CHAPTER 3

DECIDING THE USABILITY OF CONNECTOR MODELS FOR A
MAMDANI FUZZY LOGIC SYSTEMS

Fuzzy logic provides a framework to model uncertainty, the human way of thinking, reasoning, and the perception process. A fuzzy expert system is simply an expert system that uses a collection of fuzzy membership functions and rules, instead of Boolean logic, to reason about data. Fuzzy rules are the cornerstone of fuzzy logic systems. The antecedent (the rule's premise) describes to what degree the rule applies, while the rule's consequent assigns a membership function to each of one or more output variables.

The process involved in developing fuzzy expert systems is

- Specify the problem and define linguistic variables.
- Determine fuzzy sets.
- Elicit and construct fuzzy rules.
- Encode the fuzzy sets, fuzzy rules and procedures to perform fuzzy inference into the expert system.
- Evaluate and tune the system.

Fuzzy systems replace human operator's actions in the design of fuzzy control, fuzzy information and decision making. A detailed study of fuzzy systems and its application is discussed in Bezdek [3] and Mendel [48]. In terms of inference process there are two main classes of fuzzy inference system (FIS) the Mamdani type FIS and the Takagi-Sugeno type FIS. Mamdani type FIS is taken for study in this chapter. The various types of implication operators and aggregation operators, which are necessary to process the fuzzy rules are discussed in Turksen [72], Weber [76] and Yager [78]. Fuzzy systems modeling and
universal approximation of fuzzy rule based modeling is described in Krienovich et al., [41]. Fuzzy system has certain disadvantages like the validation process of the membership functions, the possible expensive computation in the fuzzy inference process and some unexpected difficulties and complexity in abstracting the fuzzy rules within numerical data. The main question often asked by fuzzy system designers is which one to use and under what conditions. The choice of implication operators, antecedent connectors, cannot be characterized by the type of the problem at hand, but it can be related to the properties of the problem.

**Hybrid systems**

Hybrid systems combining fuzzy logic, neural networks, and genetic algorithms are proving to be effective in a wide variety of real world problems. In this work, the emphasis of the hybrid systems is given to the neural-fuzzy system, which is the most common and popular combination in hybrid systems. Some of the applications of neuro fuzzy systems and soft computing model for machine intelligence can be found in Jang et al., [35].

Fuzzy inference or control system is applied in different problems and the membership functions representing the linguistic inference rules need to be tuned to create correct inference results. If there exists predefined rule system with suitable membership functions, implication operators, antecedent connector model, then they can be used to find the correct parameter for the clusters in decision support systems. Study of appropriate antecedent connector model is an important criterion for a fuzzy system to produce proper output otherwise the decision output of the system is a failure. It is extremely important to choose t-norm and t-conorm operator properly. The reason being different operators lead to radically different result in fuzzy systems.
For neuro-fuzzy logic systems that needs to be optimized during tuning procedure requires the firing degree to be separable. Tuning methods like back propagation algorithm involves the partial derivatives of the membership functions appearing in the objective functions. The firing degree must be expressed in a closed form and a piece wise-differentiable function of the membership function parameters and the parameter of the antecedent connector model in order that the fuzzy rules are usable. Hongwie Wu and Mendel [30] examined few antecedent connector models that are usable for a Mamdani fuzzy logic systems, the article adopts the ‘compensatory and’ and ‘soft ordered weighted averaging’ (SOWA) operators, with t-norm as product and min and t-conorm as max. All the connector models were usable for singleton fuzzy logic systems, but, for non singleton fuzzy logic systems only one antecedent connector model, namely, multiplicative compensatory ‘and’ with product t-norm model was usable when Mamdani product implication was applied and when Mamdani minimum implication was taken none of the antecedent connector models were usable. In this chapter, for the compensatory and and SOWA aggregation operators different combination of the t-norms and t-conorms were taken and tested for their usability. The t-norms considered are product and bounded difference, and the t-conorms considered are algebraic sum and bounded sum. Combination of the t-norms and t-conorms under study are given in Table 3.1.

3.1 Fuzzy set operations and membership functions

Union, intersection and complement are the basic operations on crisp sets. In fuzzy logic, the standard definitions of these operators are as follows:
For every x in the universe X of fuzzy sets A, B and C
**Union:** The union of two fuzzy sets $A$ and $B$ is a fuzzy set $C$, written as $C = A \cup B$, whose membership function is defined by

$$\mu_C(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \lor \mu_B(x).$$

**Intersection:** The intersection of two fuzzy sets $A$ and $B$ is a fuzzy set $C$, written as $C = A \cap B$, whose membership function is defined by

$$\mu_C(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \land \mu_B(x).$$

**Complement:** The complement of the fuzzy set $A$, denoted by $\bar{A}$, is defined by

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x).$$

Although the definitions above are classic set operators, they are not the only way to define reasonable and consistent operations on fuzzy sets.

**Fuzzy relation:** A fuzzy relation represents a degree of presence or absence of association, interaction or interconnectedness between the elements of two or more sets. Some examples of (binary) fuzzy relations are: $x$ is much larger than $y$, $y$ is very close to $x$, $z$ is much greater than $y$. Let $X$ and $Y$ be two universes of discourse.

A fuzzy relation $R(X,Y)$ is a fuzzy set in the product space $X \times Y$, i.e., it is a fuzzy subset of $X \times Y$, and is characterized by the membership function $\mu_R(x,y)$ where

$$R(X,Y) = \{(x,y), \mu_R(x,y) / (x,y) \in X \times Y\}.$$

**Composition of fuzzy relations:** Given two relations $P(X,Y)$ and $Q(Y,Z)$ and their associated membership functions $\mu_P(x,y)$ and $\mu_Q(x,y)$, the composition of these two relations is denoted by $R(X,Z) = P(X,Y) \circ Q(Y,Z)$ (or simply $R = P \circ Q$) $R(X,Z)$ a subset of $X \times Z$ defined by the membership function
\[ \mu_R (x, z) = \mu_{P \circ Q} (x, z) = \sup_y \{ T(\mu_P (x, y), \mu_Q (y, z)) \} \]
\[ = \sup_y \{ \mu_P (x, y) \ast \mu_Q (y, z) \} \]

(3.1)

This is also called sup star composition because of the use of symbol \ast for t-norm \( T \). The most commonly used sup star compositions are \( sup \; min \) and \( sup \; product \).

3.2 Fuzzy logic
Just as fuzzy set theory borrows notion from crisp set theory, fuzzy logic begins by borrowing notions from crisp logic. The IF-Then statement in fuzzy logic has the form

"If \( x \) is \( A \), then \( y \) is \( B \),"

where \( x \in X \), \( y \in Y \) and the membership function is \( \mu_{A \rightarrow B} (x, y) \in [0,1] \)

\( \mu_{A \rightarrow B} (x, y) \) measures the degree of truth of the implication relation between \( x \) and \( y \). In fuzzy logic, modus ponens is extended to generalized modus ponens

Premise 1: “\( x \) is \( A \)”
Premise 2: “\( If \; x \; is \; A \; then \; y \; is \; B \)”
Consequence : “\( y \) is \( B \)”
where \( A \) and \( B \) are fuzzy sets.

In fuzzy logic, a rule is fired so long as there is a non zero degree of similarity between the first premise and the antecedent of the rule, and the result of such rule firing is a consequent that has nonzero degree of similarity of the rule consequent. Generalized modus ponens is a fuzzy composition where the first fuzzy relation is merely the fuzzy set \( A \). Consequently membership function of the output \( \mu_B (y) \) is obtained using sup-star composition,
\[\mu_B(y) = \sup_{x \in A} \mu_A(x) \cdot \mu_{A \rightarrow B}(x, y)\] (3.2)

**Fuzzifier**

Fuzzification is the operation that maps a crisp object to a fuzzy set. Fuzzifiers are generally singleton and non singleton fuzzifier.

