Decomposition Of $\pi g\alpha$-Sets

- Introduction
- $\pi g\alpha$-Locally Closed Sets
- $\pi G\alpha$-LC Continuous And $\pi G\alpha$-LC Irresolute Functions
- Decomposition Of $\pi g\alpha$-Continuity
CHAPTER V

DECOMPOSITION OF $\pi g_{\alpha}$-SETS

5.1 Introduction

The notion of a locally closed set in a topological space was studied by many topologists [58, 82, 171]. Thereafter Balachandran [15], Arockia Rani[6], Nasef [113] and Park [147] studied the weaker forms of locally closed sets. Noiri[129], Ganster and Reilly[59], Nashef [2] established decomposition of $\alpha$-continuity, $A$-continuity, $\alpha$-continuity and semi-continuity respectively. In this chapter, we introduce three new classes of sets called $\pi G_{\alpha}$-$LC(X,\tau)$, $\pi G_{\alpha}$-$LC^*(X,\tau)$, $\pi G_{\alpha}$-$LC^{**}(X,\tau)$ sets along with their respective continuity and irresoluteness. The notions of $C_\pi$-sets, $C_{\pi^*}$-sets and $K_\pi$-sets, $K_{\pi^*}$-sets are used to obtain decompositions of $\pi g$-continuity, $\pi g$-open maps, contra-$\pi g$-continuity and decompositions of $\pi g_{\alpha}$-continuity, $\pi g_{\alpha}$-open maps, contra-$\pi g_{\alpha}$-continuity respectively.

5.2 $\pi g_{\alpha}$-Locally Closed Sets

In this section we define $\pi g_{\alpha}$-locally closed sets which contain the class of $\alpha$-$LC$ sets and study some of their properties.

Definition 5.2.1: A subset $S$ of $(X,\tau)$ is called

a) $\pi g_{\alpha}$-locally closed (briefly a $\pi g_{\alpha}$-lc set) if $S = A \cap B$ where $A$ is $\pi g_{\alpha}$-open and $B$ is $\pi g_{\alpha}$-closed in $X$.

b) a $\pi g_{\alpha}$-lc* set if there exist a $\pi g_{\alpha}$-open set $A$ and a closed set $B$ of $X$ such that $S = A \cap B$.

c) a $\pi g_{\alpha}$-lc** set if there exist an open set $A$ and a $\pi g_{\alpha}$-closed set $B$ of $X$ such that $S = A \cap B$.

The collection of all $\pi g_{\alpha}$-lc sets, $\pi g_{\alpha}$-lc* sets and $\pi g_{\alpha}$-lc** sets of $(X,\tau)$ will be denoted by $\pi G_{\alpha}$-$LC(X,\tau), \pi G_{\alpha}$-$LC^*(X,\tau)$ and $\pi G_{\alpha}$-$LC^{**}(X,\tau)$ respectively.
Proposition 5.2.2: i) If $A \in \text{LC}(X, \tau)$, then $A \in \pi G\alpha \text{-LC}(X, \tau)$.

ii) If $A \in \text{LC}(X, \tau)$, then $A \in \pi G\alpha \text{-LC}^*(X, \tau)$ and $\pi G\alpha \text{-LC}^{**}(X, \tau)$.

iii) If $A \in \pi G\alpha \text{-LC}^*(X, \tau)$, then $A \in \pi G\alpha \text{-LC}(X, \tau)$.

iv) If $A \in \alpha \text{-LC}(X, \tau)$, then $A \in \pi G\alpha \text{-LC}(X, \tau)$.

v) If $A \in \alpha \text{-LC}^*(X, \tau)$, then $A \in \pi G\alpha \text{-LC}^*(X, \tau)$.

vi) If $A \in \alpha \text{-LC}^{**}(X, \tau)$, then $A \in \pi G\alpha \text{-LC}^{**}(X, \tau)$.

Proof: Obvious.

Remark 5.2.3: Converse of the above need not be true as seen in the following examples.

Example 5.2.4:

i) Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}\}$. Then $\text{LC}(X) = \{\phi, X, \{a\}, \{b, c, d\}\}$. $\pi G\alpha \text{-LC}(X) = \text{P}(X)$. This shows that a $\pi G\alpha$-locally closed set need not be locally closed.

ii) Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}\}$. Then $\text{LC}(X, \tau) = \{\phi, X, \{a, b\}, \{c\}\}$. Then

Example 5.2.5: a) Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, c, d\}\}$. Then

i) $\{a, b, d\} \in \pi G\alpha \text{-LC}(X, \tau)$ but $\{a, b, d\} \notin \pi G\alpha \text{-LC}^*(X, \tau)$.

ii) $\{a, b, c\} \in \pi G\alpha \text{-LC}^{**}(X, \tau)$ but $\{a, b, c\} \notin \pi G\alpha \text{-LC}^*(X, \tau)$.

b) Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. Then

i) $\{a, b\} \in \pi G\alpha \text{-LC}(X, \tau)$ but $\{a, b\} \notin \alpha \text{-LC}(X, \tau)$.

ii) $\{c\} \in \pi G\alpha \text{-LC}^*(X, \tau)$ but $\{c\} \notin \alpha \text{-LC}^*(X, \tau)$.

iii) $\{c\} \in \pi G\alpha \text{-LC}^{**}(X, \tau)$ but $\{c\} \notin \alpha \text{-LC}^{**}(X, \tau)$.

Remark 5.2.6: The above discussions are summarized in the following diagram
Proposition 5.2.7: a) Let \((X, \tau)\) be a \(\pi\alpha\)-space. Then

i) \(\pi\alpha-LC(X, \tau) = LC(X, \tau)\).

ii) \(\pi\alpha-LC(X, \tau) \subset GLC(X, \tau)\).

iii) \(\pi\alpha-LC(X, \tau) \subset \alpha-LC(X, \tau)\).

b) If \(\pi\alpha-O(X, \tau) = GO(X, \tau)\), then \(\pi\alpha-LC(X, \tau) = GLC(X, \tau)\).

c) If \(X\) is a \(\pi\alpha-T_{1/2}\) space, then \(\pi\alpha-LC(X, \tau) = \alpha-LC(X, \tau)\).

d) If \(X\) is a \(\pi\alpha\)-space, then \(\pi\alpha-LC(X, \tau) = \pi\alpha-LC*(X, \tau) = \pi\alpha-LC**(X, \tau)\).

Proof: a) i) Since every \(\pi\alpha\)-open set is open and every \(\pi\alpha\)-closed set is closed in \(X\), we have \(\pi\alpha-LC(X, \tau) \subset LC(X, \tau)\) and hence \(\pi\alpha-LC(X, \tau) = LC(X, \tau)\) .

ii) and iii) Since \(LC(X, \tau) \subset GLC(X, \tau)\) and \(LC(X, \tau) \subset \alpha-LC(X, \tau)\) for any space \(X\) and from i) the proof follows.

b) Let \(A \in \pi\alpha-LC(X, \tau)\). Then \(A = P \cap Q\) where \(P\) is \(\pi\alpha\)-open and \(Q\) is \(\pi\alpha\)-closed in \(X\). By hypothesis, \(P\) is \(g\)-open and \(Q\) is \(g\)-closed. Therefore \(A \in GLC(X, \tau)\) and \(\pi\alpha-LC(X, \tau) \subset GLC(X, \tau)\). Obviously \(GLC(X, \tau) \subset \pi\alpha-LC(X, \tau)\).

