \in RO(Y,\sigma)$. By (1),

$$f^{-1}(Y-F) = X-f^{-1}(F) \in \pi G\alpha C(X).$$

This implies $f^{-1}(F) \in \pi G\alpha O(X,\tau)$.

2 $\Rightarrow$ 1: Let $V \in RO(Y,\sigma)$. Then $Y-V \in RC(Y,\sigma)$. By (2),

$$f^{-1}(Y-V) = X-f^{-1}(V) \in \pi G\alpha O(X).$$

This implies $f^{-1}(V) \in \pi G\alpha C(X,\tau)$.

2 $\Rightarrow$ 3: Let $F$ be any regular closed set in $Y$ containing $f(x)$. Then $f^{-1}(F) \in \pi G\alpha O(X,\tau)$ and $x \in f^{-1}(F)$ by (2). Take $U = f^{-1}(F)$. Then $f(U) \subset F$.

3 $\Rightarrow$ 2: Let $F \in RC(Y,\sigma)$ and $x \in f^{-1}(F)$. From (3), there exists a $\pi G\alpha$-open set $U_x$ in $X$ containing $x$ such that $U_x \subset f^{-1}(F)$. We have $f^{-1}(F) = \bigcup\{ U_x : x \in f^{-1}(F) \}$. Thus $f^{-1}(F)$ is $\pi G\alpha$-open.

3 $\Rightarrow$ 4: Let $V$ be a regular open set in $Y$ not containing $f(x)$. Then $Y-V$ is a regular closed set containing $f(x)$. By (3) there exists a $\pi G\alpha$-open set $U$ in $X$ containing $x$ such that $f(U) \subset Y-V$. Hence $U \subset f^{-1}(Y-V) \subset X-f^{-1}(V)$ and then $f^{-1}(V) \subset X-U$. Take $K = X-U$. We obtain a $\pi G\alpha$-closed set $K$ in $X$ not containing $x$ such that $f^{-1}(V) \subset K$.

4 $\Rightarrow$ 3: Let $F$ be a regular closed set in $Y$ containing $f(x)$. Then $Y-F$ is a regular open set in $Y$ not containing $f(x)$. By (4) there exists a $\pi G\alpha$-closed set $K$ in $X$ not containing $x$ such that $f^{-1}(Y-F) \subset K$. That is $X-f^{-1}(F) \subset K$ implies $X-K \subset f^{-1}(F)$ and hence
f (X-K) ⊆ F. Take U = X-K. Then U is a rcα-open set in X containing x such that f (U) ⊆ F.

1 ⇒ 5: Let G be an open subset of Y. Since int(cl(G)) is regular open, then by (1)
f⁻¹ (int(cl (G))) ∈ πGαC(X,τ) .

5 ⇒ 1: Let V ∈ RO(Y,σ) Then V is open in Y. By 5, f⁻¹ (int cl(V)) ∈ πGαC(X,τ)
implies f⁻¹ (V) ∈ πGαC(X,τ) .

2 ⇔ 6: similar as 1 ⇔ 5 .

Remark 4.3.3: The following diagram holds

![Diagram of 5 types of continuity]

None of the implications is reversible for almost contra-πgα-continuity as shown by the following examples.

Example 4.3.4:1) Let X = \{a,b,c\}, τ = \{ϕ,\{a\},\{b\},\{a,b\},\{a,c\},X\} and

σ = \{ϕ,\{a\},\{b\},\{a,b\},X\}. Then the identity function f: (X,τ) → (X,σ) is almost contra-πgα-continuous but not regular set-connected.

2) Let X = \{a,b,c,d\}, τ = \{ϕ,\{a\},\{d\},\{a,c\},\{a,d\},\{a,c,d\},X\} and

σ = \{ϕ,\{a\},\{a,b\},\{a,c,d\}\}. Then the identity function f: (X,τ) → (X,σ) is almost contra-πgα-continuous but not contra-πgα-continuous.

3) Let X = \{a,b,c\}, τ = \{ϕ,\{a\},\{b\},X\} and σ = \{ϕ,\{a\},\{a,b\},X\}. Then the identity function f: (X,τ) → (X,σ) is contra-πgα-continuous but not contra continuous.
Theorem 4.3.5: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is an almost contra-$\pi Ga$-continuous function and $A$ is an open subset of $X$, then the restriction $f/A : A \rightarrow Y$ is almost contra-$\pi Ga$-continuous.

Proof: Let $F \in RC(Y)$. Since $f$ is almost contra-$\pi Ga$-continuous, $f^{-1}(F) \in \pi GaO(X)$. Since $A$ is open, it follows that $(f/A)^{-1}(F) = A \cap f^{-1}(F) \in \pi GaO(A)$. Therefore $f/A$ is an almost contra-$\pi Ga$-continuous function.

Remark 4.3.6: Every restriction of an almost contra-$\pi Ga$-continuous function is not necessarily almost contra-$\pi Ga$-continuous.

Example 4.3.7: Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$ and $\sigma = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$. The identity function $f : (X, \tau) \rightarrow (X, \sigma)$ is almost contra-$\pi Ga$-continuous. But if $A = \{a, b, c\}$ where $A$ is not open in $(X, \tau)$ and $\tau_A = \{\phi, \{a, b, c\}, \{a\}, \{c\}, \{a, c\}\}$ is the relative topology on $A$ induced by $\tau$, then $f/A : (A, \tau_A) \rightarrow (X, \sigma)$ is not almost contra-$\pi Ga$-continuous. Note that $\{a, b, d\}$ is regular closed in $(X, \sigma)$ but $(f/A)^{-1}(\{a, b, d\}) = A \cap \{a, b, d\} = \{a, b\}$ is not $\pi Ga$-open in $(A, \tau_A)$.

Definition 4.3.8: A collection $\{U_\alpha : \alpha \in I\}$ of subsets of $X$ is called a $\pi$-cover if $U_\alpha$ is $\pi Ga$-closed and regular open for each $\alpha \in I$.

Theorem 4.3.9: Suppose that $\pi GaO(X, \tau)$ sets are closed under arbitrary unions. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function and $\{U_\alpha : \alpha \in I\}$ be a $\pi$-cover of $X$. If for each $\alpha \in I$, $f / U_\alpha$ is almost contra-$\pi Ga$-continuous, then $f : (X, \tau) \rightarrow (Y, \sigma)$ is almost contra-$\pi Ga$-continuous function.