**Singleton fuzzifier:** It is most popular fuzzifier. A singleton fuzzifier maps an object to the singleton fuzzy set centered at the object itself. In this fuzzification scheme an observation \( x_1 \) is transformed into a fuzzy set being a singleton with support \( \{x_1\} \), thus \( \mu_A(x) \) is zero everywhere except at \( x = x_1 \). Using the fact that unity and zero are respectively the neutral and the null element with respect to any t-norm, it gives the output as,

\[
\mu_B(y) = 1 \cdot \mu_{A \rightarrow B}(x_1, y) = \mu_{A \rightarrow B}(x_1, y) \\
= 1 - \mu_{A \wedge B}(x_1, y) \\
= 1 - \left\{ \mu_A(x_1) \cdot \left[ 1 - \mu_B(y) \right] \right\} \\
= 1 - \left\{ \mu_A(x_1) \cdot \left[ 1 - \mu_B(y) \right] \right\} 
\] (3.3)

**Non Singleton fuzzifier:** A non-singleton fuzzifier maps an object into a non-singleton fuzzy set generally centered at the object itself.

\[
\mu_B(y) = \sup_{x \in A} \left[ \mu_A(x) \cdot \mu_{A \rightarrow B}(x, y) \right] 
\] (3.4)

The Mamdani minimum implication and Larsen product implications are the most widely used inferences of fuzzy logic.
3.3 Fuzzy rule based systems

FIS model can easily incorporate more variables than the common mathematical models and this will give more realistic solution than the conventional models.

A fuzzy rule base consists of a collection of IF-Then rules which can be expressed as

\[ R^l : \text{If } x_1 \text{ is } F^l_1 \text{ and } x_2 \text{ is } F^l_2 \text{ and } \ldots \text{ and } x_p \text{ is } F^l_p \text{ then } y \text{ is } G^l \]

where \( R^l (l = 1, 2, \ldots, m) \) is the \( l \)-th rule, \( F^l_i (i = 1, 2, \ldots, p) \) is an antecedent of \( R^l \). \( F^l_1 \) and \( G^l \) are fuzzy sets in the universe of discourse \( X_k \) and \( Y \) respectively which are subsets of real numbers \( R \). \( x = [x_1, x_2, \ldots, x_p] \in X_1 \times X_2 \times \ldots \times X_p \) and \( y \in Y \) are linguistic variables.

In a fuzzy inference engine, fuzzy logic principles are used to combine fuzzy IF-Then rules from the fuzzy rule base into a mapping from fuzzy input sets in \( X = X_1 \times X_2 \times \ldots \times X_p \) to fuzzy sets in \( Y \). The multiple antecedents are connected by "and's" and subsequently by t-norms. Each rule is interpreted as a fuzzy implication

\[ \mu_{R^l}(x, y) = \mu_{A \rightarrow B}(x, y) \]  

where

\[ \mu_{R^l}(x, y) = \mu_{F^l_1}(x_1) \cdot \ldots \cdot \mu_{F^l_p}(x_p) \cdot \mu_{G^l}(y) \]  

\[ = T[\mu_{F^l_1} \times \mu_{F^l_2} \times \ldots \times \mu_{F^l_p}(x) \cdot \mu_{G^l}(y)] \]  

\[ F^l_1 \times F^l_2 \times \ldots \times F^l_p = A, G^l = B. \]

The membership function of the input \( A_x \) of \( R^l \) is given by

\[ \mu_{A_x}(x) = T[\mu_{A_1}(x_1), \ldots, \mu_{A_p}(x_p)] \]
Each rule $R^i$ obtains a fuzzy set $B^i = A_x \circ R^i$ such that

$$\mu_{B^i}(y) = \mu_{A_x \circ R^i}(y) = \sup_{x \in A_x} \mu_{A_x}(x) \mu_{R^i}(x,y).$$

$$= T\{\sup_x T[\mu_{A_x}(x), \mu_{R^i}(x_1), \ldots, \mu_{R^i}(x_p), \mu_{G^i}(y)]\} \quad (3.9)$$

$$= T\{f^i(x), \mu_{G^i}(y)\} \quad (3.10)$$

where $f^i(x) = \sup_x T[\mu_{A_x}(x), \mu_{R^i}(x_1), \ldots, \mu_{R^i}(x_p), \mu_{G^i}(y)] \quad (3.11)$

$f^i(x)$ is the firing degree of $R^i$.

### 3.4 Aggregation process

Aggregation is the process of unification of the outputs of all rules. The membership functions of all rule consequents previously scaled and combined them into a single fuzzy set. The input of the aggregation process is the list of clipped or scaled consequent membership functions, and the output is one fuzzy set for each output variable.

The t-norms generalize the conjunctive ‘and’ and the t-conorms generalize the disjunctive ‘or’. Interesting properties satisfied by various t-norms $T$ and its dual t-conorms $S$ are discussed in Detyniecki et al., [17]. Zimmermann and Zysno [87] identified that both the operators lack the compensatory behavior and it seems crucial in the aggregation process. In order to get closer to the human aggregation process, Zimmermann and Zysno [87] proposed an operator on $[0, 1]$ called as compensatory operators that are based on t-norms and t-conorms. Later Yager and Filev [80] introduced a special family of aggregation operators called as soft ordered weighted averaging operators (S-OWA). In this chapter the compensatory operator and S-OWA operator are taken as the aggregation operators.
3.4.1 Compensatory operator

There are two types of compensatory operators, namely exponential compensatory operator and convex linear compensatory operator. The compensatory operators are either multiplicative or additive combination of t-norm $T$ and t-conorm $S$.

**Exponential compensatory operator (Multiplicative compensatory and)**

$$\varphi(a_1,a_2,\ldots,a_n) = T((a_1,a_2,\ldots,a_n))^{1-\gamma}S((a_1,a_2,\ldots,a_n))^\gamma$$  \hspace{1cm} (3.12)

**Convex linear compensatory operator (Additive compensatory and)**

$$\varphi(a_1,a_2,\ldots,a_n) = (1-\gamma)T((a_1,a_2,\ldots,a_n)) + \gamma S((a_1,a_2,\ldots,a_n))$$  \hspace{1cm} (3.13)

The t-norm $T$ and t-conorm $S$ are not necessarily dual, $\gamma \in [0,1]$ is a parameter characterizing the degree of compensation and $a_1,a_2,\ldots,a_n \in [0,1]$.

3.4.2 Soft ordered weighted averaging operator (S-OWA)

The S-OWA operators are either S-OWA t-norm or SOWA t-conorm.

**S-OWA t-norm operator:**

The S-OWA t-norm is a multiplicative or additive combination of arithmetic average and t-norm.

$$\varphi(a_1,a_2,\ldots,a_p) = \left(\frac{1}{p} \sum_{i=1}^{p} a_i \right)^{1-\gamma} [T(a_1,a_2,\ldots,a_p)]^\gamma$$  \hspace{1cm} (3.14)

$$\varphi(a_1,a_2,\ldots,a_p) = (1-\gamma) \left(\frac{1}{p} \sum_{i=1}^{p} a_i \right) + \gamma [T(a_1,a_2,\ldots,a_p)]$$  \hspace{1cm} (3.15)
**S-OWA t-conorm operator:**

The S-OWA t-conorm is a multiplicative or additive combination of arithmetic average and t-conorm.

\[
\varphi(a_1, a_2, \ldots, a_p) = \left( \frac{1}{p} \sum_{i=1}^{p} a_i \right)^{1-\gamma} \left[ S(a_1, a_2, \ldots, a_p) \right]^\gamma
\]  

(3.16)

\[
\varphi(a_1, a_2, \ldots, a_p) = (1-\gamma) \left( \frac{1}{p} \sum_{i=1}^{p} a_i \right) + \gamma \left[ S(a_1, a_2, \ldots, a_p) \right]
\]  

(3.17)

The t-norms and t-conorms taken for the compensatory operators and S-OWA operators are given in Table 3.1 and Fig 3.1.

![Diagram of Aggregation Operators]