Hence \(\pi\alpha-LC(X, \tau) = GLC(X, \tau)\) .

c) Follows from definition 2.3.14 and from the fact that every \(\alpha\)-open set is \(\pi\alpha\)-open.

d) Obvious.
**Remark 5.2.8:** Converse of the above Proposition 5.2.7 (b),(c) does not hold as seen in the following example.

**Example 5.2.9:** Let \( X = \{a,b,c,d\} \), \( \tau = \{\emptyset,X,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,b,c\}\} \) then
\[
\pi G_\alpha LC(X,\tau) = \alpha LC(X,\tau) = GLC(X,\tau) = P(X).
\]
But
\[
GO(X) = \{\emptyset,X,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,b,c\}\} \neq \pi G_\alpha O(X).
\]
\[
\alpha O(X) = \{\emptyset,X,\{a\},\{b\},\{c\},\{b,d\},\{a,b,c\},\{a,b,d\},\{b,c,d\}\} \neq \pi G_\alpha O(X).
\]

**Remark 5.2.10:** For subsequent results in this chapter we assume that \( \pi G_\alpha C(X,\tau) \) is closed under finite intersections.

The hypothesis in Proposition 5.2.7 d) can be weakened as follows.

**Proposition 5.2.11:** If \( \pi G_\alpha O(X,\tau) \subset LC(X,\tau) \), then
\[
\pi G_\alpha LC(X,\tau) = \pi G_\alpha LC^*(X,\tau) = \pi G_\alpha LC^{**}(X,\tau).
\]

**Proof:** Let \( A \in \pi G_\alpha LC(X) \). Then \( A = P \cap Q \) where \( P \) is \( \pi g_\alpha \)-open and \( Q \) is \( \pi g_\alpha \)-closed. Since \( \pi G_\alpha O(X,\tau) \subset LC(X,\tau) \) implies \( \pi G_\alpha C(X,\tau) \subset LC(X,\tau) \), we have \( Q \) is locally closed. Let \( Q = M \cap N \) where \( M \) is open and \( N \) is closed. So \( A = (P \cap M) \cap N \) where \( P \cap M \) is \( \pi g_\alpha \)-open and \( N \) is closed. Hence \( A \in \pi G_\alpha LC^*(X) \). For any space \( X \), \( \pi G_\alpha LC^*(X) \subset \pi G_\alpha LC(X) \). Thus \( \pi G_\alpha LC(X) = \pi G_\alpha LC^*(X) \). Let \( B \in \pi G_\alpha LC(X) \). Then \( B = P \cap Q \) where \( P \) is \( \pi g_\alpha \)-open and \( Q \) is \( \pi g_\alpha \)-closed. Since \( \pi G_\alpha O(X,\tau) \subset LC(X,\tau) \) implies \( P \) is locally closed, we have \( P = M \cap N \) where \( M \) is open and \( N \) is closed. So \( A = M \cap (N \cap Q) \) where \( M \) is open and \( N \cap Q \) is \( \pi g_\alpha \)-closed. Hence \( B \in \pi G_\alpha LC^{**}(X) \). For any space \( X \), \( \pi G_\alpha LC^{**}(X) \subset \pi G_\alpha LC(X) \). Thus \( \pi G_\alpha LC(X,\tau) = \pi G_\alpha LC^{**}(X,\tau) \).

Now, we obtain a characterization for \( \pi G_\alpha LC^*(X,\tau) \) sets as follows:

**Theorem 5.2.12:** For a subset \( S \) of \((X,\tau)\) the following are equivalent:

1. \( S \in \pi G_\alpha LC^*(X,\tau) \).
2. \( S = P \cap cl(S) \) for some \( \pi g_\alpha \)-open set \( P \).
3. \( cl(S) - S \) is \( \pi g_\alpha \)-closed.
4. $S \cup (X-\text{cl}(S))$ is $\pi\alpha$-open.

**Proof**: 1$\Rightarrow$2: Let $S \in \pi\alpha-\text{LC}^*(X,\tau)$. Then there exist a $\pi\alpha$-open set $P$ and a closed set $F$ in $(X,\tau)$ such that $S = P \cap F$. Since $S \subseteq P$ and $S \subseteq \text{cl}(S)$, we have $S \subseteq P \cap \text{cl}(S)$.

Also, $S \subseteq F$ and $F$ is closed implies $P \cap \text{cl}(S) \subseteq P \cap F = S$. Hence $S = P \cap \text{cl}(S)$.

2$\Rightarrow$1: Since $P$ is $\pi\alpha$-open and $\text{cl}(S)$ is closed, $S = P \cap \text{cl}(S) \in \pi\alpha-\text{LC}^*(X,\tau)$.

2$\Rightarrow$3: Let $S = P \cap \text{cl}(S)$ for some $\pi\alpha$-open set $P$. We have $\text{cl}(S) - S = \text{cl}(S) \cap P^c$ which is $\pi\alpha$-closed.

3$\Rightarrow$2: Assume $\text{cl}(S) - S$ is $\pi\alpha$-closed. Let $P = X - (\text{cl}(S) - S)$. Then $P$ is $\pi\alpha$-open and $S = P \cap \text{cl}(S)$.

3$\Rightarrow$4: Let $F = \text{cl}(S) - S$. Then $F$ is $\pi\alpha$-closed, by assumption.

$X - F = X \cap \text{cl}(S) - S = S \cup (X - \text{cl}(S))$. Since $X - F$ is $\pi\alpha$-open, we have that $S \cup (X - \text{cl}(S))$ is $\pi\alpha$-open.

4$\Rightarrow$3: Let $U = S \cup (X - \text{cl}(S))$. Then $U$ is $\pi\alpha$-open. This implies

$X = X - (S \cup (X - \text{cl}(S))) = (X - S) \cap \text{cl}(S) = \text{cl}(S) - S$ is $\pi\alpha$-closed.

**Remark 5.2.13**: It is not true that $S \in \pi\alpha-\text{LC}^*(X,\tau)$ if and only if $S \subseteq \text{int}(S \cup (X - \text{cl}(S)))$. Let $S = \{b,c\}$ be a subset of the topological space $(X,\tau)$ given in Example 5.2.5(a). Then $S \not\subseteq \text{int}(S \cup (X - \text{cl}(S)))$ but $S \in \pi\alpha-\text{LC}^*(X,\tau)$.

**Definition 5.2.14**: A topological space $(X,\tau)$ is called $\pi\alpha$-submaximal if every dense subset in $(X,\tau)$ is $\pi\alpha$-open.

**Proposition 5.2.15**: a) Let $(X,\tau)$ be a topological space. If $X$ is submaximal, then it is $\pi\alpha$-submaximal.

b) A topological space $(X,\tau)$ is $\pi\alpha$-submaximal if and only if $\pi\alpha-\text{LC}^*(X,\tau) = P(X)$.