Proof: Let $V \in RC(Y)$. Then $f / U_\alpha$ is almost contra-$\pi Ga$-continuous function implies $(f / U_\alpha)^{-1}(V) \in \pi GaO(U_\alpha)$. Since $U_\alpha \in \pi GaC(X)$ and is regular open, it follows $(f / U_\alpha)^{-1}(V) \in \pi GaO(X)$ for each $\alpha \in I$. Then $f^{-1}(V) = \cup \{(f / U_\alpha)^{-1}(V) : \alpha \in I\} \in \pi GaO(X)$.

Thus $f$ is an almost contra-$\pi Ga$-continuous function.
Theorem 4.3.10: Let \( f : X \rightarrow Y \) be a function and let \( g : X \rightarrow X \times Y \) be the graph of \( f \) defined by \( g(x) = (x, f(x)) \) for every \( x \in X \). If \( g \) is almost contra-\( \pi \)g\( \alpha \)-continuous, then \( f \) is almost contra-\( \pi \)g\( \alpha \)-continuous.

Proof: Let \( V \in \text{RC}(Y) \), then
\[
X \times V = X \times \text{cl}(\text{int}(V)) = \text{cl}(\text{int}(X)) \times \text{cl}(\text{int}(V)) = \text{cl}(\text{int}(X \times V)).
\]
Therefore \( X \times V \in \text{RC}(X \times Y) \). Since \( g \) is almost contra-\( \pi \)g\( \alpha \)-continuous,
\[
g^{-1}(X \times V) \in \pi \text{G\( \alpha \)}(X).\]
This implies \( f^{-1}(V) = g^{-1}(X \times V) \in \pi \text{G\( \alpha \)}(X) \). Thus \( f \) is almost contra-\( \pi \)g\( \alpha \)-continuous.

Lemma 4.3.11: Suppose that \( \pi \text{G\( \alpha \)}(X, \tau) \) sets are closed under arbitrary unions. A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is almost \( \pi \)g\( \alpha \)-continuous if and only if for each \( x \in X \) and each regular open set \( V \) of \( Y \) containing \( f(x) \) there exists \( U \in \pi \text{G\( \alpha \)}(X, x) \) such that \( f(U) \subseteq V \).

Theorem 4.3.12: Suppose that \( \pi \text{G\( \alpha \)}(X, \tau) \) sets are closed under arbitrary unions. Let \( Y \) be extremally disconnected. Then a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is almost contra-\( \pi \)g\( \alpha \)-continuous if and only if it is almost \( \pi \)g\( \alpha \)-continuous.

Proof: Let \( x \in X \) and \( V \) be any regular open set of \( Y \) containing \( f(x) \). Since \( Y \) is extremally disconnected, \( V \) is clopen and hence \( V \) is regular closed. By Theorem 4.3.2, there exists \( U \in \pi \text{G\( \alpha \)}(X, x) \) such that \( f(U) \subseteq V \). Then lemma 4.3.11 implies \( f \) is almost \( \pi \)g\( \alpha \)-continuous. Conversely, let \( F \) be any regular closed set of \( Y \). Since \( Y \) is extremally disconnected, \( F \) is also regular open and \( f^{-1}(F) \) is \( \pi \)g\( \alpha \)-open in \( X \). Hence \( f \) is almost contra-\( \pi \)g\( \alpha \)-continuous.

Theorem 4.3.13: Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be two functions. Then the following properties hold:

i) If \( f \) is almost contra-\( \pi \)g\( \alpha \)-continuous and \( g \) is regular set-connected, then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is almost contra-\( \pi \)g\( \alpha \)-continuous and almost \( \pi \)g\( \alpha \)-continuous.

ii) If \( f \) is almost contra-\( \pi \)g\( \alpha \)-continuous and \( g \) is perfectly continuous, then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is \( \pi \)g\( \alpha \)-continuous and contra-\( \pi \)g\( \alpha \)-continuous.

63
iii) If \( f \) is contra-\( \pi g \alpha \)-continuous and \( g \) is regular set-connected, then \( g \circ f : (X, \tau) \to (Z, \eta) \) is almost contra-\( \pi g \alpha \)-continuous and almost \( \pi g \alpha \)-continuous.

**Proof:** Let \( V \in RO(Z) \). Then \( g \) is regular set-connected implies \( g^{-1}(V) \) is clopen in \( Y \).
Since \( f \) is almost contra-\( \pi g \alpha \)-continuous, \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is \( \pi g \alpha \)-open and \( \pi g \alpha \)-closed. Therefore \( g \circ f \) is almost contra-\( \pi g \alpha \)-continuous and almost \( \pi g \alpha \)-continuous.

**Proof of** ii) and iii) can be obtained similarly.

**Theorem 4.3.14**: If \( f : (X, \tau) \to (Y, \sigma) \) is a surjective \( M-\pi g \alpha \)-open map and \( g : (Y, \sigma) \to (Z, \eta) \) is a function such that \( g \circ f : (X, \tau) \to (Z, \eta) \) is almost contra-\( \pi g \alpha \)-continuous, then \( g \) is almost contra-\( \pi g \alpha \)-continuous.

**Proof**: Let \( V \) be any regular closed set in \( Z \). Since \( g \circ f \) is almost contra-\( \pi g \alpha \)-continuous, \( (g \circ f)^{-1}(V) \in \pi G \alpha O(X) \). Since \( f \) is a surjective \( M-\pi g \alpha \)-open map, \( f((g \circ f)^{-1}(V)) = f(f^{-1}(g^{-1}(V))) = g^{-1}(V) \) is \( \pi g \alpha \)-open in \( Y \). Therefore \( g \) is almost contra-\( \pi g \alpha \)-continuous.

**Theorem 4.3.15**: If \( f : (X, \tau) \to (Y, \sigma) \) is a surjective, \( M-\pi g \alpha \)-closed map and \( g : (Y, \sigma) \to (Z, \eta) \) is a function such that \( g \circ f : (X, \tau) \to (Z, \eta) \) is almost contra-\( \pi g \alpha \)-continuous, then \( g \) is almost contra-\( \pi g \alpha \)-continuous.

