Chapter 6
Chapter 6

NONLOCAL PROBLEMS FOR DELAY INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

6.1 Introduction

The nonlocal Cauchy problem was first considered by Byszewski. In L.Byszewski [18] discussed the theorems about the existence and uniqueness of the solutions of a semilinear equation with nonlocal conditions. In [14] M.Benchohra, S.K.Ntouyas studied the controllability results for semilinear evolution inclusions with nonlocal conditions. K.Deng [30] discussed about the exponential delay of solutions of semilinear parabolic equations with nonlocal initial conditions. Y.Lin and J.H. Liu [69] studied the solutions of the semilinear integrodifferential equations with nonlocal Cauchy problem. An existence and mild solution to nonautonomous integrodifferential equations with nonlocal conditions in a Banach space has been done by Lin and Ezzinbi [64]. In addition J.Liang, J.H. Liu [70] discussed the existence of solution of nonlocal Cauchy problems governed by compact operator families. In [80] S.K.Ntouyas, P.Ch. Tsamatos proved the global existence for semilinear evolution
equation with nonlocal conditions. In [109] Zuomao Yan studied the existence mild solution for nonlocal problems for delay integrodifferential equations in Banach spaces. J. Pruss [88] studied the Volterra type equations through resolvent operators. In [43, 44], R. Grimmer studied the integrodifferential equations through integrated semigroup under using resolvent operators.

In this chapter we will investigate the existence of mild solution for the following first order delay integrodifferential equations with nonlocal conditions

$$x'(t) = A(t)(x(t) + \int_0^t H(t, s)(x(s))ds) + f(t, x(\sigma_1(s)), \int_0^t K(t, s, x(\sigma_2(s)))ds), \quad t \in J$$  \hspace{1cm} (6.1.1)

with non local conditions

$$x(0) + g(x) = x_0$$  \hspace{1cm} (6.1.2)

Where $J = [0, b]$, the unknown function $x(\cdot)$ takes the values in the Banach space $X$ and $x_0 \in X$. Here $A(t)$ is a closed linear operator on $X$ with dense domain $D(A)$ which is independent of $t$. $H(t, s); t, s \in J$, is a bounded operator in $X$. $f : J \times X \times X \to X; g : C(J; X) \to X$ and $K : J \times J \times X \to X$ are given functions.

### 6.2 Preliminaries and Hypotheses

Let $(X, \| \cdot \|)$ be a Banach space $C(J, X)$ in the Banach space of continuous functions from $J$ to $X$ with the norm

$$\|x\|_\infty = \sup \left\{ \|x(t)\| : t \in J \right\}$$

and $B(X)$ denotes the Banach space of bounded linear operators from $X$ to $X$.

A measurable function $x : J \to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable $D(J; X)$ denotes the Banach space of measurable function $x : J \to X$ which are
Bochner integrable normed by
\[ \|x\|_D = \int_0^b \|x(t)\|dt \quad \text{for all } x \in L^2(t) \]

**Definition 6.2.1.** A resolvant operator for (6.1.1) – (6.1.2) is a bounded operator valued function \( R(t, s) \in B(X) : 0 \leq s \leq t \leq b \), the space of bounded linear operator on \( X \) having the following properties:

(i) \( R(t, s) \) is strongly continuous in \( s \) and \( t \). \( R(t, t) = I \), the identity operator on \( X \)

\[ \|R(t, s)\| \leq Me^{\lambda(t-s)} \] is strongly continuous in \( s, t \) on \( X \).

(ii) \( R(t, s)Y \subseteq Y \). \( R(t, s) \) is strongly continuous in \( s, t \) on \( Y \).

(iii) For \( y \in Y \), \( R(t, s)y \) is continuously differentiable in \( s \) and \( t \) and for \( 0 \leq s \leq t \leq b \)

\[ \frac{\partial R(t, s)}{\partial t} y = A(t)R(t, s)y + \int_s^t H(t, \tau)R(\tau, s)A(s)y d\tau \]

\[ \frac{\partial R(t, s)}{\partial s} y = -R(t, s)A(s)y - \int_s^t H(t, \tau)A(\tau)R(\tau, s)y d\tau \]

with \( \frac{\partial R(t, s)}{\partial t} y \) and \( \frac{\partial R(t, s)}{\partial s} y \) are strongly continuous on \( 0 \leq s \leq t \leq b \).

Here \( R(t, s) \) can be extracted from the evolution operator of the generator \( A(t) \). The resolvant operator is similar to the evolution operator for nonautonomous differential equations in Banach space.