*Figure 3.1 Different combination of t-norms and t-conorms*
<table>
<thead>
<tr>
<th>Model</th>
<th>Operator</th>
<th>The t norm T and t conorm S in use</th>
<th>Type of the operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi^1(a_1,\ldots,a_p)$</td>
<td>$\left(\prod_{i=1}^{p}a_i\right)^{1-\gamma}\left(\frac{\sum_{i=1}^{p}a_i - \prod_{i=1}^{p}a_i}{1-\gamma}\right)^{\gamma}$</td>
<td>T-Product S- Algebraic sum</td>
<td>Compensatory sum</td>
</tr>
<tr>
<td>$\Phi^2(a_1,\ldots,a_p)$</td>
<td>$\left(\prod_{i=1}^{p}a_i\right)^{1-\gamma}\left(\min\left(\sum_{i=1}^{p}a_i,1\right)\right)^{\gamma}$</td>
<td>T-Product S- Bounded sum</td>
<td></td>
</tr>
<tr>
<td>$\Phi^3(a_1,\ldots,a_p)$</td>
<td>$(1-\gamma)\left(\prod_{i=1}^{p}a_i\right) + \gamma\left(\sum_{i=1}^{p}a_i - \prod_{i=1}^{p}a_i\right)$</td>
<td>T-Product S-Algebraic sum</td>
<td>Compensatory sum</td>
</tr>
<tr>
<td>$\Phi^4(a_1,\ldots,a_p)$</td>
<td>$(1-\gamma)\left(\prod_{i=1}^{p}a_i\right) + \gamma\min\left(\sum_{i=1}^{p}a_i,1\right)$</td>
<td>T-Product S- Bounded sum</td>
<td>Additive</td>
</tr>
<tr>
<td>$\Phi^5(a_1,\ldots,a_p)$</td>
<td>$\max\left(\sum_{i=1}^{p}a_i - 1,0\right)\left(\sum_{i=1}^{p}a_i - 1\right)^{\gamma}$</td>
<td>T-Bounded difference S- Bounded sum</td>
<td></td>
</tr>
<tr>
<td>$\Phi^6(a_1,\ldots,a_p)$</td>
<td>$\max\left(\sum_{i=1}^{p}a_i - 1,0\right) + \gamma\left(\sum_{i=1}^{p}a_i,1\right)$</td>
<td>T-Bounded difference S- Algebraic sum</td>
<td></td>
</tr>
<tr>
<td>$\Phi^7(a_1,\ldots,a_p)$</td>
<td>$(1-\gamma)\max\left(\sum_{i=1}^{p}a_i,1\right) + \gamma\min\left(\sum_{i=1}^{p}a_i,1\right)$</td>
<td>T-Bounded difference S- Bounded sum</td>
<td>Compensatory Additive</td>
</tr>
<tr>
<td>$\Phi^8(a_1,\ldots,a_p)$</td>
<td>$(1-\gamma)\max\left(\sum_{i=1}^{p}a_i,1\right) + \gamma\left(\sum_{i=1}^{p}a_i - \prod_{i=1}^{p}a_i\right)$</td>
<td>T-Bounded difference S- Algebraic sum</td>
<td></td>
</tr>
</tbody>
</table>
3.5 Usability of antecedent connector models

The firing degree in (3.11) can be derived by applying t-norms for the multiple antecedents, input of the membership function, for sup-star composition and in the Mamdani implication. In this chapter the t-norm $T$ for implementing Mamdani implication and sup-star composition are assumed to be the same. But the t-norm $T$ for the linguistic connector word ‘and’ connecting the multiple antecedents in each rule is represented as aggregation operators namely compensatory models as well as soft ordered weighted averaging operators (S-OWA) (i.e.). The antecedent part of each rule is then modeled as

$$\mu_{\phi_x, \phi_x, \ldots, \phi_x}(x) = \phi[\mu_{\phi_1}(x_1), \ldots, \mu_{\phi_p}(x_p)]$$  \hspace{1cm} (3.18)

So the firing degree in (3.11) which now depends on $\phi$ is denoted as $f^I(x / \phi)$

$$f^I(x / \phi) = \sup_x T\{\theta[\mu_{A_{x_1}}(x_1), \ldots, \mu_{A_{x_p}}(x_p)], \phi[\mu_{\phi_1}(x_1), \ldots, \mu_{\phi_p}(x_p)]\}$$  \hspace{1cm} (3.19)
The following discussion is restricted to a multi-input single-output (MISO) problem since any multi-output (MIMO) problem can be approached as the parallel of MISO solutions. In this chapter the Mamdani fuzzy logic systems is taken for discussion.

**Definition Separable** Let $x_{\text{max}}^i = [x_{1,\text{max}}^i, ..., x_{p,\text{max}}^i]$ be a vector in $X_1 \times X_2 \times \ldots \times X_p$ wherein (3.19)

$$T(T[\mu_{A_{x_1}}(x_1), ..., \mu_{A_{x_p}}(x_p)], \varphi[\mu_{F_1}(x_1), ..., \mu_{F_p}(x_p)])$$

achieves its supremum $x_i$.

i.e. the firing degree in (3.19) can be expressed as

$$f_1(x/\varphi) = T(T[\mu_{A_{x_1}}(x_{1,\text{max}}^i), ..., \mu_{A_{x_p}}(x_{p,\text{max}}^i)], \varphi[\mu_{F_1}(x_{1,\text{max}}^i), ..., \mu_{F_p}(x_{p,\text{max}}^i)])$$

(3.20)

The firing degree is called separable if the procedure of searching for $x_{\text{max}}^i$ in $X_1 \times X_2 \times \ldots \times X_p$ can be partitioned into separate procedures of searching for $x_{\text{max}}^i$ in $X_i$ ($i = 1, 2, ..., p$).

For *singleton fuzzy logic systems (SFLS)*, the firing degree in (3.20) simplifies to

$$f_1(x/\varphi) = \varphi[\mu_{F_1}(x_1), ..., \mu_{F_p}(x_p)]$$

since $T[\mu_{A_{x_1}}(x_1), ..., \mu_{A_{x_p}}(x_p)] = 1$

where t-norm $T$ is commonly taken as *min* or *product*.

For *non singleton fuzzy logic systems (NSFLS)*, the firing degree $f_1(x/\varphi)$ is obtained from (3.19).
When the model $\emptyset$ used for the connective word ‘and’ is a t norm either product or minimum then $x_{i,\text{max}} = \arg\sup_{x_i} T[pA_{x_i}(x_i), p_{f_i}(x_i)]$. This makes the firing degree separable. $x'_{i,\text{max}}$ for all $i = 1, 2, \ldots, p$ can be obtained in parallel and hence get the value of $x'_{\text{max}}$ over $X_1 \times X_2 \times \ldots \times X_p$. Thus the model $\emptyset$ chosen for the linguistic connector word ‘and’ must lead to separable firing degree.

If a FLS is to be optimized during tuning procedure and if the tuning method like backpropogation algorithm is used then it requires that the objective function should be expressed in a closed form and piece-wise differentiable function of the membership function parameters and the parameter of $\emptyset$.

Hongwie Wu and Mendel [30] proposed two design criterion for a connector model $\emptyset$ to be usable. The predesign criterion is that the model $\emptyset$ should lead to separable firing degree and the design criterion is that the firing degree should be a closed-form and piecewise-differentiable function of the membership function parameters and the parameter of $\emptyset$.

**Definition Usable** A connector model is called usable if the model $\emptyset$ chosen for connector word ‘and’ leads to separable firing degree and the firing degree can be expressed as a closed form and piece-wise differentiable function of the membership function parameters and of the parameter of $\emptyset$.

Both separability and piecewise differentiability are necessary conditions for a connector model $\emptyset$ to be usable.

### 3.5.1 Singleton fuzzy logic systems

For singleton fuzzy logic systems since $x'_{i,\text{max}} = x'_i$, $i = 1, 2, \ldots, p$ all the models in Table 3.1 are usable. A singleton firing degree is always separable for any model.
for the connector word ‘and’. For each model given in Table 3.1 the firing degree is a closed form function of the parameters and is piecewise differentiable with respect to the membership function parameters. Hence it follows that all the models in Table 3.1 are usable for a singleton fuzzy logic systems. Table 3.2 gives the partial derivatives of the singleton firing degree, with respect to membership function parameter and with respect to the parameter characterizing ϕ for the connector word ‘and’ considered in Table 3.1.

Table 3.2 Firing degree for singleton fuzzy logic systems and partial derivatives with respect to the antecedent membership function parameters and the parameter characterizing ϕ

<table>
<thead>
<tr>
<th>Firing degree for singleton fuzzy logic systems and its partial derivatives</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f^I(x/\phi') = \left[ \prod_{i=1}^{p} \mu_{Fi}(x_i) \right]^{1-\gamma} \left[ \sum_{i=1}^{p} \mu_{Fi}(x_i) - \prod_{i=1}^{p} \mu_{Fi}(x_i) \right]^{\gamma} )</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial \theta_k} f^I(x/\phi') = \frac{f^I(x/\phi')}{\mu_{Fi}(x_k)} \frac{\partial}{\partial \theta_k} \mu_{Fi}(x_k) + \frac{\gamma f^I(x/\phi') \left( \frac{\partial}{\partial \theta_k} \mu_{Fi}(x_k) - \prod_{i \neq k} \mu_{Fi}(x_i) \frac{\partial}{\partial \theta_k} \mu_{Fi}(x_k) \right)}{\sum_{i=1}^{p} \mu_{Fi}(x_i) - \prod_{i=1}^{p} \mu_{Fi}(x_i)} )</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial \gamma} f^I(x/\phi') = f^I(x/\phi') \left[ -\ln \prod_{i=1}^{p} \mu_{Fi}(x_i) + \ln \left( \sum_{i=1}^{p} \mu_{Fi}(x_i) - \prod_{i=1}^{p} \mu_{Fi}(x_i) \right) \right] )</td>
</tr>
</tbody>
</table>
\[ f^1(\frac{x}{\sqrt{\phi^2}}) = \left[ \prod_{i=1}^{p} \mu_{\phi_i}(x_i) \right]^{1-\gamma} \min \left[ \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \right] \]