**Proof**: Similar to that of Theorem 4.3.14.

**Definition 4.3.16**: A space \( X \) is said to be
a) \( \pi G \alpha \)-closed if every \( \pi g \alpha \)-closed cover of \( X \) has a finite subcover.
b) countably \( \pi G \alpha \)-closed if every countable cover of \( X \) by \( \pi g \alpha \)-closed sets has a finite subcover.
c) countably \( \pi G \alpha \)-compact if every countable cover of \( X \) by \( \pi g \alpha \)-open sets has a finite subcover.
d) \( \pi G \alpha C \)-Lindelof if every cover of \( X \) by \( \pi g \alpha \)-closed sets has a countable subcover.
e) nearly compact[165] if every regular open cover of \( X \) has a finite subcover.
f) nearly countably compact[54,166] if every countable cover of \( X \) by regular open sets has a finite subcover.
g) nearly Lindelof[49] if every cover of \( X \) by regular open sets has a countable subcover.
b) S-closed [176] if every regular closed cover of $X$ has a finite subcover.
i) countably S-closed compact [31] if every countable cover of $X$ by regular closed sets has a finite subcover.

j) S-Lindelof [97] if every cover of $X$ by regular closed sets has a countable subcover.

**Theorem 4.3.17**: Let $f : (X,t) \rightarrow (Y,\sigma)$ be an almost contra-$\pi\alpha$-continuous surjection. Then the following statements hold:
a) If $X$ is $\pi\alpha$-closed, then $Y$ is nearly compact.
b) If $X$ is $\pi\alpha$-Lindelof, then $Y$ is nearly Lindelof.
c) If $X$ is countably-$\pi\alpha$-closed, then $Y$ is nearly countably compact.
d) If $X$ is $\pi\alpha$-compact, then $Y$ is S-closed.
e) If $X$ is $\pi\alpha$-Lindelof, then $Y$ is S-Lindelof.
f) If $X$ is countable $\pi\alpha$-compact, then $Y$ is countably S-closed compact.

**Proof**: a) Let $\{V_{\alpha} : \alpha \in I\}$ be any regular open cover of $Y$. Then $f$ is almost contra-$\pi\alpha$-continuous implies $\{f^{-1}(V_{\alpha}) : \alpha \in I\}$ is a $\pi\alpha$-closed cover of $X$. Since $X$ is $\pi\alpha$-closed, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_{\alpha}) : \alpha \in I_0\}$. Thus we have $Y = \bigcup \{V_{\alpha} : \alpha \in I_0\}$ and $Y$ is nearly compact.

Proof of b) and c) is similar to that of a).

d) Let $\{V_{\alpha} : \alpha \in I\}$ be any regular closed cover of $Y$. Since $f$ is almost contra-$\pi\alpha$-continuous, $\{f^{-1}(V_{\alpha}) : \alpha \in I\}$ is a $\pi\alpha$-open cover of $X$ and by assumption there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_{\alpha}) : \alpha \in I_0\}$. Thus we have $Y = \bigcup \{V_{\alpha} : \alpha \in I_0\}$. Hence $Y$ is S-closed.

Proof of e) and f) is similar to that of d).

**Definition 4.3.18**: A space $X$ is said to be

a) mildly $\pi\alpha$-compact if every $\pi\alpha$-clopen cover of $X$ has a finite subcover.
b) mildly countably-$\pi\alpha$-compact if every $\pi\alpha$-clopen countable cover of $X$ has a finite subcover.
c) mildly $\pi\alpha$-Lindelof if every $\pi\alpha$-clopen cover of $X$ has a countable subcover.
Theorem 4.3.19: Let \( f:(X,x) \to (Y,\alpha) \) be almost contra-\( \pi \alpha \)-continuous and almost \( \pi \alpha \)-continuous surjection. Then

a) If \( X \) is mildly \( \pi \alpha \)-compact, then \( Y \) is nearly compact.

b) If \( X \) is mildly countably-\( \pi \alpha \)-compact, then \( Y \) is nearly countably compact.

c) If \( X \) is mildly \( \pi \alpha \)-Lindelöf, then \( Y \) is nearly Lindelöf.

Proof: a) Let \( V \in RO(Y) \). Since \( f \) is almost contra-\( \pi \alpha \)-continuous and almost \( \pi \alpha \)-continuous, \( f^{-1}(V) \) is \( \pi \alpha \)-closed and \( \pi \alpha \)-open in \( X \) respectively. Hence \( f^{-1}(V) \) is \( \pi \alpha \)-clopen in \( X \). Let \( \{V_\alpha : \alpha \in I\} \) be any regular open cover of \( Y \). Then \( \{f^{-1}(V_\alpha) : \alpha \in I\} \) is a \( \pi \alpha \)-clopen cover of \( X \). Since \( X \) is mildly \( \pi \alpha \)-compact, there exists a finite subset \( I_0 \) of \( I \) such that \( X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\} \). Since \( f \) is surjective, we obtain \( Y = \bigcup \{V_\alpha : \alpha \in I_0\} \). Hence \( Y \) is nearly compact.

Proof of b) and c) is similar to that of a).

Definition 4.3.20: A topological space \( X \) is called \( \pi \alpha \)-ultra connected if every two non-void \( \pi \alpha \)-closed subsets of \( X \) intersect.

Theorem 4.3.21: a) If a function \( f:(X,\tau) \to (Y,\sigma) \) is almost contra-\( \pi \alpha \)-continuous and almost-\( \pi \)-continuous, then \( f \) is regular set-connected.

b) If \( X \) is \( \pi \alpha \)-ultra connected and \( f:(X,\tau) \to (Y,\sigma) \) is almost contra-\( \pi \alpha \)-continuous and surjective, then \( Y \) is hyperconnected.

c) If \( f : (X,\tau) \to (Y,\sigma) \) is an almost contra-\( \pi \alpha \)-continuous surjection and \( X \) is \( \pi \alpha \)-connected, then \( Y \) is connected.

d) If \( f : (X,\tau) \to (Y,\sigma) \) is an almost contra-\( \pi \alpha \)-continuous injection and \( Y \) is weakly Hausdorff, then \( X \) is \( \pi \alpha \)-T\(_1\).