**Definition 6.2.2.** A continuous function \( x(t) \) is said to be a mild solution of the nonlocal Cauchy problem (6.1.1) – (6.1.2) if

\[ x(t) = R(t, 0)[x_0 - g(x)] + \int_0^t R(t, s)f(s, x(\sigma_1(s))), \int_0^s K(s, \tau, x(\sigma_2(\tau))d\tau)ds \]

is satisfied.

We need the following fixed point theorem due to Schaefer’s [6-d].

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Theorem 6.2.1. Let $E$ be a normed linear space, let $f : E \to E$ be a completely continuous operator, that is: it is continuous and the image of any bounded set is contained in a compact set and let

$$Z(f) = \left\{ x \in E : x = \lambda f x \text{ for some } 0 < \lambda < 1 \right\}.$$ 

Then either $Z(f)$ is unbounded or $f$ has a fixed point.

We assume the following hypotheses:

$(H_1)$ There exists a resolvent operator $R(t, s)$ which is compact and continuous in the uniform operator topology for $t > s$. Further, there exists a constant $M_1 > 0$ such that

$$\|R(t, s)\| \leq M_1.$$ 

$(H_2)$ For each $t \in J$, the function $f(t, \cdot, \cdot) : X \times X \to X$ is continuous and for each $x \in X$ and the function $f(\cdot, x(\sigma_1(t)), \int_0^x K(t, s, x(\sigma_2(x)))ds) : J \to X$ is strongly measurable.

$(H_3)$ There exist an integrable function $k_1 : J \times J \to [0, \infty)$ such that

$$\|K(t, s, x)\| \leq k_1(t, s)\Omega_0(\|x\|)$$

for every $t, s \in J : x \in X$. Where $\Omega_0 : [0, \infty) \to [0, \infty)$ is a continuous nondecreasing function.

$(H_4)$ There exists an integrable function $k_2 : J \to [0, \infty)$ such that

$$\|f(t, x, y)\| \leq k_2(t)\Omega_1(\|x\| + \|y\|)$$

for any $t \in J : x, y \in X$. Where $\Omega_1 : [0, \infty) \to [0, \infty)$ is a continuous non decreasing function.

$(H_5)$ The function $g : C(J : X) \to X$ is completely continuous and there exists a constant $M_2 > 0$ such that $\|g(x)\| \leq M_2$ for $x \in X$. 

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(H6) The function
\[ m(t) = \max \left\{ M_1, k_2(t), k_1(t, t), \int_0^t \frac{\partial}{\partial t} k_1(t, s) ds \right\} \]
satisfies
\[ \int_0^b \tilde{m}(s) ds < \int_c^\infty \frac{ds}{2\Omega_0(s) + \Omega_1(s)} \]
where \( c = M_1[\|x_0\| + M_2] \).

6.3 EXISTENCE OF SOLUTION

Theorem 6.3.1. If the hypotheses \((H_1) - (H_6)\) are satisfied then the problem \((6.1.1) - (6.1.2)\)
has a mild solution on \(J\).

Proof. Consider the Banach space \(Z = C(J; X)\). We establish the existence of a mild solution of the problem \((6.1.1) - (6.1.2)\) by applying the Schaefer's fixed point theorem. Let us execute a priori bounds for the operator equation.

\[ x(t) = \lambda \Phi_x(t), \quad 0 < \lambda < 1. \tag{6.3.1} \]

where \( \Phi : Z \to Z \) is defined as

\[ \Phi_x(t) = R(t, 0)[x_0 - g(x)] + \int_0^t R(t, s)f(s, x(\sigma_1(s)), \int_0^s K(s, \tau, x(\sigma_2(\tau))) d\tau) ds \tag{6.3.2} \]

from (6.3.1) and (6.3.2) we have

\[ \|x(t)\| = \|\lambda \Phi_x(t)\| \]

\[ \leq \|R(t, 0)[x_0 - g(x)]\| + \int_0^t R(t, s)f(s, x(\sigma_1(s)), \int_0^s K(s, \tau, x(\sigma_2(\tau))) d\tau) ds \|
\]

\[ \leq \|R(t, 0)[x_0 - g(x)]\| + \| \int_0^t R(t, s)f(s, x(\sigma_1(s)), \int_0^s K(s, \tau, x(\sigma_2(\tau))) d\tau) ds \|
\]

\[ \leq M_1 \|x\| + M_2 + M_1 \int_0^t k_2(s) \Omega_1(\|x(s)\|) + \int_0^s k_1(s, \tau) \Omega_0(\|x(\tau)\|) d\tau) ds \]