\[ \frac{\partial}{\partial \theta_k} f^1(\frac{x}{\sqrt{\phi^2}}) = \left[ \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \right]^{\gamma-1} + \gamma \mu_{\phi_k}(x_k) \frac{\partial}{\partial \theta_k} \mu_{\phi_1}(x_k), \text{if } \min \left[ \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \right] = 1 \]

\[ = \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \]

\[ \frac{\partial}{\partial \gamma} f^1(\frac{x}{\sqrt{\phi^2}}) = f^1(\frac{x}{\sqrt{\phi^2}}) \left[ - \sum_{i=1}^{p} \ln \mu_{\phi_i}(x_i) \right] + \ln \min \left[ \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \right] \]

\[ f^1(\frac{x}{\sqrt{\phi^3}}) = (1-\gamma) \prod_{i=1}^{p} \mu_{\phi_i}(x_i) + \gamma \left( \sum_{i=1}^{p} \mu_{\phi_i}(x_i) - \prod_{i=1}^{p} \mu_{\phi_i}(x_i) \right) \]

\[ \frac{\partial}{\partial \theta_k} f^1(\frac{x}{\sqrt{\phi^3}}) = \left[ 1 - 2\gamma \prod_{i=1}^{p} \mu_{\phi_i}(x_i) + \gamma \right] \frac{\partial}{\partial \theta_k} \mu_{\phi_1}(x_k) \]

\[ = \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \]

\[ f^1(\frac{x}{\sqrt{\phi^4}}) = (1-\gamma) \prod_{i=1}^{p} \mu_{\phi_i}(x_i) + \gamma \min \left[ \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \right] \]

\[ \frac{\partial}{\partial \theta_k} f^1(\frac{x}{\sqrt{\phi^4}}) = \left[ (1-\gamma) \prod_{i=1}^{p} \mu_{\phi_i}(x_i) + \gamma \right] \frac{\partial}{\partial \theta_k} \mu_{\phi_1}(x_k), \text{if } \min \left[ \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \right] = 1 \]

\[ = \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \]

\[ \frac{\partial}{\partial \gamma} f^1(\frac{x}{\sqrt{\phi^5}}) = - \prod_{i=1}^{p} \mu_{\phi_i}(x_i) + \min \left[ \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \right] \]

\[ f^1(\frac{x}{\sqrt{\phi^5}}) = \max \left( \sum_{i=1}^{p} \mu_{\phi_i}(x_i) - 1, 0 \right)^{1-\gamma} \max \left( \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \right) \]

If \( \max \left( \sum_{i=1}^{p} \mu_{\phi_i}(x_i) - 1, 0 \right) = \sum_{i=1}^{p} \mu_{\phi_i}(x_i) - 1 \implies \sum_{i=1}^{p} \mu_{\phi_i}(x_i) - 1 \geq 0 \text{ (i.e.) } \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \geq 1 \)

So \( \min \left( \sum_{i=1}^{p} \mu_{\phi_i}(x_i) \right) = 1 \) Hence \( f^1(\frac{x}{\sqrt{\phi^5}}) = \left( \sum_{i=1}^{p} \mu_{\phi_i}(x_i) - 1 \right)^{1-\gamma} \)

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\[ \frac{\partial}{\partial \theta_k} f^l \left( x_{\phi^5} \right) = \frac{(1-\gamma) f^l \left( x_{\phi^5} \right) \mu_{h_k}(x_k)}{\sum_{i=1}^{p} \mu_{f_i}(x_i) - 1} \frac{\partial}{\partial \theta_k} \mu_{f_i}(x_i) \]

If \( \max \left( \sum_{i=1}^{p} \mu_{f_i}(x_i) - 1,0 \right) = 0 \)

\[ f^l \left( x_{\phi^5} \right) = \left( \sum_{i=1}^{p} \mu_{f_i}(x_i) \right)^\gamma \]

\[ \frac{\partial}{\partial \theta_k} f^l \left( x_{\phi^5} \right) = \frac{\gamma f^l \left( x_{\phi^5} \right) \partial}{\sum_{i=1}^{p} \mu_{f_i}(x_i)} \frac{\partial}{\partial \theta_k} \mu_{f_i}(x_i) \]

\[ \frac{\partial}{\partial \gamma} f^l \left( x_{\phi^5} \right) = -f^l \left( x_{\phi^5} \right) \ln \left( \sum_{i=1}^{p} \mu_{f_i}(x_i) \right) + 1 \text{ or } \frac{\partial}{\partial \gamma} f^l \left( x_{\phi^5} \right) = f^l \left( x_{\phi^5} \right) \ln \left( \sum_{i=1}^{p} \mu_{f_i}(x_i) \right) \]

\[ f^l \left( x_{\phi^5} \right) = \max \left( \sum_{i=1}^{p} \mu_{f_i}(x_i) - 1,0 \right) \left( \sum_{i=1}^{p} \mu_{f_i}(x_i) - \prod_{i=1}^{p} \mu_{f_i}(x_i) \right)^\gamma \]

\[ \frac{\partial}{\partial \theta_k} f^l \left( x_{\phi^5} \right) = f^l \left( x_{\phi^5} \right) \left[ \frac{(1-\gamma) \left( \sum_{i=1}^{p} \mu_{f_i}(x_i) - 1 \right) + \gamma \left( \prod_{i=1}^{p} \mu_{f_i}(x_i) \right)}{\sum_{i=1}^{p} \mu_{f_i}(x_i) - 1} \right] \frac{\partial}{\partial \theta_k} \mu_{f_i}(x_i) \]

\[ \frac{\partial}{\partial \gamma} f^l \left( x_{\phi^5} \right) = -f^l \left( x_{\phi^5} \right) \prod_{i=1}^{p} \mu_{f_i}(x_i) \]
\[ f'(x/\varphi^\gamma) = (1 - \gamma) \max \left( \sum_{i=1}^{p} \mu_{fi}(x_i) - 1, 0 \right) + \gamma \min \left( \sum_{i=1}^{p} \mu_{fi}(x_i), 1 \right) \]

**Case 1**

If \( \max \left( \sum_{i=1}^{p} \mu_{fi}(x_i) - 1, 0 \right) = \sum_{i=1}^{p} \mu_{fi}(x_i) - 1 \Rightarrow \sum_{i=1}^{p} \mu_{fi}(x_i) > 1 \)

So \( \min \left( \sum_{i=1}^{p} \mu_{fi}(x_i), 1 \right) = 1 \)

\[ \frac{\partial}{\partial \varphi^\gamma} f'(x/\varphi^\gamma) = (1 - \gamma) \frac{\partial}{\partial \varphi^\gamma} \mu_{fi}(x_i) \]

**Case 2**

If \( \max \left( \sum_{i=1}^{p} \mu_{fi}(x_i) - 1, 0 \right) = 0 \Rightarrow \sum_{i=1}^{p} \mu_{fi}(x_i) < 1 \)

So \( \min \left( \sum_{i=1}^{p} \mu_{fi}(x_i), 1 \right) = \sum_{i=1}^{p} \mu_{fi}(x_i) \)

\[ \frac{\partial}{\partial \varphi^\gamma} f'(x/\varphi^\gamma) = \gamma \frac{\partial}{\partial \varphi^\gamma} \mu_{fi}(x_i) \]

or

\[ \frac{\partial}{\partial \varphi^\gamma} f'(x/\varphi^\gamma) = - \left( \sum_{i=1}^{p} \mu_{fi}(x_i) - 1 \right) \]

\[ \frac{\partial}{\partial \varphi^\gamma} f'(x/\varphi^\gamma) = \sum_{i=1}^{p} \mu_{fi}(x_i) \]

\[ f'(x/\varphi^\delta) = (1 - \gamma) \max \left( \sum_{i=1}^{p} \mu_{fi}(x_i) - 1, 0 \right) + \gamma \left( \sum_{i=1}^{p} \mu_{fi}(x_i) - \prod_{i=1}^{p} \mu_{fi}(x_i) \right) \]