Proof: a) Let \( V \in RO(Y) \). Since \( f \) is almost contra-\( \pi \alpha \)-continuous and almost-\( \pi \)-continuous, \( f^{-1}(V) \) is \( \pi \alpha \)-closed and \( \pi \)-open. \( f^{-1}(V) \) is clopen by Lemma 2.2.9. Hence \( f \) is regular set-connected.

b) Assume that \( Y \) is not hyperconnected. Then there exists an open set \( V \) such that \( V \) is not dense in \( Y \). Then there exist disjoint non-empty regular open subsets \( B_1 \) and \( B_2 \) in \( Y \).
namely $B_1 = \text{int}(\text{cl}(V))$ and $B_2 = Y - \text{cl}(V)$. Since $f$ is almost contra-$\pi \alpha$-continuous and surjective, $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$ are disjoint non-empty $\pi \alpha$-closed subsets of $X$ which is a contradiction to the fact that $X$ is $\pi \alpha$-ultra connected. Hence $Y$ is hyperconnected.

c) Suppose that $Y$ is not connected. Then there exist non-empty disjoint open sets $V_1$ and $V_2$ such that $Y = V_1 \cup V_2$. Then $V_1$ and $V_2$ are clopen in $Y$. Since $f$ is almost contra-$\pi \alpha$-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are $\pi \alpha$-open in $X$. Moreover $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$ which is a contradiction to the fact that $X$ is $\pi \alpha$-connected. Hence $Y$ is connected.

d) Suppose that $Y$ is weakly Hausdorff. For any two distinct points $x$ and $y$ in $X$, there exist $V, W \in \text{RC}(Y)$ such that $f(x) \in V$, $f(y) \in W$, $f(x) \not\in W$, $f(y) \not\in V$. Since $f$ is almost contra-$\pi \alpha$-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $\pi \alpha$-open subsets of $X$ such that $x \in f^{-1}(V)$, $y \in f^{-1}(W)$, $y \not\in f^{-1}(V)$ and $x \not\in f^{-1}(W)$. This shows that $X$ is $\pi \alpha$-$T_1$.

**Theorem 4.3.22:**

a) If $f: (X, \tau) \rightarrow (Y, \sigma)$ is an almost contra-$\pi \alpha$-continuous injection and $Y$ is ultra Hausdorff, then $X$ is $\pi \alpha$-$T_2$.

b) If $f: (X, \tau) \rightarrow (Y, \sigma)$ is an almost contra-$\pi \alpha$-continuous closed injection and $Y$ is ultra normal, then $X$ is $\pi \alpha$-normal.

**Proof:** Similar to that of Theorem 4.2.25.

**4.4 Functions With $\pi \alpha$-Closed Graphs**

In this section the concept of $\pi \alpha$-closed graphs, contra-$\pi \alpha$-closed graphs and $\pi \alpha$-regular graphs for functions between topological spaces are investigated.

Recall that for a function $f: (X, \tau) \rightarrow (Y, \sigma)$, the subset $\{(x, f(x)): x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of $f$ and is denoted by $G(f)$. 

67
Definition 4.4.1: For a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) the graph \( G(f) \) is said to be a \( \pi g \alpha \)-closed graph if for each \((x, y) \in X \times Y - G(f)\) there exist \( U \in \pi G \alpha O(X, x) \), \( V \in \pi G \alpha O(Y, y) \) such that \((U \times V) \cap G(f) = \emptyset\).

Lemma 4.4.2: The function \( f : (X, \tau) \rightarrow (Y, \sigma) \) has a \( \pi g \alpha \)-closed graph if and only if for each \((x, y) \in X \times Y - G(f)\) there exist \( U \in \pi G \alpha O(X, x) \), \( V \in \pi G \alpha O(Y, y) \) such that \( f(U) \cap V = \emptyset \).

Proof: It follows from definition and the fact that for any two subsets \( U \subset X \) and \( V \subset Y \), \((U \times V) \cap G(f) = \emptyset\) if and only if \( f(U) \cap V = \emptyset \).

Lemma 4.4.3: Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be given. Then \( G(f) \) is closed if and only if for each \((x, y) \in X \times Y - G(f)\) there exist an open set \( U \) in \( X \) and an open set \( V \) in \( Y \) such that \( f(U) \cap V = \emptyset \).

Lemma 4.4.4: Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be given. Then \( G(f) \) is \( \alpha \)-closed if and only if for each \((x, y) \in X \times Y - G(f)\) there exist \( U \in \alpha O(X, x) \) and \( V \in \alpha O(Y, y) \) in \( Y \) such that \( f(U) \cap V = \emptyset \).

Theorem 4.4.5: a) Every closed graph is a \( \pi g \alpha \)-closed graph.

b) Every \( \alpha \)-closed graph is a \( \pi g \alpha \)-closed graph.

Proof: a) Let \( G(f) \) be closed. Then for each \((x, y) \in X \times Y - G(f)\) there exist open set \( U \) in \( X \) and open set \( V \) in \( Y \) such that \( f(U) \cap V = \emptyset \). Since every open set is \( \pi g \alpha \)-open, every closed graph is \( \pi g \alpha \)-closed graph.

b) Follows from the definition and from the fact that every \( \alpha \)-open set is \( \pi g \alpha \)-open.

Remark 4.4.6: Converse of the above is not true as seen in the following example.

Example 4.4.7: Let \( X = \{a, b\}, Y = \{a, b, c, d\}, \tau = \{\phi, X, \{a\}, \{b\}\} \) and \( \sigma = \{\phi, \{c, d\}, Y\} \) respectively. Let \( f : X \rightarrow Y \) be the mapping defined by \( f(a) = a, f(b) = b \). Then \( G(f) \) is \( \pi g \alpha \)-closed but is neither a closed graph nor an \( \alpha \)-closed graph.

Remark 4.4.8: Functions having \( \pi g \alpha \)-closed graph need not be \( \pi g \alpha \)-continuous.
Example 4.4.9: Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \text{discrete topology}$. Let $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity mapping. Then $G(f)$ is $\pi\sigma\alpha$-closed but $f$ is not $\pi\sigma\alpha$-continuous.