Let \( v(t) = M_1 \|x\| + M_2 + M_1 \int_0^t k_2(s) \Omega_1(\|x(s)\|) + \int_0^s k_1(s, \tau) \Omega_0(\|x(\tau)\|) d\tau) ds \)
Then we have

\[ \|x(t)\| \leq v(t) \quad \text{and} \quad v(0) = M_1(\|x\| + M_2) = C \]

Differentiating \( v(t) \) we get,

\[
v'(t) = M_1k_2(t)\Omega_1(\|x(s)\|) + \int_0^t k_1(t, s)\Omega_0(\|x(s)\|) ds \leq M_1k_2(t)\Omega_1(v(t)) + \int_0^t k_1(t, s)\Omega_0(v(s)) ds \]

since \( v(t) \) is increasing and let

\[
w(t) = v(t) + \int_0^t k_1(t, s)\Omega_0(v(s)) ds \]

then, \( w(0) = v(0) = c \) and \( v(t) \leq w(t) \), differentiating \( w(t) \) we have

\[
w'(t) = v'(t) + k_1(t, t)\Omega_0(v(t)) + \int_0^t \frac{\partial}{\partial t} k_1(t, s)\Omega_0(v(s)) ds \\
\leq M_1k_2(t)\Omega_1(v(t)) + \int_0^t k_1(t, s)\Omega_0(v(s)) ds + k_1(t, t)\Omega_0(v(t)) \\
+ \int_0^t \frac{\partial}{\partial t} k_1(t, s)\Omega_0(v(s)) ds \leq \tilde{m}(t)\left\{ 2\Omega_0(w(t)) + \Omega_1(w(t)) \right\}
\]

This implies

\[
\int_{w(0)}^{\nu(t)} \frac{ds}{2\Omega_0(s) + \Omega_1(s)} \leq \int_0^b \tilde{m}(s) ds < \int_c^\infty \frac{ds}{2\Omega_0(s) + \Omega_1(s)} \quad 0 \leq t \leq b. \quad (6.3.3)
\]

Inequality (6.3.3) implies that there is a constant \( N \) such that \( v(t) \leq N; t \in J \) and hence

\[ \|x\| = \sup \left\{ |x(t)| : t \in J \right\} \leq N \]

where \( N \) depends only on \( b, \tilde{m}, \Omega_0 \text{ and } \Omega_1. \)

We shall now prove that the operator \( \Phi : Z \to Z \) is a completely continuous operator. Let \( B_r = \left\{ x \in Z : \|x\| \leq r \right\} \) for \( r \geq 1. \) We first show that \( \Phi \) maps \( B_r \) into an equicontinuous
family. Let \( x \in B_r \) and \( t_1, t_2 \in [0, b] \). Then if \( 0 < t_1 < t_2 < b \)

\[
\| (\Phi x)(t_1) - (\Phi x)(t_2) \|
\]

\[
= \| R(t_1, 0)[x_0 - g(x)] - R(t_2, 0)[x_0 - g(x)] \|
\]

\[
\quad + \left( \int_{0}^{t_1} R(t_1, s)f(s, x(\sigma_1(s)), \int_{0}^{s} K(s, \tau, x(\sigma_2(\tau)))d\tau)ds \right)
\]

\[
\quad - \left( \int_{0}^{t_2} R(t_2, s)f(s, x(\sigma_1(s)), \int_{0}^{s} K(s, \tau, x(\sigma_2(\tau)))d\tau)ds \right)
\]

\[
\leq \| R(t_1, 0)[x_0 - g(x)] - R(t_2, 0)[x_0 - g(x)] \|
\]

\[
\quad + \left( \int_{0}^{t_1} R(t_1, s)f(s, x(\sigma_1(s)), \int_{0}^{s} K(s, \tau, x(\sigma_2(\tau)))d\tau)ds \right)
\]

\[
\quad - \left( \int_{0}^{t_2} R(t_2, s)f(s, x(\sigma_1(s)), \int_{0}^{s} K(s, \tau, x(\sigma_2(\tau)))d\tau)ds \right)
\]

\[
\leq \| R(t_1, 0) - R(t_2, 0) \| \| x_0 - g(x) \|
\]

\[
\quad + \left( \int_{0}^{t_1} \| R(t_1, s) - R(t_2, s) \| \| f(s, x(\sigma_1(s)), \int_{0}^{s} K(s, \tau, x(\sigma_2(\tau)))d\tau) \| ds \right)
\]

\[
\quad + \left( \int_{0}^{t_2} \| R(t_2, s) - R(t_2, s) \| \| f(s, x(\sigma_1(s)), \int_{0}^{s} K(s, \tau, x(\sigma_2(\tau)))d\tau) \| ds \right)
\]

The right hand side is independent of \( x \in B_r \) and tends to zero as \( t_2 - t_1 \to 0 \), since the
compactness of $R(t,s)$ for $t > s$, implies the continuity in the uniform operator topology. Thus $\Phi$ maps $B_r$ into an equicontinuous family of functions. We also have $\Phi B_r$ is uniformly bounded.