If \( \max \left( \sum_{i=1}^{p} \mu_{fi}(x_i) - 1, 0 \right) = \sum_{i=1}^{p} \mu_{fi}(x_i) - 1 \)

\[ \frac{\partial}{\partial \varphi^\delta} f'(x/\varphi^\delta) = \left[ 1 - \gamma \prod_{i=1}^{p} \mu_{fi}(x_i) \right] \frac{\partial}{\partial \varphi^\delta} \mu_{fi}(x_i) \]

or

\[ \frac{\partial}{\partial \varphi^\delta} f'(x/\varphi^\delta) = \gamma \left[ 1 - \prod_{i=1}^{p} \mu_{fi}(x_i) \right] \frac{\partial}{\partial \varphi^\delta} \mu_{fi}(x_i) \]

\[ \frac{\partial}{\partial \varphi^\delta} f'(x/\varphi^\delta) = - \prod_{i=1}^{p} \mu_{fi}(x_i) \text{ if } \max \left( \sum_{i=1}^{p} \mu_{fi}(x_i) - 1, 0 \right) = \sum_{i=1}^{p} \mu_{fi}(x_i) - 1 \]

\[ \frac{\partial}{\partial \varphi^\delta} f'(x/\varphi^\delta) = \sum_{i=1}^{p} \mu_{fi}(x_i) - \prod_{i=1}^{p} \mu_{fi}(x_i) \]
\[
\begin{align*}
\ell^\prime\left(\frac{X}{\varphi^\alpha}\right) &= \left(\frac{1}{p} \sum_{i=1}^{p} \mu_{F_i}(x_i)\right)^{1-\gamma} \max\left[\sum_{i=1}^{p} \mu_{F_i}(x_i) - 1, 0\right] \\
\frac{\partial}{\partial \theta_k} \ell^\prime\left(\frac{X}{\varphi^\alpha}\right) &= \frac{1}{p} \left(\sum_{i=1}^{p} \mu_{F_i}(x_i)\right)^{-\gamma} \left[\sum_{i=1}^{p} \mu_{F_i}(x_i) - 1\right]^{-1} \sum_{i=1}^{p} \mu_{F_i}(x_i) - \gamma \frac{\partial}{\partial \theta_k} \mu_{F_i}(x_k) \\
\frac{\partial}{\partial \gamma} \ell^\prime\left(\frac{X}{\varphi^\alpha}\right) &= \ell^\prime\left(\frac{X}{\varphi^\alpha}\right) \left[\sum_{i=1}^{p} \mu_{F_i}(x_i) - 1 - \ln\left(\frac{1}{p} \sum_{i=1}^{p} \mu_{F_i}(x_i)\right)\right] \\
\ell^\prime\left(\frac{X}{\varphi^{\alpha_0}}\right) &= \left[1 - \frac{1}{p} \sum_{i=1}^{p} \mu_{F_i}(x_i)\right] + \gamma \max\left[\sum_{i=1}^{p} \mu_{F_i}(x_i) - 1, 0\right] \\
\frac{\partial}{\partial \theta_k} \ell^\prime\left(\frac{X}{\varphi^{\alpha_0}}\right) &= \left(1 - \frac{1}{p} \mu_{F_i}(x_i)\right) + \gamma \frac{\partial}{\partial \theta_k} \mu_{F_i}(x_k) \\
or\ &
\frac{\partial}{\partial \theta_k} \ell^\prime\left(\frac{X}{\varphi^{\alpha_0}}\right) &= \left(1 - \frac{1}{p} \mu_{F_i}(x_i)\right) + \gamma \frac{\partial}{\partial \theta_k} \mu_{F_i}(x_k) \\
\frac{\partial}{\partial \gamma} \ell^\prime\left(\frac{X}{\varphi^{\alpha_0}}\right) &= \sum_{i=1}^{p} \mu_{F_i}(x_i) \left[1 - \frac{1}{p}\right] - 1 \\
or\ &
\frac{\partial}{\partial \gamma} \ell^\prime\left(\frac{X}{\varphi^{\alpha_0}}\right) &= -\frac{1}{p} \sum_{i=1}^{p} \mu_{F_i}(x_i) \\
\ell^\prime\left(\frac{X}{\varphi^{\alpha_1}}\right) &= \left[\sum_{i=1}^{p} \mu_{F_i}(x_i)\right] \left[\sum_{i=1}^{p} \mu_{F_i}(x_i) - \sum_{i=1}^{p} \mu_{F_i}(x_i)\right] \\
\frac{\partial}{\partial \alpha_i} \ell^\prime\left(\frac{X}{\varphi^{\alpha_1}}\right) &= \ell^\prime\left(\frac{X}{\varphi^{\alpha_1}}\right) \sum_{i=1}^{p} \mu_{F_i}(x_i) \left[\sum_{i=1}^{p} \mu_{F_i}(x_i) - \sum_{i=1}^{p} \mu_{F_i}(x_i)\right] \frac{\partial}{\partial \alpha_i} \mu_{F_i}(x_i) \\
or\ &
\frac{\partial}{\partial \alpha_i} \ell^\prime\left(\frac{X}{\varphi^{\alpha_1}}\right) &= \ell^\prime\left(\frac{X}{\varphi^{\alpha_1}}\right) \sum_{i=1}^{p} \mu_{F_i}(x_i) \left[\sum_{i=1}^{p} \mu_{F_i}(x_i) - \sum_{i=1}^{p} \mu_{F_i}(x_i)\right] \frac{\partial}{\partial \alpha_i} \mu_{F_i}(x_i) \\
\frac{\partial}{\partial \gamma} \ell^\prime\left(\frac{X}{\varphi^{\alpha_1}}\right) &= \frac{\partial}{\partial \gamma} \left[-\frac{1}{p} \ln\sum_{i=1}^{p} \mu_{F_i}(x_i) + \ln\left(\sum_{i=1}^{p} \mu_{F_i}(x_i) - \sum_{i=1}^{p} \mu_{F_i}(x_i)\right)\right] \\
\ell^\prime\left(\frac{X}{\varphi^{\alpha_2}}\right) &= \left[1 - \frac{1}{p} \sum_{i=1}^{p} \mu_{F_i}(x_i)\right]^{-1} \min\left[\sum_{i=1}^{p} \mu_{F_i}(x_i) l, 1\right]^\gamma \\
\ell^\prime\left(\frac{X}{\varphi^{\alpha_2}}\right) &= \left(\frac{1}{p}\right)^\gamma \sum_{i=1}^{p} \mu_{F_i}(x_i) \text{ if } \min\left[\sum_{i=1}^{p} \mu_{F_i}(x_i) l, 1\right] = \sum_{i=1}^{p} \mu_{F_i}(x_i) 
\end{align*}
\]

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3.5.2 Non singleton fuzzy logic systems

For NSFLS the discussion is carried for two different cases (i) product t norm model is used for both sup-star composition and for Mamdani implication and (ii) minimum t-norm model is used for both the sup-star composition and Mamdani implication.
3.5.2.1 Implementation of product t-norm model for sup-star composition and Mamdani implication

The firing degree in (3.19) in this case takes the form

$$f^l(X/\varphi) = \sup_x \{ \prod_{i=1}^{p} \mu_{\Lambda_{x_i}}(x_i) \times \varphi[\mu_{\Pi_1}(x_1), \ldots, \mu_{\Pi_p}(x_p)] \} \quad (3.21)$$

In order to derive a condition to determine whether or not $\varphi$ leads to separable firing degree, the following theorem is proved by Hongwei Wu and Mendel [30].

**Theorem 3.1** If the firing degree in (3.21) is separable then the product t-norm model must satisfy

$$\frac{\partial^2}{\partial a_i \partial a_j} \ln \varphi_p(a_1, a_2, \ldots, a_p) = 0 \quad \forall i, j \in \{1, 2, \ldots, p\} \land i \neq j, a_1, a_2, \ldots, a_p \in [0, 1] \quad (3.22)$$

Theorem 3.1 is used to test whether the models defined in Table 3.1 leads to separable firing degree.

**Corollary 3.1**

If a product t-norm model for multiplicative compensatory and has the structure

$$\varphi(a_1, \ldots, a_p) = \left( \prod_{i=1}^{p} a_i \right)^{1-\gamma} g(a_1, \ldots, a_p) \quad (3.23)$$

then $\varphi$ does not lead to separable firing degree if

$$\frac{\partial}{\partial a_i} \ln g(a_1, \ldots, a_p) \neq \frac{(1-\gamma)}{a_i} \quad (3.24)$$

provided $g(a_1, \ldots, a_p)$ is not a constant.

