CHAPTER - II
ADVANCED PROBABLISTIC MODELS

A. INTRODUCTION

In general, better results can be obtained by fitting models on continuous trading. This kind of approach can be studied by various research scholars on the theory of economic behavior. A clear cut analysis based on the martingales with respect to $\sigma$ - fields gives better results in the study of portfolio management.

The earlier assumption on the share market analysis is based on the markov property assumption. This model to some extent helps us predict the risk involved in a particular industry and thereby helps us to settle down with maximum gain under prescribed limiting property. In general better results can be obtained by fitting models on continuous trading. This kind of approach can be studied by various research scholars on the theory of economic behavior like Modigliani, Millar and others.

A special type of stochastic process which is based on conditional expectations as sequence of random variables called Martingales has become a better tool to study continuous trading. This type of perspective helps one to have two types of options; one on the sampling and the other on the stopping process. The market can be regarded as a, complete one, in the sense that lower risk on one is compensated by higher profit on the other for investors. This series study is taken up by Harrison and Pliska and we find that the martingale theory plays an important role in optional sampling and optional investment on various shares yielding low and high returns. We explain in detail the different stages of improvement on this model and their implications on the consumer’s satisfaction.

A set of events form a collection of sets in the sample space which is closed under arbitrary union, finite intersection, complements and they form a $\sigma$ -field. So we are left with the consideration of martingale with respect to $\sigma$ - fields.
2.1. Martingales with Respect to \( \sigma \)-Fields

Until now we have always considered conditional expectations to be expectations computed under conditional distributions. This is mostly satisfactory for expressions of the form \( E[ X \mid Y_0, \ldots, Y_n ] \), where \( X, Y_0, \ldots, Y_n \) possess a joint continuous density or are jointly discrete random variables. However, the analysis extended to the more complex expressions like \( E[ X \mid Y_0, Y_1, \ldots ] \) or \( E[ X \mid Y(u), 0 \leq u \leq t ] \) becomes more delicate.

The alternative and more modern approach is to define and evaluate conditional expectation, not with respect to a finite family of random variables, as we have done so far, but with respect to certain collections, called \( \sigma \)-fields of events. This suggests in a natural way a definition of a martingale with respect to a sequence of \( \sigma \)-fields [42].

**Definition 2.1:**

The probability measure, a function \( P \) defined on \( \mathcal{F} \) and satisfying

(a) \( 0 = P(\emptyset) \leq P(A) < P(\Omega) = 1 \), for \( A \in \mathcal{F} \) (\( \emptyset \)-the empty set),

(b) \( P(A_i \cup A_2) = P(A_1) + P(A_2) - P[A_1 \cap A_2] \), for \( A_i \in \mathcal{F}, i = 1,2, \) and

(c) \( P\left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} P[A_n] \),

if \( A_n \in \mathcal{F} \) are mutually disjoint (\( A_i \cap A_j = \emptyset, i \neq j \)).

2.2. Modigliani and Miller

What is cost of capital to a firm in a world in which funds are used to acquire assets whose yields are uncertain; and in which capital can be gained by different media, ranging from pure debt instruments, representing money fixed claims to pure equity, issues, giving holders only the right to a pro-rata share in the uncertain venture?
In much of his analysis the economic theorist at least tends to sidestep the essence of this cost of capital problem by proceeding as though physical assets like bonds could be regarded as yielding sure streams. Only recently have economists begun to face seriously the problem of the cost capital cum risk. Modigliani and Miller [53] have derived the following simple rule for optimal investment policy by the firm. Regardless of the financing used, the marginal cost of capital to a firm is equal to the capitalization rate for an unlevered stream in the class to which the firm belongs.

To establish they have considered the three major financing alternatives open to the firm—bonds, retained earnings, and common stock issues and showed in each case an investment is worth undertaking if and only if the rate of return on investment is longer than the average cost of capital.

The analysis developed here was essentially a comparative statistics not a dynamic analysis. Such analysis as those posed by expected changes in rate of return and in average cost of capital over time has not been treated.

2.3. Black and Scholes

Black and Scholes [8] made dazzling observation that, in the idealized market, investors can duplicate the cash flow (or pay off stream) from a call option by cleverly managing a portfolio that contains only stock and bond. Since the possession of this portfolio is completely equivalent to possession of their call option, the market value of its securities at time zero is the unique rational value for the option.