Remark 4.4.10: A $\pi\sigma\alpha$-continuous function need not have a $\pi\sigma\alpha$-closed graph as shown by the following example.

Example 4.4.11: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\emptyset, X, \{a, b\}\}$ and $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity mapping. Then $f$ is $\pi\sigma\alpha$-continuous but $G(f)$ is not $\pi\sigma\alpha$-closed.

Remark 4.4.12: Examples 4.4.9 and 4.4.11 show that $\pi\sigma\alpha$-closed graph and $\pi\sigma\alpha$-continuous functions are independent concepts.

Theorem 4.4.13: a) Let $f : (X, x) \rightarrow (Y, \sigma)$ be a $\pi\sigma\alpha$-irresolute surjection where $X$ is an arbitrary topological space and $Y$ is $\pi\sigma\alpha$-$T_2$. Then $G(f)$ is $\pi\sigma\alpha$-closed.

b) Let $f : (X, x) \rightarrow (Y, \sigma)$ be a $\pi\sigma\alpha$-continuous surjection where $X$ is an arbitrary topological space and $Y$ is $T_2$. Then $G(f)$ is $\pi\sigma\alpha$-closed.

Proof: a) Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Since $Y$ is $\pi\sigma\alpha$-$T_2$, there exist $\pi\sigma\alpha$-open sets $U, V \subset Y$ such that $f(x) \in U$, $y \in V$ and $U \cap V = \emptyset$. Since $f$ is $\pi\sigma\alpha$-irresolute, $W = f^{-1}(U) \in \pi\sigma\alpha\sigma\sigma(O(X, x))$. Hence $f(W) = f(f^{-1}(U)) \subset U$. This implies $f(W) \cap V = \emptyset$. Hence by Lemma 4.4.2, $G(f)$ is $\pi\sigma\alpha$-closed.

b) Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Since $Y$ is $T_2$, there exist open sets $U$ and $V$ containing $f(x)$ and $y$ respectively such that $U \cap V = \emptyset$. Since $f$ is $\pi\sigma\alpha$-continuous, $f^{-1}(U) = W \in \pi\sigma\alpha\sigma\sigma(O(X, x))$. Since $f$ is a surjection, $f(W) = f(f^{-1}(U)) \subset U$. Hence $f(W) \cap V = \emptyset$. By Lemma 4.4.2, $G(f)$ is $\pi\sigma\alpha$-closed.

Theorem 4.4.14: a) Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be any surjection with $G(f)$ $\pi\sigma\alpha$-closed. Then $Y$ is $\pi\sigma\alpha$-$T_1$.

b) Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be injective with $G(f)$ $\pi\sigma\alpha$-closed. Then $X$ is $\pi\sigma\alpha$-$T_1$.

Proof: a) Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since $f$ is surjective, there exists $x_1 \in X$ such that $f(x_1) = y_2$. Now $(x_1, y_1) \in X \times Y - G(f)$. Since $G(f)$ is $\pi\sigma\alpha$-closed, there exist
\( U_1 \in \pi\text{G}_\alpha\text{O}(X,x_1) \) and \( V_1 \in \pi\text{G}_\alpha\text{O}(Y,y_1) \) such that \( f(U_1) \cap V_1 = \phi \). Now \( x_1 \in U_1 \) implies \( f(x_1) = y_2 \in f(U_1) \). \( y_2 \in f(U_1) \) and \( f(U_1) \cap V_1 = \phi \) implies \( y_2 \notin V_1 \). Again since \( f \) is surjective, there exist a point \( x_2 \in X \) such that \( f(x_2) = y_1 \).

Now \((x_2, y_2) \in X \times Y - \text{G}(f)\). Since \( \text{G}(f) \) is \( \pi\text{g}_\alpha \)-closed, there exist \( U_2 \in \pi\text{G}_\alpha\text{O}(X, x_2) \) and \( V_2 \in \pi\text{G}_\alpha\text{O}(Y, y_2) \) such that \( f(U_2) \cap V_2 = \phi \). Now \( x_2 \in U_2 \) implies \( f(x_2) = y_1 \in f(U_2) \). Now \( y_1 \in f(U_2) \) and \( f(U_2) \cap V_2 = \phi \) implies \( y_1 \notin V_2 \). Thus we obtain sets \( V_1, V_2 \in \pi\text{G}_\alpha\text{O}(Y) \) such that \( y_1 \in V_1 \) but \( y_2 \notin V_1 \) while \( y_2 \in V_2, y_1 \notin V_2 \). Hence \( Y \) is \( \pi\text{g}_\alpha \)-\( T_2 \).

**b)** Let \( x_1, x_2 \) be distinct points of \( X \). Since \( f \) is injective, \( f(x_1) \neq f(x_2) \).

Therefore \((x_1, f(x_2)) \in X \times Y - \text{G}(f)\). Since \( \text{G}(f) \) is \( \pi\text{g}_\alpha \)-closed, by Lemma 4.4.2 there exist \( U_1 \in \pi\text{G}_\alpha\text{O}(X, x_1) \) and \( V_1 \in \pi\text{G}_\alpha\text{O}(Y, f(x_2)) \) such that \( f(U_1) \cap V_1 = \phi \). \( f(x_2) \in V_1 \) and \( f(U_1) \cap V_1 = \phi \) implies \( f(x_2) \notin f(U_1) \) and so \( x_2 \notin U_1 \). Similarly for \((x_2, f(x_1)) \in X \times Y - \text{G}(f)\) there exist \( U_2 \in \pi\text{G}_\alpha\text{O}(X, x_2) \), \( V_2 \in \pi\text{G}_\alpha\text{O}(Y, f(x_1)) \) such that \( f(U_2) \cap V_2 = \phi \). Therefore \( f(x_1) \notin f(U_2) \) and so \( x_1 \notin U_2 \). Hence we obtain \( \pi\text{g}_\alpha \)-open sets \( U_1 \) and \( U_2 \) in \( X \) respectively such that \( x_1 \in U_1 \) but \( x_2 \notin U_1 \) and \( x_2 \in U_2 \) but \( x_1 \notin U_2 \). Thus \( X \) is \( \pi\text{g}_\alpha \)-\( T_1 \).