**Proof:**

For $\varphi(a_1, \ldots, a_p) = \left( \prod_{i=1}^{p} a_i \right)^{1-\gamma} g(a_1, \ldots, a_p)$
\[
\ln \varphi(a_1, \ldots, a_p) = (1 - \gamma) \ln \left( \prod_{i=1}^{p} a_i \right) + \ln g(a_1, \ldots, a_p)
\]

\[
= (1 - \gamma) \sum_{i=1}^{p} \ln a_i + \ln g(a_1, \ldots, a_p)
\]

\[
\frac{\partial \ln \varphi(a_1, \ldots, a_p)}{\partial a_i} = (1 - \gamma) \frac{1}{a_i} + \frac{\partial \ln g(a_1, \ldots, a_p)}{\partial a_i} \quad (3.25)
\]

The first term of (3.25) cannot be cancelled by the second term, due to (3.24).

This implies (3.25) is a function of \((a_1, a_2, \ldots, a_p)\).

Therefore (3.22) does not hold good.

Hence the firing degree with connector model \( \varphi \) as in (3.23) is not separable.

Testing for \( \varphi^1(a_1, a_2, \ldots, a_p) = \left( \prod_{i=1}^{p} a_i \right)^{1-\gamma} \left( \sum_{i=1}^{p} a_i - \prod_{i=1}^{p} a_i \right)^{\gamma} \)

\[
\varphi^1(a_1, a_2, \ldots, a_p) = \left( \prod_{i=1}^{p} a_i \right)^{1-\gamma} \left( \sum_{i=1}^{p} a_i - \prod_{i=1}^{p} a_i \right)^{\gamma}
\]

Here \( g(a_1, a_2, \ldots, a_p) = \left( \sum_{i=1}^{p} a_i - \prod_{i=1}^{p} a_i \right)^{\gamma} \)

\[
\ln \varphi^1 = (1 - \gamma) \ln \left( \prod_{i=1}^{p} a_i \right) + \gamma \ln \left( \sum_{i=1}^{p} a_i - \prod_{i=1}^{p} a_i \right)
\]

\[
\ln \varphi^1 = (1 - \gamma) \sum_{i=1}^{p} \ln a_i + \gamma \ln \left( \sum_{i=1}^{p} a_i - \prod_{i=1}^{p} a_i \right)
\]

\[
\frac{\partial \ln g(a_1, a_2, \ldots, a_p)}{\partial a_i} = \gamma \times \frac{1}{\left( \sum_{i=1}^{p} a_i - \prod_{i=1}^{p} a_i \right)} \times \left( 1 - \frac{p}{\prod_{k=1}^{p} a_k} \right) = -(1 - \gamma) \frac{1}{a_i}
\]

So according to corollary 3.1 the model \( \varphi^1 \) does not lead to separable firing degree.

Testing for \( \varphi^2 = \left( \prod_{i=1}^{p} a_i \right)^{1-\gamma} \left( \min \left( \sum_{i=1}^{p} a_i, 1 \right) \right)^{\gamma} \)

Here \( g(a_1, a_2, \ldots, a_p) = \left( \min \left( \sum_{i=1}^{p} a_i, 1 \right) \right)^{\gamma} \)
Case 1

If \( \min_i \left( \sum_{i=1}^p a_i, 1 \right) = \sum_{i=1}^p a_i \Rightarrow \sum_{i=1}^p a_i < 1 \)

\[
\ln g(a_1, a_2, \ldots, a_p) = \gamma \ln \left( \sum_{i=1}^p a_i \right)
\]

\[
\frac{\partial}{\partial a_i} \ln g(a_1, a_2, \ldots, a_p) = \gamma \times \frac{1}{\sum_{i=1}^p a_i} \Rightarrow \frac{-(1-\gamma)}{a_i}
\]

Hence for case 1 the model \( \varphi^2 \) is not separable.

Case 2

If \( \sum_{i=1}^p a_i \leq 1 \) then, \( \min_i \left( \sum_{i=1}^p a_i, 1 \right) = 1 \)

So \( g(a_1, a_2, \ldots, a_p) = 1 \) is a constant

\[
\varphi^2(a_1, a_2, \ldots, a_p) = \left( \prod_{i=1}^p a_i \right)^{1-\gamma}
\]

The firing degree in (3.21) becomes

\[
f^i(X/\varphi^2) = \sup_x \left\{ \prod_{i=1}^p \mu_{A_{x_i}}(x_i) \times \left( \prod_{i=1}^p \mu_{F_i}(x_i) \right)^{1-\gamma} \right\}
\]

\[
= \sup_x \left\{ \prod_{i=1}^p \mu_{A_{x_i}}(x_i) \times \left( \prod_{i=1}^p \mu_{F_i}^{1-\gamma}(x_i) \right) \right\}
\]

\[
= \sup_x \left\{ \prod_{i=1}^p \mu_{A_{x_i}}(x_i) \mu_{F_i}^{1-\gamma}(x_i) \right\}
\]

The procedure of searching for \( x_{i,\text{max}} \) in \( X_1 \times \ldots \times X_p \) can be divided into separate

procedure of obtaining \( x_{i,\text{max}} \).

Corollary 3.2
If a product t norm model for additive compensatory and has the structure

$$\varphi(a_1,\ldots,a_p) = (1 - \gamma) \prod_{i=1}^{p} a_i + g(a_1,\ldots,a_p)$$

(3.26)

then \( \varphi \) does not lead to separable firing degree if

$$\frac{\partial g(a_1,\ldots,a_p)}{\partial a_i} \neq -(1 - \gamma) \prod_{k \neq i} a_k$$

(3.27)

provided \( g(a_1,\ldots,a_p) \) is not a constant.

**Proof:**

$$\varphi(a_1,\ldots,a_p) = (1 - \gamma) \prod_{i=1}^{p} a_i + g(a_1,\ldots,a_p)$$

$$\ln(\varphi(a_1,\ldots,a_p)) = \left\{ (1 - \gamma) \prod_{i=1}^{p} a_i + g(a_1,\ldots,a_p) \right\}$$

$$\frac{\partial \varphi(a_1,\ldots,a_p)}{\partial a_i} = \left[ (1 - \gamma) \prod_{k \neq i} a_k + \frac{\partial g(a_1,\ldots,a_p)}{\partial a_i} \right]$$

(3.28)

Since (3.28) is a function of \( a_1,\ldots,a_p \) and according to (3.27), (3.26) do not lead to separable firing degree and hence do not satisfy (3.22).

**Testing** for \( \varphi^3 = (1 - \gamma) \prod_{i=1}^{p} a_i + \gamma \left( \sum_{i=1}^{p} a_i - \prod_{i=1}^{p} a_i \right) \)

Here \( g(a_1,a_2,\ldots,a_p) = \gamma \left( \sum_{i=1}^{p} a_i - \prod_{i=1}^{p} a_i \right) \)

$$\frac{\partial g}{\partial a_i} = \gamma \left( 1 - \prod_{k \neq i} a_k \right) \neq -(1 - \gamma) \prod_{k \neq i} a_k$$

Hence the model \( \varphi^3 \) is not separable.
Testing for \( \phi^4 = (1 - \gamma) \prod_{i=1}^{p} a_i + \gamma \min \left( \sum_{i=1}^{p} a_i, 1 \right) \)

\[ g(a_1, \ldots, a_p) = \gamma \min \left( \sum_{i=1}^{p} a_i, 1 \right) \]

Case 1

If \( \min \left( \sum_{i=1}^{p} a_i, 1 \right) = \sum_{i=1}^{p} a_i \),

\[ \frac{\partial g}{\partial a_i} = \gamma \neq -(1 - \gamma) \prod_{k \neq i} a_k \]

So according to Corollary 3.2 \( \phi^4 \) is not usable.

Case 2

If \( \min \left( \sum_{i=1}^{p} a_i, 1 \right) = 1 \)

\[ \phi^4 = (1 - \gamma) \prod_{i=1}^{p} a_i + \gamma \]

\[ f^1(X/\phi^4) = \sup_x \left\{ \prod_{i=1}^{p} \mu_{A_i X_i}(x_i) \times \left( (1 - \gamma) \prod_{i=1}^{p} \mu_{\xi_i}(x_i) + \gamma \right) \right\} \]

\[ f^1(X/\phi^4) = \sup_x \left\{ (1 - \gamma) \prod_{i=1}^{p} \mu_{A_i X_i}(x_i) \mu_{\xi_i}(x_i) + \left( \gamma \prod_{i=1}^{p} \mu_{A_i X_i}(x_i) \right) \right\} \]

The procedure of searching for \( x_{\max}^1 \) in \( X_1 \times \ldots \times X_p \) can be divided into separate procedure of obtaining \( x_{i,\max}^1 = \arg \sup_{x_i} \left[ \mu_{A_i X_i}(x_i) \mu_{\xi_i}(x_i) \right] \). Hence \( \phi^4 \) leads to separable firing degree when \( \sum_{i=1}^{p} a_i > 1 \).