**Proof**: a) Obvious.

b) **Necessity**: Let $S \in P(X)$ and $U = S \cup (X - \text{cl}(S))$. Then $\text{cl}(U) = X$. $U$ is dense in $X$ and $X$ is $\pi\alpha$-submaximal implies $U$ is $\pi\alpha$-open. By Theorem 5.2.12, $S \in \pi\alpha-\text{LC}^*(X,\tau)$.

**Sufficiency**: Let $S$ be a dense subset of $(X,\tau)$. Then $S \cup (X - \text{cl}(S)) = S \cup \varnothing = S$. Now
$S \in \mathcal{P}(X)$ implies $S \in \pi G\alpha-LC^*(X,\tau)$. By Theorem 5.2.12, $S \cup (X-\text{cl}(S)) = S$ is $\pi g\alpha$-open. Hence $(X,\tau)$ is $\pi g\alpha$-submaximal.

Remark 5.2.16: Converse of Proposition 5.2.15 a) is not true as seen in the following example.

Example 5.2.17: Let $X = \{a,b,c\}, \tau = \{\phi, X,\{a\},\{b,c\}\}$. Let $A = \{a,b\}$. Then $A$ is dense in $X$ such that $A$ is $\pi g\alpha$-open but not open.

Proposition 5.2.18: For a subset $S$ of $(X,\tau)$ if $S \in \pi G\alpha-LC**(X,\tau)$, then there exists an open set $P$ such that $S = P \cap \text{cl}(S)$ where $\text{cl}(S)$ is the $\pi g\alpha$-closure of $S$.

Proof: Let $S \in \pi G\alpha-LC**(X,\tau)$. Then there exist an open set $P$ and a $\pi g\alpha$-closed set $F$ of $(X,\tau)$ such that $S = P \cap F$. Since $S \subset P$ and $S \subset \text{cl}(S)$, we have $S \subset P \cap \text{cl}(S)$. Since $\text{cl}(S) \subset F$, we have $P \cap \text{cl}(S) \subset P \cap F \subset S$. Thus $S = P \cap \text{cl}(S)$.

Theorem 5.2.19: Let $A$ and $B$ be any two subsets of $(X,\tau)$.

a) If $A \in \pi G\alpha-LC(X,\tau)$ and $B$ is $\pi g\alpha$-open or $\pi g\alpha$-closed, then $A \cap B \in \pi G\alpha-LC(X,\tau)$.

b) If $A \in \pi G\alpha-LC**(X,\tau)$ and $B$ is closed or open, then $A \cap B \in \pi G\alpha-LC**(X,\tau)$.

Proof: a) $A \in \pi G\alpha-LC(X,\tau)$ implies $A \cap B = (G \cap F) \cap B$ for some $\pi g\alpha$-open set $G$ and $\pi g\alpha$-closed set $F$. If $B$ is $\pi g\alpha$-open then $A \cap B = (G \cap B) \cap F \in \pi G\alpha-LC(X,\tau)$. If $B$ is $\pi g\alpha$-closed, then $A \cap B = G \cap (B \cap F) \in \pi G\alpha-LC(X,\tau)$.

b) If $A \in \pi G\alpha-LC**(X,\tau)$, then there exist an open set $G$ and a $\pi g\alpha$-closed set $F$ of $(X,\tau)$ such that $A \cap B = (G \cap F) \cap B$. If $B$ is open, then $A \cap B = (G \cap B) \cap F \in \pi G\alpha-LC**(X,\tau)$. If $B$ is closed, then $A \cap B = G \cap (F \cap B) \in \pi G\alpha-LC**(X,\tau)$.

Theorem 5.2.20: If $A \in \pi G\alpha-LC^*(X,\tau)$ and $B \in \pi G\alpha-LC^*(X,\tau)$, then $A \cap B \in \pi G\alpha-LC^*(X,\tau)$.

Proof: If $A, B \in \pi G\alpha-LC^*(X,\tau)$ then by Theorem 5.2.12, there exist $\pi g\alpha$-open sets $P$ and $Q$ such that $A = P \cap \text{cl}(A)$ and $B = Q \cap \text{cl}(B)$. $P \cap Q$ is also $\pi g\alpha$-open. Then $A \cap B = (P \cap Q) \cap (A \cap \text{cl}(A) \cap \text{cl}(B)) \in \pi G\alpha-LC^*(X,\tau)$.
Proposition 5.2.21: Let $A$ and $Z$ be any two subsets of $(X, \tau)$ and let $A \subset Z$. If $Z$ is regular open and $\pi_{\alpha}$-closed in $(X, \tau)$ and if $A \in \pi_{\alpha_{*}}LC^{*}(Z, \tau / Z)$, then $A \in \pi_{\alpha_{*}}LC^{*}(X, \tau)$.

**Proof:** If $A \in \pi_{\alpha_{*}}LC^{*}(Z, \tau / Z)$ then by Theorem 5.2.12, there is a $\pi_{\alpha}$-open set $G$ in $(Z, \tau / Z)$ such that $A = G \cap cl_{Z}(A)$ where $cl_{Z}(A) = Z \cap cl(A)$. By Proposition 2.2.19, $G$ is $\pi_{\alpha}$-open in $X$. We have $A = (G \cap Z) \cap cl(A) \in \pi_{\alpha_{*}}LC^{*}(X, \tau)$.

Remark 5.2.22: The following examples show that one of the assumptions in the above theorem. That is, $Z$ is regular open in $(X, \tau)$ cannot be removed.

Example 5.2.23: Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}, \{c, d\}, \{b, c, d\}\}$. Let $Z = A = \{a, b, d\}$.

$\tau / Z = \{\emptyset, \{b\}, \{d\}, \{b, d\}, Z\}$ where $Z$ is not regular open in $X$. Then $A \in \pi_{\alpha_{*}}LC^{*}(Z, \tau / Z)$ but $A \notin \pi_{\alpha_{*}}LC^{*}(X, \tau)$.

Theorem 5.2.24: Let $A$ and $Z$ be any two subsets of $(X, \tau)$ and let $A \subset Z$ such that $Z$ is $\pi_{\alpha}$-closed and regular open in $X$. Then

1) if $A \in \pi_{\alpha_{*}}LC(Z, \tau / Z)$, then $A \in \pi_{\alpha_{*}}LC(X, \tau)$.

2) if $A \in \pi_{\alpha_{*}}LC^{**}(Z, \tau / Z)$, then $A \in \pi_{\alpha_{*}}LC^{**}(X, \tau)$.

**Proof:** 1) Let $A \in \pi_{\alpha_{*}}LC(Z, \tau / Z)$. Then $A = G \cap F$ where $G$ is $\pi_{\alpha}$-open and $F$ is $\pi_{\alpha}$-closed in $(Z, \tau / Z)$. Then by Proposition 2.2.19, $G$ and $F$ are $\pi_{\alpha}$-open and $\pi_{\alpha}$-closed sets in $(X, \tau)$ respectively. Hence $A = G \cap F \in \pi_{\alpha_{*}}LC(X, \tau)$.

2) Let $A \in \pi_{\alpha_{*}}LC^{**}(Z, \tau / Z)$. Then $A = G \cap F$ where $G$ is open and $F$ is $\pi_{\alpha}$-closed in $(Z, \tau / Z)$. Then by Proposition 2.2.19, $G$ is open and $F$ is $\pi_{\alpha}$-closed in $(X, \tau)$. Hence $A = G \cap F \in \pi_{\alpha_{*}}LC^{**}(X, \tau)$.