**Theorem 4.4.15:** Let \( f:(X,\tau) \rightarrow (Y,\sigma) \) be any \( M\pi\text{g}_\alpha \)-open surjection with \( \text{G}(f) \) \( \pi\text{g}_\alpha \)-closed. Then \( Y \) is \( \pi\text{g}_\alpha \)-\( T_2 \).

**Proof:** Let \( y_1, y_2 \in Y \) such that \( y_1 \neq y_2 \). Since \( f \) is surjective, there exists \( x_1 \in X \) such that \( f(x_1) = y_2 \). Then \((x_1, y_1) \in X \times Y - \text{G}(f)\). Since \( \text{G}(f) \) is \( \pi\text{g}_\alpha \)-closed, by Lemma 4.4.2, there exist \( U \in \pi\text{G}_\alpha\text{O}(X, x_1) \) and \( V \in \pi\text{G}_\alpha\text{O}(Y, y_1) \) such that \( f(U) \cap V = \phi \). Since \( f \) is \( M\pi\text{g}_\alpha \)-open, \( f(U) \) is \( \pi\text{g}_\alpha \)-open in \( Y \). Now \( x_1 \in U \) implies \( f(x_1) = y_2 \in f(U) \). Therefore there exist \( V \in \pi\text{G}_\alpha\text{O}(Y, y_1) \) and \( f(U) \in \pi\text{G}_\alpha\text{O}(Y, y_2) \) such that \( f(U) \cap V = \phi \). Hence \( Y \) is \( \pi\text{g}_\alpha \)-\( T_2 \) space.

**Theorem 4.4.16:** If \( f:(X,\tau) \rightarrow (Y,\sigma) \) is injective, \( \pi\text{g}_\alpha \)-irresolute with a \( \pi\text{g}_\alpha \)-closed graph, then \( X \) is \( \pi\text{g}_\alpha \)-\( T_2 \).

**Proof:** Let \( x_1, x_2 \) be two distinct points of \( X \). Since \( f \) is injective, \( f(x_1) \neq f(x_2) \).

Therefore \((x_1, f(x_2)) \in X \times Y - \text{G}(f)\). Since \( \text{G}(f) \) is \( \pi\text{g}_\alpha \)-closed, by Lemma 4.2.2 there exist
$U \in \pi G_{\alpha}O(X, x_1)$ and $V \in \pi G_{\alpha}O(Y, f(x_2))$ such that $f(U) \cap V = \emptyset$. That is $U \cap f^{-1}(V) = \emptyset$. Since $f$ is $\pi g_{\alpha}$- irresolute, $f^{-1}(V) \in \pi G_{\alpha}O(X, x_2)$. Hence there exist $\pi g_{\alpha}$-open sets $U$ and $f^{-1}(V)$ in $X$ containing $x_1$ and $x_2$ respectively such that $U \cap f^{-1}(V) = \emptyset$. Therefore $X$ is $\pi g_{\alpha}$-$T_2$.

**Corollary 4.4.17**: If $f:(X,\tau)\rightarrow(Y,\sigma)$ is bijective, $M$-$\pi g_{\alpha}$-open, $\pi g_{\alpha}$- irresolute and $G(f)$ is $\pi g_{\alpha}$-closed, then both $X$ and $Y$ are $\pi g_{\alpha}$-$T_2$.

**Proof**: Follows from Theorems 4.4.15 and 4.4.16.

**Theorem 4.4.18**: Suppose $\pi G_{\alpha}O(X, \tau)$ is closed under arbitrary unions. If for the function $f:(X,\tau)\rightarrow(Y,\sigma)$ $Y$ is $\pi G_{\alpha}O$-compact and $G(f)$ is $\pi g_{\alpha}$-closed in $X \times Y$, then $f$ is $\pi g_{\alpha}$-continuous.

**Proof**: Let $x \in X$. Let $V$ be open in $Y$ and $y \in Y-V$. Then $(x, y) \in X \times Y - G(f)$. Since $G(f)$ is $\pi g_{\alpha}$-closed, there exist $U_y \in \pi G_{\alpha}O(X, x)$ and $V_y \in \pi G_{\alpha}O(Y, y)$ such that $f(U_y) \cap V_y = \emptyset$. This holds for every $y \in Y-V$. Clearly $\mathcal{C} = \{ V_y : y \in Y-V \}$ is a cover of $Y-V$ by $\pi g_{\alpha}$-open sets. Now $Y$ is $\pi G_{\alpha}O$-compact and $Y-V$ is $\pi g_{\alpha}$-closed. Then by Theorem 2.5.2 a) $Y-V$ is $\pi G_{\alpha}O$-compact relative to $Y$. So $\mathcal{C}$ has a finite subfamily $\{ V_{y_i} : i = 1 \ldots n \}$ such that $Y-V \subset \cup \{ V_{y_i} : i = 1 \ldots n \}$. Let $\{ U_{y_i} : i = 1 \ldots n \}$ be the corresponding sets of $\pi G_{\alpha}O(X, x)$ satisfying $f(U_{y_i}) \cap V_{y_i} = \emptyset$.

Set $U = \cap \{ U_{y_i} : i = 1 \ldots n \}$. Now $U \in \pi G_{\alpha}O(X)$. If $\alpha \in U$, then $f(\alpha) \not\in V_{y_i}$ for all $i = 1 \ldots n$. This implies $f(\alpha) \not\in \cup \{ V_{y_i} : i = 1 \ldots n \}$ so that $f(\alpha) \not\in Y-V$ and hence $f(\alpha) \in V$. Since $\alpha$ is arbitrary, it follows that $f(U) \subset V$ and hence $f$ is $\pi g_{\alpha}$-continuous.

**Definition 4.4.19**: A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is sub contra-$\pi g_{\alpha}$-continuous provided there exists an open base $\mathcal{B}$ for the topology on $Y$ such that $f^{-1}(V)$ is $\pi g_{\alpha}$-closed in $X$ for every $V \in \mathcal{B}$.

**Theorem 4.4.20**: If $f:(X,\tau)\rightarrow(Y,\sigma)$ is a sub contra-$\pi g_{\alpha}$-continuous function and $Y$ is $T_1$, then $G(f)$ is $\pi g_{\alpha}$-closed.
Proof: Let \((x, y) \in X \times Y - G(f)\). Then \(y \neq f(x)\). Let \(\mathcal{B}\) be an open base for the topology on \(Y\). Since \(f\) is sub contra-\(\pi\)-\(\alpha\)-continuous, \(f^{-1}(V)\) is \(\pi\)-\(\alpha\)-closed in \(X\) for every \(V \in \mathcal{B}\).