Corollary 3.3

If the connector model with bounded difference t-norm for multiplicative compensatory and takes the form \( \phi = \max \left( \sum_{i=1}^{p} a_i - 1, 0 \right)^{1 - \gamma} \) \( g(a_1, \ldots, a_p) \) (3.29)
then \( \varphi \) does lead to separable firing degree if
\[
\frac{\partial \ln g(a_1, \ldots, a_p)}{\partial a_i} 
= -\frac{(1 - \gamma)}{\sum_{i=1}^{p} a_i - 1}
\] (3.30)
provided \( \sum_{i=1}^{p} a_i > 1 \).

**Proof:**
\[
\varphi = \left( \max \left( \sum_{i=1}^{p} a_i - 1, 0 \right) \right)^{1-\gamma} g(a_1, \ldots, a_p)
\]

For \( \sum_{i=1}^{p} a_i > 1 \)
\[
\ln \varphi = (1 - \gamma) \ln \left( \sum_{i=1}^{p} a_i - 1 \right) + \ln g(a_1, \ldots, a_p)
\]
\[
\frac{\partial \ln \varphi(a_1, \ldots, a_p)}{\partial a_i} = (1 - \gamma) \frac{1}{\left( \sum_{i=1}^{p} a_i - 1 \right)} + \frac{\partial \ln g(a_1, \ldots, a_p)}{\partial a_i}
\] (3.31)

As in the discussion of corollary 3.1 and corollary 3.2 the model does not lead to separable firing degree if (3.30) does not hold good.
Testing for $\varphi^5 = \left\{ \max_{i=1}^{p} \left( \sum_{i=1}^{p} a_i - 1, 0 \right) \right\}^{1-\gamma} \left\{ \min_{i=1}^{p} \left( \sum_{i=1}^{p} a_i, 1 \right) \right\}^\gamma$

$\ln \varphi^5 = (1-\gamma) \ln \max_{i=1}^{p} \left( \sum_{i=1}^{p} a_i - 1, 0 \right) + \gamma \ln \min_{i=1}^{p} \left( \sum_{i=1}^{p} a_i, 1 \right)$

$\sum_{i=1}^{p} a_i - 1 > 0 \Rightarrow \sum_{i=1}^{p} a_i > 1$

$max_{i=1}^{p} \left( \sum_{i=1}^{p} a_i - 1, 0 \right) = \sum_{i=1}^{p} a_i - 1$

then $\min_{i=1}^{p} \left( \sum_{i=1}^{p} a_i, 1 \right) = 1$

$\ln g(a_1, \ldots, a_p) = 0$

$\frac{\partial \ln g(a_1, \ldots, a_p)}{\partial a_i} \neq -(1-\gamma) \frac{1}{\sum_{i=1}^{p} a_i - 1}$

$\varphi^6_{\max} = \left\{ \max_{i=1}^{p} \left( \sum_{i=1}^{p} a_i - 1, 0 \right) \right\}^{1-\gamma} \left\{ \sum_{i=1}^{p} a_i - \prod_{i=1}^{p} a_i \right\}^\gamma$

$\ln \varphi^6_{\max} = (1-\gamma) \ln \max_{i=1}^{p} \left( \sum_{i=1}^{p} a_i - 1, 0 \right) + \gamma \ln \left\{ \sum_{i=1}^{p} a_i - \prod_{i=1}^{p} a_i \right\}$

$= (1-\gamma) \ln \sum_{i=1}^{p} a_i - 1 + g(a_1, \ldots, a_p)$

$g(a_1, \ldots, a_p) = \gamma \ln \left\{ \sum_{i=1}^{p} a_i - \prod_{i=1}^{p} a_i \right\}$

$\frac{\partial \ln g(a_1, \ldots, a_p)}{\partial a_i} = \frac{\gamma}{\sum_{i=1}^{p} a_i - \prod_{i=1}^{p} a_i} \left( 1 - \prod_{k \neq i} a_k \right) \neq - (1-\gamma) \frac{1}{\sum_{i=1}^{p} a_i - 1}$

Condition (3.30) is not true for $\varphi^5$ and $\varphi^6$. Hence both the models are not separable.
**Corollary 3.4**

If a t norm model for additive compensatory and has the structure

\[ \varphi = (1 - \gamma) \max \left( \sum_{i=1}^{p} a_i - 1, 0 \right) + g(a_1, \ldots, a_p) \quad (3.32) \]

then \( \varphi \) does not lead to separable firing degree if

\[ \frac{\partial g(a_1, \ldots, a_p)}{\partial a_i} \neq -(1 - \gamma) \quad (3.33) \]

Provided \( \sum_{i=1}^{p} a_i > 1 \) and if \( \sum_{i=1}^{p} a_i \leq 1 \) then either \( \frac{\partial g(a_1, \ldots, a_p)}{\partial a_i} \) should be a constant

or \( g(a_1, \ldots, a_p) \) should be a function independent of \( a_j \) for \( i \neq j \).

**Proof:**

If \( \max \left( \sum_{i=1}^{p} a_i - 1, 0 \right) = \sum_{i=1}^{p} a_i - 1 \)

From (3.32) which means \( \varphi \) does not lead to separable firing degree if

\[ \frac{\partial g(a_1, \ldots, a_p)}{\partial a_i} \neq -(1 - \gamma) . \quad (3.34) \]

On the other hand if \( \max \left( \sum_{i=1}^{p} a_i - 1, 0 \right) = 0 \) then

\[ \frac{\partial \varphi(a_1, \ldots, a_p)}{\partial a_i} = \frac{\partial g(a_1, \ldots, a_p)}{\partial a_i} , \quad \text{implies (3.22) holds if} \]

\[ \frac{\partial g(a_1, \ldots, a_p)}{\partial a_i} \] is a constant or is function of \( a_i \) only for \( i \neq j \).
Testing for $\phi^8 = (1-\gamma)\max\left(\sum_{i=1}^{p} a_i, -1.0\right) + \gamma\left(\sum_{i=1}^{p} a_i, -\prod_{i=1}^{p} a_i\right)$

Case 1 $\max\left(\sum_{i=1}^{p} a_i, -1.0\right) = \sum_{i=1}^{p} a_i, -1$

$\phi^8 = (1-\gamma)\left(\sum_{i=1}^{p} a_i, -1\right) + \gamma\left(\sum_{i=1}^{p} a_i, -\prod_{i=1}^{p} a_i\right)$

$g(a_1,\ldots,a_p) = \gamma\left(\sum_{i=1}^{p} a_i, -\prod_{i=1}^{p} a_i\right)$

$\frac{\partial \phi^8}{\partial a_i} = (1-\gamma) + \gamma\left(1 - \prod_{k\neq i} a_k\right)$

$\frac{\partial g(a_1,\ldots,a_p)}{\partial a_i} \neq -(1-\gamma)$, so according to corollary 3.4 $\phi^8$ do not lead to separable firing degree.

Case 2 If $\max\left(\sum_{i=1}^{p} a_i, -1.0\right) = 0$

$\phi^8 = g(a_1,\ldots,a_p) = \gamma\left(\sum_{i=1}^{p} a_i, -\prod_{i=1}^{p} a_i\right)$

Condition (3.34) is not true for $\phi^8$. Hence do not lead to separable firing degree.

Corollary 3.5

If the model $\phi$ is of the form $\phi = \left\{\prod_{p=1}^{l} a_i\right\}^{1-\gamma} g(a_1, a_2, \ldots, a_p)$ then it does not lead to separable firing degree if

$$
\frac{\partial \ln(g(a_1, a_2, \ldots, a_p))}{\partial a_i} \neq -\frac{1-\gamma}{p} \frac{1}{\left\{\prod_{p=1}^{l} a_i\right\}}
$$

(3.35)

Proof:

$\phi = \left\{\prod_{p=1}^{l} a_i\right\}^{1-\gamma} g(a_1, a_2, \ldots, a_p)$
\[
\ln \varphi = (1 - \gamma) \ln \left( \frac{1}{p} \sum_{i=1}^{p} a_i \right) + \ln g(a_1, a_2, \ldots, a_p)
\]

\[
\frac{\partial \ln \varphi}{\partial a_i} = (1 - \gamma) \frac{1}{p} \frac{1}{\sum_{i=1}^{p} a_i} + \frac{\partial \ln g(a_1, a_2, \ldots, a_p)}{\partial a_i}
\]

It is seen that

\[
\frac{\partial \ln g(a_1, a_2, \ldots, a_p)}{\partial a_i} \neq \frac{1 - \gamma}{1} \frac{1}{p} \left( \frac{1}{\sum_{i=1}^{p} a_i} \right)
\]

then \( \varphi \) does not lead to separable firing degree.