Now we shall prove that $\Phi B_r$ is compact. Since we have shown that $\Phi B_r$ is equicontinuous collection, by the Arzela-Ascoli theorem it suffices to show that $\Phi$ maps $B_r$ into a precompact set in $X$.

Let $0 < t \leq b$ be fixed and let $\epsilon$ be a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$(\Phi x)(t) = R(t,0)[x_0 - g(x)] + \int_0^{t-\epsilon} R(t,s)f\left(s, x(\sigma_1(s))\right) \int_0^s K(s, \tau, x(\sigma_2(\tau)))d\tau ds.$$ 

Since $R(t,s)$ is a compact operator, the set

$$Y_r(t) = \left\{(\Phi x)(t) : x \in B_r \right\}$$

is precompact in $X$ for every $0 < \epsilon < t$. Moreover for every $x \in B_k$, we have

$$\| (\Phi x)(t) - (\Phi x)(t) \| \leq \int_0^t \| R(t,s) \| \left\| f\left(s, x(\sigma_1(s))\right) \int_0^s K(s, \tau, x(\sigma_2(\tau)))d\tau \right\| ds$$

$$\leq M_1 \int_0^t k_2(s)\Omega_1 \left( r + \int_0^s k_1(s, \tau)\Omega_0(\tau)d\tau \right)ds.$$
Therefore there are precompact sets arbitrarily close to the set \( \{ (\Phi x)(t) : x \in B_r \} \). Hence the set \( \{ (\Phi x)(t) : x \in B_r \} \) is precompact in \( X \).

Now let us prove that \( \Phi : Z \to Z \) is continuous. Let \( \{ x_n \}_{0}^{\infty} \subseteq Z \) with \( x_n \to x \) in \( Z \). Then there is an integer \( q \) such that \( ||x_n(t)|| \leq q \) for all \( n \) and \( t \in J \). Hence \( x_n \in B_q \) and \( x \in B_q \). By \( (H_2) \)

\[
f(t, x_n(\sigma_1(t)), \int_{0}^{t} K(t, s, x_n(\sigma_2(s)))ds) \to f(t, x(\sigma_1(t)), \int_{0}^{t} K(t, s, x(\sigma_2(s)))ds).
\]
for each \( t \in J \) and since,

\[
||f(t, x_n(\sigma_1(t)), \int_{0}^{t} K(t, s, x_n(\sigma_2(s)))ds) - f(t, x(\sigma_1(t)), \int_{0}^{t} K(t, s, x(\sigma_2(s)))ds)|| \leq 2k_2(t)\Omega_1 \left( q + \int_{0}^{t} k_1(t, s)\Omega_0(q)ds \right)
\]

By dominated convergence theorem

\[
||\Phi x_n - \Phi x|| = ||R(t, 0)[x_0 - g(x_n)] + \int_{0}^{t} R(t, s)f(s, x_n(\sigma_1(s)), \int_{0}^{s} K(s, \tau, x_n(\sigma_2(\tau)))d\tau)ds

- R(t, 0)[x_0 - g(x)] - \int_{0}^{t} R(t, s)f(s, x(\sigma_1(s)), \int_{0}^{s} K(s, \tau, x(\sigma_2(\tau)))d\tau)ds||
\]

\[
\leq M_1||g(x_n) - g(x)|| + M_1 \int_{0}^{t} ||f(s, x_n(\sigma_1(s)), \int_{0}^{s} K(s, \tau, x_n(\sigma_2(\tau)))d\tau)||ds

- \int_{0}^{t} f(s, x(\sigma_1(s)), \int_{0}^{s} K(s, \tau, x(\sigma_2(\tau)))d\tau)||ds
\]

\[
\to 0 \text{ as } n \to \infty.
\]

Since \( x_n \to x \) as \( n \to \infty \) and \( ||\Phi x_n - \Phi x|| \to 0 \).

Therefore \( \Phi \) is continuous. Thus the proof of the theorem is completed.

The set \( Z(\Phi) = \{ x \in Z : x = \lambda \Phi x, \lambda \in (0, 1) \} \) is bounded and by Schaefer’s theorem the operator \( \Phi \) has a fixed point in \( Z \). This means that any fixed point of \( \Phi \) is a mild solution of (6.1.1) - (6.1.2) on \( J \). satisfying \( (\Phi x)(t) = x(t) \).