Since \(Y\) is \(T_1\), there exists a \(V \in \mathcal{B}\) such that \(y \in V\) and \(f(x) \notin V\). Then 
\[(x, y) \in (X - f^{-1}(V)) \times V \subseteq X \times Y - G(f)\]. Hence \(G(f)\) is \(\pi\)-\(\alpha\)-closed.

Corollary 4.4.21: If \(f: (X, \tau) \to (Y, \sigma)\) is contra-\(\pi\)-\(\alpha\)-continuous and \(Y\) is \(T_1\), then \(G(f)\) is \(\pi\)-\(\alpha\)-closed.

Proof: Follows from the fact that every contra-\(\pi\)-\(\alpha\)-continuous function is sub contra-\(\pi\)-\(\alpha\)-continuous.

Definition 4.4.22: The graph \(G(f)\) of a function \(f: (X, \tau) \to (Y, \sigma)\) is said to be contra-\(\pi\)-\(\alpha\)-closed if for each \((x, y) \in X \times Y - G(f)\), there exist \(U \in \pi\text{GoO}(X, x)\) and \(V \in C(Y, y)\) such that \((U \times V) \cap G(f) = \emptyset\).

Lemma 4.4.23: The graph \(G(f)\) of a function \(f: (X, \tau) \to (Y, \sigma)\) is said to be contra-\(\pi\)-\(\alpha\)-closed in \(X \times Y\) if and only if for each \((x, y) \in X \times Y - G(f)\), there exist \(U \in \pi\text{GoO}(X, x)\) and \(V \in C(Y, y)\) such that \(f(U) \cap V = \emptyset\).

Theorem 4.4.24: If \(f: (X, \tau) \to (Y, \sigma)\) is contra-\(\pi\)-\(\alpha\)-continuous and \(Y\) is Urysohn, then \(G(f)\) is contra-\(\pi\)-\(\alpha\)-closed in \(X \times Y\).

Proof: Let \((x, y) \in X \times Y - G(f)\). Then \(y \neq f(x)\) and there exist open sets \(V, W\) such that \(f(x) \in V, y \in W\) and \(\text{cl}(V) \cap \text{cl}(W) = \emptyset\). Since \(f\) is contra-\(\pi\)-\(\alpha\)-continuous, there exists \(U \in \pi\text{GoO}(X, x)\) such that \(f(U) \subseteq \text{cl}(V)\). Therefore \(f(U) \cap \text{cl}(W) = \emptyset\). This shows that \(G(f)\) is contra-\(\pi\)-\(\alpha\)-closed.

Theorem 4.4.25: If \(f: (X, \tau) \to (Y, \sigma)\) is \(\pi\)-\(\alpha\)-continuous and \(Y\) is \(T_1\), then \(G(f)\) is contra-\(\pi\)-\(\alpha\)-closed in \(X \times Y\).

Proof: Let \((x, y) \in X \times Y - G(f)\) then \(y \neq f(x)\). Since \(Y\) is \(T_1\) and \(y \neq f(x)\), there exists an open set \(V\) of \(Y\) such that \(f(x) \in V\) and \(y \notin V\). Since \(f\) is \(\pi\)-\(\alpha\)-continuous, there exists \(U \in \pi\text{GoO}(X, x)\), such that \(f(U) \subseteq V\). Therefore \(f(U) \cap (Y - V) = \emptyset\) and \(Y - V \in C(Y, y)\). This implies \(G(f)\) is contra-\(\pi\)-\(\alpha\)-closed in \(X \times Y\).
Theorem 4.4.26: Suppose $\pi G_{aC}(X,\tau)$ is closed under arbitrary intersections. If $f: (X,\tau) \to (Y,\sigma)$ has a contra-$\pi g_{a}$-closed graph, then the inverse image of a strongly $S$-closed set $A$ of $Y$ is $\pi g_{a}$-closed in $X$.

**Proof:** Assume that $A$ is a strongly $S$-closed set of $Y$ and $x \notin f^{-1}(A)$. For each $a \in A$, $(x, a) \notin G(f)$. By Lemma 4.4.23, there exist $U_a \in \pi G_{aO}(X,x)$ and $V_a \in C(Y,a)$ such that $f(U_a) \cap V_a = \phi$. Since $\{A \cap V_a: a \in A\}$ is a closed cover of the subspace $A$, there exists a finite subset $A_0 \subset A$ such that $A \subset \cup\{V_a: a \in A_0\}$. Set $U = \cap\{U_a: a \in A_0\}$. Then $U$ is $\pi g_{a}$-open and $f(U) \cap A = \phi$. Therefore $U \cap f^{-1}(A) = \phi$ and by Lemma 2.2.25, $x \notin \pi g_{a}-cl(f^{-1}(A))$. Hence $f^{-1}(A)$ is $\pi g_{a}$-closed in $X$.

Theorem 4.4.27: Suppose $\pi G_{aC}(X,\tau)$ is closed under arbitrary intersections. Let $Y$ be a strongly $S$-closed space. If $f: (X,\tau) \to (Y,\sigma)$ has a contra-$\pi g_{a}$-closed graph, then $f$ is contra-$\pi g_{a}$-continuous.

**Proof:** Let $U$ be an open set of $Y$ and $\{V_i: i \in I\}$ be a cover of $U$ by closed sets $V_i$ of $U$. For each $i \in I$, there exists a closed set $K_i$ of $Y$ such that $V_i = K_i \cap U$. Then the family $\{K_i: i \in I\} \cup \{(Y-U)\}$ is a closed cover of $Y$. By assumption, there exists a subset $I_0 \subset I$ such that $Y = \{K_i: i \in I_0\} \cup \{(Y-U)\}$. Therefore, we obtain $U = \cup\{V_i: i \in I_0\}$. This shows every open set in $Y$ is strongly $S$-closed in $Y$. Now, for any open set $U$ by Theorem 4.4.26, $f^{-1}(U)$ is $\pi g_{a}$-closed in $X$. Hence $f$ is contra-$\pi g_{a}$-continuous.