Proposition 5.2.25: Let $A, B \in \pi_{\alpha_{*}}LC^{*}(X, \tau)$. If $A$ and $B$ are separated in $(X, \tau)$, then $A \cup B \in \pi_{\alpha_{*}}LC^{*}(X, \tau)$.

**Proof:** Since $A, B \in \pi_{\alpha_{*}}LC^{*}(X, \tau)$ by Theorem 5.2.12, there exist $\pi_{\alpha}$-open sets $P$ and $Q$ of $(X, \tau)$ such that $A = P \cap cl(A)$ and $B = Q \cap cl(B)$. Put $U = P \cap (X - cl(B))$ and $V = Q \cap (X - cl(A))$. Then $U$ and $V$ are $\pi_{\alpha}$-open subsets of $(X, \tau)$. Then $A = U \cup cl(A)$,
\[ B = V \cap \text{cl}(B), U \cap \text{cl}(B) = \emptyset, V \cap \text{cl}(A) = \emptyset \text{ hold. Consequently.} \]
\[ A \cup B = (U \cup V) \cap (\text{cl}(A \cup B)) \text{ showing that } A \cup B \in \pi\alpha-LC^*(X, \tau). \]

**Proposition 5.2.26**: Let \( \{Z_i : i \in A\} \) be a finite \( \pi \)-cover of \((X, \tau)\) and let \( A \) be a subset of \((X, \tau)\). If \( A \cap Z_i \in \pi\alpha-LC^*(Z_i, \tau / Z_i) \) for each \( i \in A \), then \( A \in \pi\alpha-LC^*(X, \tau) \).

**Proof**: For each \( i \in A \), there exist an open set \( U_i \in \tau \) and \( \pi\alpha\text{-closed set } F_i \) of \((Z_i, \tau / Z_i)\) such that \( A \cap Z_i = (U_i \cap F_i) \cap Z_i = U_i \cap (F_i \cap Z_i) \). Then \( A = \cup\{A \cap Z_i : i \in A\} = \left[ \cup\{U_i : i \in A\} \right] \cap \left[ \cup\{Z_i \cap F_i : i \in A\} \right] \) and hence by Proposition 2.2.10, \( A \in \pi\alpha-LC^*(X, \tau) \).

**Theorem 5.2.27**: Let \( X, Y \) be topological spaces which are \( T_\pi \)-spaces.

i) If \( A \in \pi\alpha-LC(X, \tau) \) and \( B \in \pi\alpha-LC(Y, \sigma) \), then \( A \times B \in \pi\alpha-LC(X \times Y, \tau \times \sigma) \).

ii) If \( A \in \pi\alpha-LC^*(X, \tau) \) and \( B \in \pi\alpha-LC^*(Y, \sigma) \), then \( A \times B \in \pi\alpha-LC^*(X \times Y, \tau \times \sigma) \).

iii) If \( A \in \pi\alpha-LC^{**}(X, \tau) \) and \( B \in \pi\alpha-LC^{**}(Y, \sigma) \), then \( A \times B \in \pi\alpha-LC^{**}(X \times Y, \tau \times \sigma) \).

**Proof**: i) Let \( A \in \pi\alpha-LC(X, \tau) \) and \( B \in \pi\alpha-LC(Y, \sigma) \).

Then there exist \( \pi\alpha\text{-open sets } V, V^1 \) and \( \pi\alpha\text{-closed sets } W, W^1 \) of \((X, \tau)\) and \((Y, \sigma)\) respectively such that \( A = V \cap W \) and \( B = V^1 \cap W^1 \). Then \( A \times B = (V \cap W) \times (V^1 \cap W^1) = (V \times V^1) \cap (W \times W^1) \) holds and hence \( A \times B \in \pi\alpha-LC(X \times Y, \tau \times \sigma) \).

Proofs of (ii) and (iii) are similar to that of (i).

### 5.3 \( \pi\alpha-LC \) Continuous And \( \pi\alpha-LC \) Irresolute Functions

In this section, we define \( \pi\alpha-LC \) continuous and \( \pi\alpha-LC \) irresolute functions and obtain pasting Lemma for \( \pi\alpha-LC^{**} \) continuous functions and \( \pi\alpha-LC^{**} \) irresolute functions.

**Definition 5.3.1**: A function \( f : (X, \tau) \to (Y, \sigma) \) is called

i) \( \pi\alpha-LC \) continuous if \( f^{-1}(V) \in \pi\alpha-LC(X, \tau) \) for every \( V \in \sigma \).

ii) \( \pi\alpha-LC^* \) continuous if \( f^{-1}(V) \in \pi\alpha-LC^*(X, \tau) \) for every \( V \in \sigma \).
iii) \( \pi \alpha - \text{LC}^{**} \text{continuous} \) if \( f^{-1}(V) \in \pi \alpha - \text{LC}^{**}(X, \tau) \) for every \( V \in \sigma \).

iv) \( \pi \alpha - \text{LC} \text{ irresolute} \) if \( f^{-1}(V) \in \pi \alpha - \text{LC}(X, \tau) \) for every \( V \in \pi \alpha - \text{LC}(Y, \sigma) \).

v) \( \pi \alpha - \text{LC}^{*} \text{ irresolute} \) if \( f^{-1}(V) \in \pi \alpha - \text{LC}^{*}(X, \tau) \) for every \( V \in \pi \alpha - \text{LC}^{*}(Y, \sigma) \).

vi) \( \pi \alpha - \text{LC}^{**} \text{ irresolute} \) if \( f^{-1}(V) \in \pi \alpha - \text{LC}^{**}(X, \tau) \) for every \( V \in \pi \alpha - \text{LC}^{**}(Y, \sigma) \).

**Proposition 5.3.2:** If \( f:(X, \tau) \rightarrow (Y, \sigma) \) is \( \pi \alpha - \text{LC} \text{ irresolute} \), then it is \( \pi \alpha - \text{LC} \text{ continuous} \).

**Proof:** Let \( V \) be open in \( Y \). Then \( V \in \pi \alpha - \text{LC}(Y, \sigma) \). By assumption, 
\( f^{-1}(V) \in \pi \alpha - \text{LC}(X, \tau) \). Hence \( f \) is \( \pi \alpha - \text{LC} \text{ continuous} \).

**Proposition 5.3.3:** Let \( f:(X, \tau) \rightarrow (Y, \sigma) \) be a function.

1) If \( f \) is \( \text{LC-continuous} \), then \( f \) is \( \pi \alpha - \text{LC}^{*} \text{ continuous} \) and \( \pi \alpha - \text{LC}^{**} \text{ continuous} \).

2) If \( f \) is \( \pi \alpha - \text{LC}^{*} \text{ continuous} \), then \( f \) is \( \pi \alpha - \text{LC} \text{ continuous} \).

3) If \( f \) is \( \pi \alpha - \text{LC}^{*} \text{ irresolute} \), then \( f \) is \( \pi \alpha - \text{LC}^{*} \text{ continuous} \).