It could be proved as in the previous case that \( \varphi^9, \varphi^{11} \) and \( \varphi^{13} \) does not lead to separable firing degree.

**Corollary 3.6**

If the connector model is of the form

\[
\varphi = (1 - \gamma) \left( \frac{1}{p} \sum_{i=1}^{p} a_i \right) + \gamma g(a_1, a_2, \ldots, a_p)
\]

will not lead to separable firing degree if

\[
\frac{\partial g(a_1, a_2, \ldots, a_p)}{\partial a_i} \neq \frac{1 - \gamma}{p}
\]

(3.36)

**Proof:**

\[
\varphi = (1 - \gamma) \left( \frac{1}{p} \sum_{i=1}^{p} a_i \right) + g(a_1, a_2, \ldots, a_p)
\]

\[
\frac{\partial \varphi}{\partial a_i} = (1 - \gamma) \frac{1}{p} + \frac{\partial g(a_1, a_2, \ldots, a_p)}{\partial a_i}
\]

Hence the SOWA model \( \varphi_{SOWA} \) will not lead to separable firing degree if

\[
\frac{\partial g(a_1, a_2, \ldots, a_p)}{\partial a_i} \neq \frac{1 - \gamma}{p}
\]

Similar to the previous case \( \varphi^{10}, \varphi^{14} \) could be proved to be not separable.
3.5.2.2 Implementation of minimum t-norm model for sup-star composition and Mamdani implication

In this section min t-norm model is used for both sup-star composition and Mamdani implication, and the additive S-OWA connector models that use either the t-norm as bounded difference or t-conorm as bounded sum, namely the models $\phi^{10}$ and $\phi^{14}$ in Table 3.1 are examined for separability.

**Theorem 3.2**

When connector model $\phi$ can be expressed as

$$
\phi\left[\mu_{x_1}(x_1), \ldots, \mu_{x_p}(x_p)\right] = \left(\frac{1-\gamma}{p} + \gamma\right) g\left(\mu_{f_1}(x_1), \ldots, \mu_{f_p}(x_p)\right) - \gamma h\left(\mu_{f_1}(x_1), \ldots, \mu_{f_p}(x_p)\right)
$$

then $\phi$ leads to separable firing degree for nonsingleton fuzzification if and only if

$$
\sup \min \left\{\mu_{x_1}(x_1), \ldots, \mu_{x_p}(x_p), \left(\frac{1-\gamma}{p} + \gamma\right) g\left(\mu_{f_1}(x_1), \ldots, \mu_{f_p}(x_p)\right) - \gamma h\left(\mu_{f_1}(x_1), \ldots, \mu_{f_p}(x_p)\right)\right\}
$$

is separable.

**Proof:**

For minimum t-norm model the firing degree in (3.21) takes the form

$$
f(x/\phi) = \sup \min \left\{\min_i \mu_{\mu_i}(x_i), \phi\right\}
$$

$$
= \sup \min \left\{\min_i \mu_{\mu_i}(x_i), \left(\frac{1-\gamma}{p} + \gamma\right) g\left(\mu_{f_1}(x_1), \ldots, \mu_{f_p}(x_p)\right) - \gamma h\left(\mu_{f_1}(x_1), \ldots, \mu_{f_p}(x_p)\right)\right\}
$$

$$
= \sup \min \left\{\min_i \mu_{\mu_i}(x_1), \ldots, \mu_{\mu_i}(x_p), \left(\frac{1-\gamma}{p} + \gamma\right) g\left(\mu_{f_1}(x_1), \ldots, \mu_{f_p}(x_p)\right) - \gamma h\left(\mu_{f_1}(x_1), \ldots, \mu_{f_p}(x_p)\right)\right\}
$$

$$
= \sup \min \left\{\mu_{\mu_i}(x_1), \ldots, \mu_{\mu_i}(x_p), \left(\frac{1-\gamma}{p} + \gamma\right) g\left(\mu_{f_1}(x_1), \ldots, \mu_{f_p}(x_p)\right) - \gamma h\left(\mu_{f_1}(x_1), \ldots, \mu_{f_p}(x_p)\right)\right\}
$$
The procedure for searching for \( x_{\text{max}} \) is separable if and only if the computation of (3.38) is separable.

Testing for the model \( \phi^{10} = (1 - \gamma) \left[ \frac{1}{p} \sum_{i=1}^{p} a_i \right] + \gamma \max \left( \sum_{i=1}^{p} a_i - 1, 0 \right) \)

\[
f^1(x / \phi^{10}) = \sup_{x} \min \left\{ \min_{\mu_{A_{x_i}}(x_i), \phi^{10}} \right\}
= \sup_{x} \min \left\{ \min_{\mu_{A_{x_i}}(x_i), (1 - \gamma) \left[ \frac{1}{p} \sum_{i=1}^{p} a_i \right] + \gamma \max \left( \sum_{i=1}^{p} a_i - 1, 0 \right)} \right\}
= \sup_{x} \min \left\{ \min_{\mu_{A_{x_i}}(x_i), (1 - \gamma) \left[ 1 \sum_{i=1}^{p} \mu_{f_i}(x_i) \right] + \gamma \max \left( \sum_{i=1}^{p} \mu_{f_i}(x_i) - 1, 0 \right)} \right\}
\]

If \( \sum_{i=1}^{p} \mu_{f_i}(x_i) > 1 \)

\[
f^1(x / \phi_{m}) = \sup_{x} \min \left\{ \min_{\mu_{A_{x_i}}(x_i), (1 - \gamma) \left[ \frac{1}{p} \sum_{i=1}^{p} \mu_{f_i}(x_i) \right] + \gamma \left( \sum_{i=1}^{p} \mu_{f_i}(x_i) - 1 \right)} \right\}
= \sup_{x} \min \left\{ \min_{\mu_{A_{x_i}}(x_i), \left( \frac{1 - \gamma}{p} + \gamma \right) \sum_{i=1}^{p} \mu_{f_i}(x_i) - \gamma} \right\}
= \sup_{x} \min_{\mu_{A_{x_i}}(x_i), \left( \frac{1 - \gamma}{p} + \gamma \right) \sum_{i=1}^{p} \mu_{f_i}(x_i) - \gamma} \left( \mu_{A_{x_i}}(x_i), \left( \frac{1 - \gamma}{p} + \gamma \right) \sum_{i=1}^{p} \mu_{f_i}(x_i) - \gamma \right)
\] (3.39)
The procedure of searching for $x_{\text{max}}^1$ in $X_1 \times \ldots \times X_p$ cannot be divided into separate procedure of searching for $x_{i,\text{max}}^i$ in $X_i$ (i=1,2,\ldots,p), because of the coupling among $x_1$, $x_2, \ldots, x_p$ in $\sum_{i=1}^{p} \mu_{i,i'}^i(x_i)$ in (3.39). Hence the connector model $\varphi^{10}$ is not usable.

3.6 Conclusion

Knowledge representation with fuzzy production rules is quite natural and simple procedure. The fuzzy model considers human's imprecise perception and decision base. Factors which can influence on exactness of fuzzy logic system has to be taken into account. Selecting operator is very important and it directly influences on result exactness. Selecting appropriate connector models and operators in rules facilitate the system output. A learning pattern that corresponds to expert opinion about desired measurement process can greatly accelerate this process. Fuzzy inference system can be calibrated by collaboration with the neural network.

For fuzzy logic systems that needs to be optimized during tuning procedure requires the firing degree to be separable. The firing degree must be expressed in a closed form and a piece wise-differentiable function of the membership function parameters and the parameter of the antecedent connector model. The compensatory and soft ordered weighted averaging operators are used as antecedent connector models and were examined which models lead to separable firing degree.

Here it is assumed the t-norm $T$ for implementing Mamdani implication and sup-star composition as same. It is seen that for a singleton fuzzy logic systems all the models defined are usable whereas when Mamdani product is used for non-singleton fuzzy logic systems only the multiplicative as well as additive
compensatory and connector models with t-norm as product and t-conorm as bounded sum are usable.

For future work various other antecedent connector models or the compensatory models that use other t-norms and t-conorms can be tested for usability. In this chapter the t-norm T for Mamdani implication and sup-star composition were adopted to be same. The work could also be extended by taking different combination of t-norms could also be implemented and tested for separabilitiy of the firing degree.