Theorem 4.4.28: Let $f: (X,\tau) \to (Y,\sigma)$ have a contra-$\pi g_{a}$-graph. If $f$ is injective, then $X$ is $\pi g_{a}$-$T_1$.

**Proof:** Let $x$ and $y$ be two distinct points of $X$. Then $(x, f(y)) \in X \times Y - G(f)$. Then there exist $U \in \pi G_{aO}(X, x)$ and $V \in C(Y, f(y))$ such that $f(U) \cap V = \phi$. Hence $U \cap f^{-1}(V) = \phi$ implies $y \notin U$. Thus $x \in U$ and $y \notin U$. Similarly, $y \in U_1 \in \pi G_{aO}(X, x)$ and $x \notin U_1$. Hence $X$ is $\pi g_{a}$-$T_1$. 

73
Definition 4.4.29: A graph $G(f)$ of a function $f: (X, \tau) \to (Y, \sigma)$ is said to be $\pi \alpha$-regular if for each $(x, y) \in X \times Y - G(f)$, there exist a $U \in \pi G \alpha C(X, x)$ and a $V \in RO(Y, y)$ such that $(U \times V) \cap G(f) = \phi$.

Lemma 4.4.30: The following properties are equivalent for a graph $G(f)$ of a function $f$:
1. $G(f)$ is $\pi \alpha$-regular.
2. For each point $(x, y) \in X \times Y - G(f)$, there exist a $U \in \pi G \alpha C(X, x)$ and a $V \in RO(Y, y)$ such that $f(U) \cap V = \phi$.

Theorem 4.4.31: If $f: (X, \tau) \to (Y, \sigma)$ is almost contra-$\pi \alpha$-continuous and $Y$ is $T_2$, then $G(f)$ is a $\pi \alpha$-regular graph in $X \times Y$.

Proof: Let $(x, y) \in X \times Y - G(f)$. It follows that $f(x) \neq y$. Since $Y$ is $T_2$, there exist open sets $V$ and $W$ containing $f(x)$ and $y$ respectively such that $V \cap W = \phi$. We have $\text{int}(\text{cl}(V)) \cap \text{int}(\text{cl}(W)) = \phi$. Since $f$ is almost contra-$\pi \alpha$-continuous, $f^{-1}(\text{int}(\text{cl}(V)))$ is $\pi \alpha$-closed in $X$ containing $x$. Take $U = f^{-1}(\text{int}(\text{cl}(V)))$. Then $f(U) \subset \text{int}(\text{cl}(V))$. Therefore $f(U) \cap \text{int}(\text{cl}(W)) = \phi$. Hence $G(f)$ is $\pi \alpha$-regular.

Theorem 4.4.32: Let $f: (X, \tau) \to (Y, \sigma)$ have a $\pi \alpha$-regular graph $G(f)$. If $f$ is injective, then $X$ is $\pi \alpha$-$T_1$.

Proof: Let $x_1$ and $x_2$ be any two distinct points of $X$. Then we have $(x_1, f(x_2)) \in X \times Y - G(f)$. By definition, there exist a $U_1 \in \pi G \alpha C(X)$ and a $V_1 \in RO(Y)$ such that $(x_1, f(x_2)) \in U_1 \times V_1$ and $f(U_1) \cap V_1 = \phi$. That is $U_1 \cap f^{-1}(V_1) = \phi$. Therefore we have $x_2 \in X - U_1$ and $x_1 \notin X - U_1$. Similarly for $(x_2, f(x_1)) \in X \times Y - G(f)$ there exist a $U_2 \in \pi G \alpha C(X)$ and a $V_2 \in RO(Y)$ such that $x_1 \in X - U_2$ and $x_2 \notin X - U_2$. Then $X - U_1, X - U_2 \in \pi G \alpha O(X)$ implies $X$ is $\pi \alpha$-$T_1$.

Theorem 4.4.33: Let $f: (X, \tau) \to (Y, \sigma)$ have a $\pi \alpha$-regular graph $G(f)$. If $f$ is surjective, then $Y$ is weakly $T_2$.

Proof: Let $y_1$ and $y_2$ be two distinct points of $Y$. Since $f$ is surjective, $f(x_1) = y_1$ for some $x_1 \in X$ and $(x_1, y_2) \in X \times Y - G(f)$. By Lemma 4.4.30, there exist a $U_1 \in \pi G \alpha C(X)$ and an $F_1 \in RO(Y)$ such that $(x_1, y_2) \in U_1 \times F_1$ and $f(U_1) \cap F_1 = \phi$. Then $y_2 \notin Y - F_1$ and
\( y_1 \in Y - F_1 \). Similarly for \((x_2, y_1) \in X \times Y - G(f)\) there exist \( U \in \pi G_\alpha C(X, x_2) \) and \( F_2 \in RO(Y, y_1) \) such that \( f(U) \cap F_2 = \emptyset \) and \( y_2 \in Y - F_2 \) and \( y_1 \notin Y - F_2 \). This implies that \( Y \) is weakly \( T_2 \).

**Theorem 4.4.34:** If \( f : (X, \tau) \to (Y, \sigma) \) is an injective almost contra-\( \pi g\alpha \)-continuous function with a regular graph, then \( X \) is \( \pi g\alpha \)-\( T_2 \).

**Proof:** Let \( x \) and \( y \) be distinct points of \( X \). Since \( f \) is injective, we have \( f(x) \neq f(y) \). Then \((x, f(y)) \in X \times Y - G(f)\). Since \( G(f) \) is \( \pi g\alpha \)-regular, by Lemma 4.4.30, there exist \( U \in \pi G_\alpha O(X, x) \) and a regular closed set \( V \) containing \( f(y) \) such that \( f(U) \cap V = \emptyset \). Since \( f \) is almost contra-\( \pi g\alpha \)-continuous, by Theorem 4.3.2, there exist \( G \in \pi G_\alpha O(X, y) \) such that \( f(G) \subseteq V \). Therefore we have \( f(U) \cap f(G) = \emptyset \). That is, \( U \cap G = \emptyset \). Hence \( X \) is \( \pi g\alpha \)-\( T_2 \).

\[ \bullet \bullet \bullet \]