**Remark 5.3.4:** Converse of the above need not be true as can be seen in the following examples.

**Examples 5.3.5:**

1) Let \( X = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} \). Let \( f:(X, \tau) \rightarrow (X, \sigma) \) be the identity mapping. Then \( f \) is \( \pi \alpha - \text{LC}^{*} \text{ continuous} \) and \( \pi \alpha - \text{LC}^{**} \text{ continuous} \) but not \( \text{LC-continuous} \).

2) Let \( X = \{a, b, c, d\}, \tau = \{\phi, X, \{b\}, \{c, d\}, \{b, c, d\}\} \). \( \sigma = \{\phi, X, \{c\}, \{a, b, d\}\} \) and \( f:(X, \tau) \rightarrow (X, \sigma) \) be the identity mapping. Then \( f \) is \( \pi \alpha - \text{LC} \text{ continuous} \) but not \( \pi \alpha - \text{LC}^{*} \text{ continuous} \) since \( \{a, b, d\} \in (X, \sigma) \) but \( \{a, b, d\} \not\in \pi \alpha - \text{LC}^{*}(X, \tau) \).

3) Let \( X = \{a, b, c, d\}, \tau = \{\phi, X, \{b\}, \{c, d\}, \{b, c, d\}\} \). \( \sigma = \{\phi, X, \{b\}\} \) and \( f:(X, \tau) \rightarrow (X, \sigma) \) be the identity mapping. Then \( f \) is \( \pi \alpha - \text{LC}^{*} \text{continuous} \) but not \( \pi \alpha - \text{LC}^{*} \text{ irresolute} \) since \( \{a, b, d\} \in \pi \alpha - \text{LC}^{*}(X, \sigma) \) but \( \{a, b, d\} \not\in \pi \alpha - \text{LC}^{*}(X, \tau) \).

**Proposition 5.3.6:** Any map defined on a door space is \( \pi \alpha - \text{LC} \text{ irresolute} \).

**Proof:** Let \( (X, \tau) \) be a door space and \( (Y, \sigma) \) be any space. Define a map \( f:(X, \tau) \rightarrow (Y, \sigma) \). Let \( A \in \pi \alpha - \text{LC}(Y, \sigma) \). Then \( f^{-1}(A) \) is either open or closed in \( (X, \tau) \). In both cases \( f^{-1}(A) \in \pi \alpha - \text{LC}(X, \tau) \). Hence \( f \) is \( \pi \alpha - \text{LC} \text{ irresolute} \).
Theorem 5.3.7: A topological space \((X, \tau)\) is \(\pi\alpha\)-submaximal if and only if every function having \((X, \tau)\) as its domain is \(\pi\alpha\)-LC* continuous.

Proof: Suppose that \(f:(X, \tau) \rightarrow (Y, \sigma)\) is a function. By Theorem 5.2.15 b), we have \(f^{-1}(V) \in P(X) = \pi\alpha\)-LC\((X, \tau)\) for each open set \(V\) of \((Y, \sigma)\). Therefore \(f\) is \(\pi\alpha\)-LC* continuous. Conversely, let every map having \((X, \tau)\) as its domain be \(\pi\alpha\)-LC* continuous. Let \(Y = \{0, 1\}\) be the Sierpinski space with topology \(\sigma = \{Y, \emptyset, \{0\}\}\). Let \(V\) be a subset of \((X, \tau)\) and \(f:(X, \tau) \rightarrow (Y, \sigma)\) be a function defined by \(f(x) = 0\) for every \(x \in V\) and \(f(x) = 1\) for every \(x \notin V\). By assumption, \(f\) is \(\pi\alpha\)-LC* continuous and hence \(f^{-1}\{0\} = V \in \pi\alpha\)-LC\((X, \tau)\). Therefore we have \(P(X) = \pi\alpha\)-LC\((X, \tau)\) and by Theorem 5.2.15 b), \((X, \tau)\) is \(\pi\alpha\)-submaximal.

Proposition 5.3.8: If \(f:(X, \tau) \rightarrow (Y, \sigma)\) is \(\pi\alpha\)-LC** continuous and a subset \(B\) is regular open, \(\pi\alpha\)-closed in \((X, \tau)\), then the restriction of \(f\) to \(B\) say \(f|B: (B, \tau|B) \rightarrow (Y, \sigma)\) is \(\pi\alpha\)-LC** continuous.

Proof: Let \(V\) be an open set of \((Y, \sigma)\). Then \(f^{-1}(V) = G \cap F\) for some open set \(G\) and \(\pi\alpha\)-closed set \(F\) of \((X, \tau)\). Now \(G \cap B \subseteq \tau|B\) and \((F \cap B)\) is a \(\pi\alpha\)-closed subset of \((B, \tau|B)\). But \((f|B)^{-1}(V) = (G \cap B) \cap (F \cap B)\). Hence \((f|B)^{-1}(V) \in \pi\alpha\)-LC\((B, \tau|B)\). This implies that \(f|B\) is \(\pi\alpha\)-LC** continuous.

We recall the definition of the combination of two functions: Let \(X = A \cup B\) and \(f: A \rightarrow Y\) and \(h: B \rightarrow Y\) be two functions. We say that \(f\) and \(h\) are compatible if \(f|A \cap B = h|A \cap B\). If \(f: A \rightarrow Y\) and \(h: B \rightarrow Y\) are compatible, then the function \(f \vee h: X \rightarrow Y\) defined as \((f \vee h)(x) = f(x)\) for every \(x \in A\), \((f \vee h)(x) = h(x)\) for every \(x \in B\) is called the combination of \(f\) and \(h\).

Pasting Lemma for \(\pi\alpha\)-LC** continuous (resp. \(\pi\alpha\)-LC**-irresolute) functions.

Theorem 5.3.9: Let \(X = A \cup B\), where \(A\) and \(B\) are \(\pi\alpha\)-closed and regular open subsets of \((X, \tau)\) and \(f: (A, \tau|A) \rightarrow (Y, \sigma)\) and \(h: (B, \tau|B) \rightarrow (Y, \sigma)\) be compatible functions.

a) If \(f\) and \(h\) are \(\pi\alpha\)-LC** continuous, then \((f \vee h): X \rightarrow Y\) is \(\pi\alpha\)-LC** continuous.

b) If \(f\) and \(h\) are \(\pi\alpha\)-LC** irresolute, then \((f \vee h): X \rightarrow Y\) is \(\pi\alpha\)-LC** irresolute.
Proof: \( a) \) Let \( V \in \sigma \). Then \((f \vee h)^{-1}(V) \cap A = f^{-1}(V) \) and \((f \vee h)^{-1}(V) \cap B = h^{-1}(V)\). By assumption, \((f \vee h)^{-1}(V) \cap A \in \pi \alpha-LC^*(A, \tau/A)\) and \((f \vee h)^{-1}(V) \cap B \in \pi \alpha-LC^*(B, \tau/B)\). Therefore by Proposition 5.2.26, \((f \vee h)^{-1}(V) \in \pi \alpha-LC^*(X, \tau)\) and hence \( f \vee h \) is \( \pi \alpha-LC^* \)-continuous.

\( b) \) Proof is similar to that of \( a) \).

Next we have the theorem concerning the composition of functions.

**Theorem 5.3.10**: Let \( f: (X, \tau) \to (Y, \sigma) \) and \( g: (Y, \sigma) \to (Z, \eta) \) be two functions. Then

\( a) \) \( g \circ f \) is \( \pi \alpha-LC \)- irresolute if \( f \) and \( g \) are \( \pi \alpha-LC \)- irresolute.

\( b) \) \( g \circ f \) is \( \pi \alpha-LC^* \)- irresolute if \( f \) and \( g \) are \( \pi \alpha-LC^* \)- irresolute.

\( c) \) \( g \circ f \) is \( \pi \alpha-LC^* \)- irresolute if \( f \) and \( g \) are \( \pi \alpha-LC^* \)- irresolute.

\( d) \) \( g \circ f \) is \( \pi \alpha-LC \)- continuous if \( f \) is \( \pi \alpha-LC \)- irresolute and \( g \) is \( \pi \alpha-LC \)- continuous.

\( e) \) \( g \circ f \) is \( \pi \alpha-LC^* \)- continuous if \( f \) is \( \pi \alpha-LC^* \)- continuous and \( g \) is continuous.

\( f) \) \( g \circ f \) is \( \pi \alpha-LC^* \)- continuous if \( f \) is \( \pi \alpha-LC^* \)- continuous and \( g \) is \( \pi \alpha-LC^* \)- continuous.

**Definition 5.3.11**: A function \( f: (X, \tau) \to (Y, \sigma) \) is called sub \( \pi \alpha-LC^* \)-continuous if there exists a basis \( B \) for \( (Y, \sigma) \) such that \( f^{-1}(U) \in \pi \alpha-LC^*(X, \tau) \) for each \( U \in B \).

**Proposition 5.3.12**: Let \( f: (X, \tau) \to (Y, \sigma) \) be a function.

\( a) \) \( f \) is sub-\( \pi \alpha-LC^* \)-continuous if and only if there is a subbasis \( C \) of \( (Y, \sigma) \) such that \( f^{-1}(U) \in \pi \alpha-LC^*(X, \tau) \) for each \( U \in C \).

\( b) \) If \( f \) is sub-LC-continuous, then \( f \) is sub-\( \pi \alpha-LC^* \)-continuous.

**Proof**: \( a) \) By assumption, there exists a basis \( B \) for \( (Y, \sigma) \) such that \( f^{-1}(U) \in \pi \alpha-LC^*(X, \tau) \) for each \( U \in B \). Since \( B \) is also a subbasis for \( (Y, \sigma) \), the proof is obvious.

Conversely, for a subbasis \( C \), let \( C_\delta = \{ A \subset Y : A \text{ is an intersection of finitely many sets belonging to } C \} \). Then \( C_\delta \) is a basis for \( (Y, \sigma) \). For \( U \in C_\delta \), \( U = \cap \{ F_i : F_i \in C, i \in \Lambda \} \) where...
\( \Lambda \) is a finite set. By assumption and Proposition 5.2.20, we have

\[
\Gamma^{-1}(U) = \bigcap \{\Gamma^{-1}(F_i) : i \in \Lambda \} \in \pi G\alpha-LC^*(X,\tau).
\]

b) follows from the Definition 5.3.11 and the fact that every LC \((X,\tau)\) is \(\pi G\alpha-LC^*(X,\tau)\).

**Remark 5.3.13:** Converse of Proposition 5.3.12 a) is not true as seen in the following example.

**Example 5.3.14:** Let \( X = Y = \{a,b,c\} \), \( \tau = \{\phi, X, \{a\}\} \) and \( \sigma \) be the topology induced by a base \( B \) of \( Y \). Let \( f : (X,\tau) \to (Y,\sigma) \) be the identity function. If \( B = \{Y,\{c\}\} \), then \( f \) is sub-\(\pi G\alpha-LC^*\) continuous but not sub LC-continuous since \( f^{-1}(\{c\}) = \{c\} \notin LC(X,\tau) \).

### 5.4 Decomposition Of \(\pi g\alpha\)-Continuity

In this section, we introduce the notions of \( C_\pi \)-sets, \( C_\pi^* \)-sets, \( K_\pi \)-sets and \( K_\pi^* \)-sets to obtain decompositions of \(\pi g\alpha\)-continuity and \(\pi g\alpha\)-continuity.

**Definition 5.4.1:** A subset \( S \) of \((X,\tau)\) is called a

1. \( C_\pi \)-set if \( S = G \cap F \) where \( G \) is \(\pi g\)-open and \( F \) is a t-set
2. \( C_\pi^* \)-set if \( S = G \cap F \) where \( G \) is \(\pi g\)-open and \( F \) is a \(\alpha^*\)-set.
3. \( K_\pi \)-set if \( S = G \cap F \) where \( G \) is \(\pi g\alpha\)-open and \( F \) is a t-set.
4. \( K_\pi^* \)-set if \( S = G \cap F \) where \( G \) is \(\pi g\alpha\)-open and \( F \) is a \(\alpha^*\)-set.

**Proposition 5.4.2:**

1. Every B set is a \( C_\pi \)-set.
2. Every B- set is a \( C_\pi^* \)-set
3. Every C- set is a \( C_\pi \)-set.
4. Every C- set is a \( C_\pi^* \)-set.
5. Every \( C_\pi \)-set is a \( C_\pi^* \)-set.
6. Every \( C_\pi \)-set is a \( K_\pi \)-set.
7. Every $C_\pi$-set is a $K_\pi$-set.
8. Every $C_\tau$-set is a $C_\tau$-set.
9. Every $C_\eta$-set is a $C_\tau$-set.
10. Every $C_\pi$-set is a $K_\pi$-set.
11. Every $K_\pi$-set is a $K_\pi$-set.

**Remark 5.4.3:** Converse of the above need not be true as seen in the following examples.

**Example 5.4.4:** Let $X = \{a,b,c\}, \tau = \{\emptyset, X, \{a,b\}\}$. Let $A = \{a,c\}$.
Then $A$ is a $C_\pi$-set and $C_\eta$-set. But $A$ is neither a $B$-set nor a $C$-set.

**Example 5.4.5:** Let $X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{a\}, \{b,c\}, \{a,b,c\}\}$. Let $A = \{c,d\}$. Then $A$ is a $C_\tau$-set, $C_\eta$-set, $C_\pi$-set and $K_\pi$-set. But $A$ is neither a $C_\pi$-set nor a $K_\pi$-set.

**Example 5.4.6:** Let $X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{a\}, \{c,d\}, \{a,c,d\}, \{d\}, \{a,d\}\}$. Let $A = \{a,b,d\}$. Then $A$ is a $K_\pi$-set and $K_\eta$-set. But $A$ is neither $C_\pi$-set, nor $C_\pi^*$-set, nor $C_\tau$-set, nor $C_\pi$-set, nor $C$-set.

**Remark 5.4.7:** $K_\pi$-set and $C_\pi$-set are independent concepts follows from Examples 5.4.6 and 5.4.5 respectively.

**Remark 5.4.8:** $K_\pi$-set and $C_\pi$-set are independent concepts follow from Examples 5.4.6 and 5.4.5 respectively.

**Proposition 5.4.9:** If $S$ is a $\pi\alpha$-open set, then
i) $S$ is a $K_\pi$-set.
ii) $S$ is a $K_\tau^*$-set.

**Remark 5.4.10:** Converse of the above need not be true as seen in the following example.
Example 5.4.11: Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). Then \( A = \{c\} \) is a \( K_T \)-set and a \( K_{\pi} \)-set but not \( \pi g\alpha \)-open.

The above discussions are summarized in the following diagram:

```
\[ \text{C-set} \]
\[ \text{Cr-set} \]
\[ \text{B-set} \]
\[ \text{K}_{\pi} \text{-set} \]
\[ \text{C}_{\pi} \text{-set} \]
\[ \text{C}_{\pi^*} \text{-set} \]
\[ \text{K}_{\pi^*} \text{-set} \]
```

Proposition 5.4.12: Let \( A \) and \( B \) be \( K_{\pi} \)-sets in \( X \). Then \( A \cap B \) is a \( K_{\pi} \)-set in \( X \).

Proof: Since \( A \) and \( B \) are \( K_{\pi} \)-sets, \( A = G_1 \cap F_1 \) and \( B = G_2 \cap F_2 \) where \( G_1 \) and \( G_2 \) are \( \pi g\alpha \)-open and \( F_1 \) and \( F_2 \) are t-sets. Since intersection of two \( \pi g\alpha \)-open sets is \( \pi g\alpha \)-open and intersection of t-sets is a t-set, it follows that \( A \cap B \) is a \( K_{\pi} \)-set in \( X \).

Remark 5.4.13: a) The union of two \( K_{\pi} \)-sets need not be a \( K_{\pi} \)-set.

b) Complement of a \( K_{\pi} \)-set need not be a \( K_{\pi} \)-set.

Example 5.4.14: In Example 5.4.5

a) \( A = \{a, c\} \) and \( B = \{d\} \) are \( K_{\pi} \)-sets. \( A \cup B = \{a, c, d\} \) is not a \( K_{\pi} \)-set.
b) $X - \{a,c\} = \{b,d\}$ is not a $K_x$-set.

Proposition 5.4.15: Let $A$ and $B$ be $C_x$-sets in $X$. Then $A \cap B$ is $C_x$-set in $X$.

Remark 5.4.16: The union of two $C_x$-sets need not be a $C_x$-set and the complement of a $C_x$-set need not be a $C_x$-set follows from Example 5.4.14.

Definition 5.4.17: A function $f: X \to Y$ is said to be
i) $C_x$-continuous if $f^{-1}(V)$ is a $C_x$-set for every open set $V$ in $Y$.
ii) $C_x$-continuous if $f^{-1}(V)$ is a $C_x$-set for every open set $V$ in $Y$.
iii) $K_x$-continuous if $f^{-1}(V)$ is a $K_x$-set for every open set $V$ in $Y$.
iv) $K_x$-continuous if $f^{-1}(V)$ is a $K_x$-set for every open set $V$ in $Y$.

Proposition 5.4.18: i) Every $C_x$-continuous function is $C_x$-continuous.
ii) Every $C_x$-continuous function is $K_x$-continuous.
iii) Every $K_x$-continuous function is $C_x$-continuous.
iv) Every $K_x$-continuous function is $K_x$-continuous.

Proof: Follows from Proposition 5.4.2 and Definition 5.4.17.

Remark 5.4.19: Converse of the above need not be true as can be seen from the following examples.

Example 5.4.20: a) Let $X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{a\}, \{b,c\}, \{a,b,c\}\}$, $\sigma = \{\emptyset, X, \{c,d\}\}$ and $f:(X, \tau) \to (X, \sigma)$ be the identity mapping. Then $f$ is $C_x$-continuous but not $C_x$-continuous.

b) Let $X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{a\}, \{c,d\}, \{a,c,d\}, \{d\}, \{a,d\}\}$, $\sigma = \{\emptyset, \{a,b,d\}, X\}$ and $f:(X, \tau) \to (X, \sigma)$ be the identity mapping. Then $f$ is $K_x$-continuous but not $C_x$-continuous.
c) Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X\}$, $\sigma = \{\emptyset, \{c\}, \{c,d\}, X\}$ and $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Then $f$ is $K^*_\tau$-continuous but not $K^*_\sigma$-continuous.

d) Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{a\}, \{a,c,d\}, \{c,d\}, \{a,d\}\}$, $\sigma = \{\emptyset, \{a\}, \{a,b,d\}, X\}$ and $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Then $f$ is $K^*_\sigma$-continuous but not $C^*_\sigma$-continuous.

Remark 5.4.21: The above discussions are summarized in the following implications:

\[
C^*_\tau\text{-continuity} \Rightarrow C^*_\sigma\text{-continuity} \\
K^*_\tau\text{-continuity} \Rightarrow K^*_\sigma\text{-continuity}
\]

Definition 5.4.22: A map $f : X \to Y$ is said to be

i) $K^*_\tau$-open if $f(U)$ is a $K^*_\tau$-set in $Y$ for each open set $U$ in $X$.

ii) $C^*_\tau$-open if $f(U)$ is a $C^*_\tau$-set in $Y$ for each open set $U$ in $X$.

iii) $C^*_\sigma$-open if $f(U)$ is a $C^*_\sigma$-set in $Y$ for each open set $U$ in $X$.

iv) $K^*_\sigma$-open if $f(U)$ is a $K^*_\sigma$-set in $Y$ for each open set $U$ in $X$.

Definition 5.4.23: A map $f : X \to Y$ is said to be

i) contra-$K^*_\tau$-continuous if $f^{-1}(V)$ is a $K^*_\tau$-set for every closed set $V$ in $Y$.

ii) contra-$C^*_\tau$-continuous if $f^{-1}(V)$ is a $C^*_\tau$-set for every closed set $V$ in $Y$.

iii) contra-$C^*_\sigma$-continuous if $f^{-1}(V)$ is a $C^*_\sigma$-set for every closed set $V$ in $Y$.

iv) contra-$K^*_\sigma$-continuous if $f^{-1}(V)$ is a $K^*_\sigma$-set for every closed set $V$ in $Y$.

Lemma 5.4.24: A subset $A$ of a space $X$ is

a) $\pi g$-open if and only if $F \subseteq \text{int}(A)$ whenever $F$ is $\pi$-closed and $F \subseteq A$ [42].

b) $\pi g p$-open if and only if $F \subseteq \text{pint}(A)$ whenever $F$ is $\pi$-closed and $F \subseteq A$ [146].

Theorem 5.4.25: A subset $S$ of $X$ is

a) $\pi g$-open if and only if it is both $\pi g p$-open and a $C^*_\tau$-set in $X$. 

91
b) \( \pi_{g}\)-open if and only if it is both \( \pi_{g}\alpha\)-open and a \( C_{\pi}\)-set in \( X \).

c) \( \pi_{g}\)-open if and only if it is both \( \pi_{g}\alpha\)-open and a \( C_{\pi}\)-set in \( X \).

**Proof :**  
**a) Necessity:** Obvious.

**Sufficiency:** Assume that \( S \) is both \( \pi_{g}\)-open and a \( C_{\pi}\)-set in \( X \). By assumption, \( S \) is a \( C_{\pi}\)-set in \( X \) implies \( S = A \cap B \) where \( A \) is \( \pi_{g}\)-open and \( B \) is a \( t\)-set. Let \( F \) be a \( \pi\)-closed set such that \( F \subset S \). Since \( S \) is \( \pi_{g}\)-open, \( F \subset S \) implies \( F \subset \text{pint}(S) \subset \text{int}(B) \). Then \( A \) is \( \pi_{g}\)-open and \( F \subset S \subset A \) implies \( F \subset \text{int}(A) \). Hence \( F \subset \text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B) = \text{int}(S) \). Hence \( S \) is \( \pi_{g}\)-open.

b) **Necessity:** Obvious.

**Sufficiency:** Let \( S \) be both \( \pi_{g}\alpha\)-open and a \( C_{\pi}\)-set in \( X \). Since \( S \) is a \( C_{\pi}\)-set, \( S = A \cap B \) where \( A \) is \( \pi_{g}\)-open and \( B \) is a \( t\)-set. Let \( F \) be a \( \pi\)-closed set such that \( F \subset S \). Since \( S \) is \( \pi_{g}\alpha\)-open, \( F \subset S \) implies \( F \subset \alpha\text{int}(S) \subset \text{int}(B) \). Then \( A \) is \( \pi_{g}\)-open and \( F \subset S \subset A \) implies \( F \subset \text{int}(A) \). Hence \( F \subset \text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B) = \text{int}(S) \).

c) **Necessity:** Obvious.

**Sufficiency:** Assume \( S \) is both \( \pi_{g}\alpha\)-open and a \( C_{\pi}\)-set in \( X \). Since \( S \) is a \( C_{\pi}\)-set, \( S = A \cap B \) where \( A \) is \( \pi_{g}\)-open and \( B \) is \( \alpha^{\pi}\)-set in \( X \). Let \( F \) be a \( \pi\)-closed set such that \( F \subset S \). Since \( S \) is \( \pi_{g}\alpha\)-open, \( F \subset S \) implies \( F \subset \alpha\text{int}(S) \subset \text{int}(B) \). Then \( A \) is \( \pi_{g}\)-open and \( F \subset S \subset A \) implies \( F \subset \text{int}(A) \). Hence \( F \subset \text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B) = \text{int}(S) \).

**Theorem 5.4.26:** A mapping \( f : X \rightarrow Y \) is

a) \( \pi_{g}\)-continuous if and only if it is both \( \pi_{g}\)-continuous and \( C_{\pi}\)-continuous .

b) \( \pi_{g}\)-continuous if and only if it is both \( \pi_{g}\alpha\)-continuous and \( C_{\pi}\)-continuous .

c) \( \pi_{g}\)-continuous if and only if it is both \( \pi_{g}\alpha\)-continuous and \( C_{\pi}^{\pi}\)-continuous .

**Proof :** Follows from Theorem 5.4.25.

**Theorem 5.4.27:** A map \( f : X \rightarrow Y \) is

a) \( \pi_{g}\)-open if and only if it is both \( \pi_{g}\)-open and \( C_{\pi}\)-open .

b) \( \pi_{g}\)-open if and only if it is both \( \pi_{g}\alpha\)-open and \( C_{\pi}\)-open .

c) \( \pi_{g}\)-open if and only if it is both \( \pi_{g}\alpha\)-open and \( C_{\pi}^{\pi}\)-open .

92
**Theorem 5.4.28**: A mapping $f : X \to Y$ is

a) contra-$\pi g$-continuous if and only if $f$ is both contra-$\pi gp$-continuous and contra-$C_\pi$-continuous.

b) contra-$\pi g$-continuous if and only if $f$ is both contra-$\pi g\alpha$-continuous and contra-$C_\pi$-continuous.

c) contra-$\pi g$-continuous if and only if $f$ is both contra-$\pi g\alpha$-continuous and contra-$C_\pi$-continuous.

**Proof**: Follows from Theorem 5.4.25.

**Lemma 5.4.29**: [155] Let $A$ and $B$ be subsets of a space $X$. If $B$ is an $\alpha^*$-set, then

$\alpha\text{int}(A \cap B) = \alpha\text{int}(A) \cap \text{int}(B)$.

**Theorem 5.4.30**: A subset $S$ of $X$ is

a) $\pi g\alpha$-open if and only if it is both $\pi gp$-open and a $K_\pi$-set.

b) $\pi g\alpha$-open if and only if it is both $\pi gp$-open and a $K_\pi^*$-set.

**Proof: a) Necessity**: Let $S$ be $\pi g\alpha$-open. For any subset $A$ of $X$,

$\text{int}(A) \subset \alpha\text{int}(A) \subset \pi\text{int}(A)$.

Let $F$ be a $\pi$-closed set such that $F \subset S$. Since $S$ is $\pi g\alpha$-open, $F \subset S$ implies $F \subset \alpha\text{int}(S) \subset \pi\text{int}(S)$ which implies $S$ is $\pi gp$-open. Since $S = S \cap X$ where $S$ is $\pi g\alpha$-open and $X$ is a t-set, $S$ is a $K_\pi$-set.

**Sufficiency**: Let $S$ be both $\pi gp$-open and a $K_\pi$-set. Since $S$ is a $K_\pi$-set, $S = A \cap B$ where $A$ is $\pi g\alpha$-open and $B$ is a t-set. Let $F$ be a $\pi$-closed set such that $F \subset S$. Since $S$ is $\pi gp$-open, $F \subset S$ implies $F \subset \pi\text{int}(S) = S \cap \text{int}(\text{cl}(S)) \subset \text{int}(B)$. Then $A$ is $\pi g\alpha$-open and $F \subset S \subset A$ implies $F \subset \alpha\text{int}(A)$. Therefore $F \subset \alpha\text{int}(A) \cap \text{int}(B) \subset \alpha\text{int}(A \cap B) \subset \alpha\text{int}(S)$.

**b) Proof**: Similar as that of (a).

**Theorem 5.4.31**: A map $f : X \to Y$ is

a) $\pi g\alpha$-continuous if and only if it is both $\pi gp$-continuous and $K_\pi$-continuous.
b) \(\pi g\alpha\)-continuous if and only if it is both \(\pi gp\)-continuous and \(K_{s-}\)-continuous

**Proof:** Follows from Theorem 5.4.30.

**Theorem 5.4.32:** A map \(f: X \to Y\) is

a) \(\pi g\alpha\)-open if and only if it is both \(\pi gp\)-open and \(K_{s-}\)-open

b) \(\pi g\alpha\)-open if and only if it is both \(\pi gp\)-open and \(K_{s-}\)-open

**Proof:** Follows from Theorem 5.4.30.

**Theorem 5.4.33:** A map \(f: X \to Y\) is

a) contra-\(\pi g\alpha\)-continuous if and only if it is both contra-\(\pi gp\)-continuous and contra-\(K_{s-}\)-continuous.

b) contra-\(\pi g\alpha\)-continuous if and only if it is both contra-\(\pi gp\)-continuous and contra-\(K_{s-}\)-continuous.

**Proof:** Follows from Theorem 5.4.30

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