2.4. Harrison and Pliska

Harrison and Pliska [38] have taken up the study of continuous trading and developed a general stochastic model of a frictionless security market with continuous trading. Within the framework of that model they discussed the option pricing formula; we can use it for the study of consumption investment problems.
2.5. Modern Theory of contingent claim valuation

Let $W = \{W_t; 0 \leq t \leq T\}$ be a standard (zero drift and unit variance) Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{F})$. Let $r$, $\mu$ and $\sigma$ be real constants with $\sigma > 0$. Define

$$S_t^0 = S_0^0 \exp(rt), \quad 0 \leq t \leq T,$$

$$S_t^1 = S_0^1 \exp(\sigma W_t + (\mu - \frac{1}{2} \sigma^2)t), \quad 0 \leq t \leq T,$$

where the initial values $S_0^0$ and $S_0^1$ are positive constants. $S^0$ and $S^1$ are the price process for risk less security and risky security respectively. These satisfy the stochastic differential equations

$$dS_t^0 = rS_t^0 dt$$

$$dS_t^1 = \sigma S_t^1 dW_t + \mu S_t^1 dt$$

where $S^1$ is a geometric Brownian with rate of return $\sigma dW_t + \mu dt$, called the return process for the stock. Consider a ticket which entitles its bearer to buy one share of stock, at the terminal date $T$, if he wishes for a specified price of $c$ units. Call option is equivalent to a payment $X = (S_T^1 - c)^+$ at time $T$.

Black and Scholes [8] asserted that there is a unique rational value for the option, independent of one's risk attitude. Define

$$f(x, t) = x \Phi(g(x, t)) - ce^{-rT} \Phi(h(x, t)), \quad \text{where}$$

$$g(x, t) = \left[ \ln \left( \frac{x}{c} \right) + (r + \frac{1}{2} \sigma^2) t \right] / \sigma \sqrt{t},$$

$$h(x, t) = g(x, t) - \sigma \sqrt{t}$$

and $\Phi(\cdot)$ is the standard normal distribution function, this unique rational value is $f(S_t^1, T)$.

The function $f(x, t)$ defined above satisfies the partial differential equation

$$\frac{\partial}{\partial t} f(x, t) = \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} f(x, t) + r x \frac{\partial}{\partial x} f(x, t) - r f(x, t)$$

with initial condition $f(x, 0) = (x - c)^+$. 

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Define stochastic processes
\[ V_t = f(S^1_t, T-t), 0 \leq t \leq T, \]
\[ \phi^1_t = \frac{\partial}{\partial x} f(S^1_t, T-t), 0 \leq t \leq T, \]
\[ \phi^0_t = (V_t - \phi^1_t S^1_t) / S^0_t, 0 \leq t \leq T. \]

The market value of the portfolio held at time \( t \) is
\[ \phi^0_t S^0_t + \phi^1_t S^1_t = V_t, 0 \leq t \leq T. \]

The initial value of the portfolio is
\[ V_0 = f(S^1_0, T) \]
and the terminal value
\[ V_T = f(S^1_T, 0) = (S^1_T - c)^+ \]
is precisely equal to the terminal value of the call option. Finally applying Ito's formula we obtain
\[ dV_t = \frac{\partial}{\partial x} f(S^1_t, T-t) dS^1_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} f(S^1_t, T-t)(dS^1_t)^2 + \frac{\partial}{\partial t} f(S^1_t, T-t) dt. \]

Then
\[ V_t - V_0 = \int_0^t \phi^0_u dS^0_u + \int_0^t \phi^1_u dS^1_u, 0 \leq t \leq T. \]

The right hand side represents the total earnings or capital gains which we realize on our holdings up to time \( t \).

2.6. Continuous Trading

We begin now with probability space \((\Omega, \mathcal{F}, \mathcal{P})\) and a filtration (increasing family of sub \( \sigma \)-algebras) \( \mathcal{F} = \{ \mathcal{F}_t; 0 \leq t \leq T \} \), satisfying the usual conditions \( \mathcal{F}_0 \) contains all null sets of \( \mathcal{P} \); \( \mathcal{F} \) is right continuous meaning that
\[ \mathcal{F}_t = \bigcap_{\sigma \leq t} \mathcal{F}_\sigma, \text{ for } 0 \leq t \leq T. \]

Let \( S = \{ S_t; 0 \leq t \leq T \} \) be a vector process whose components \( S^0, S^1, \ldots, S^K \) are adopted (\( S^k_t \in \mathcal{F}_t \) for \( 0 \leq t \leq T \)), right continuous with left limits and strictly positive.
Let $S^0_0 = 1$. We could write

$$S_t^0 = \exp \left( \int_0^t \gamma_s ds \right), 0 \leq t \leq T,$$

for some process $\gamma$, and then $\gamma_t$ would be interpreted as the riskless interest rate at time $t$. Define $\alpha_t = \log (S_t^0), 0 \leq t \leq T$, we call $\alpha$ the return process for $S^0$,

$$\beta_t = 1 / S_t^0 = \exp (-\alpha_t), \quad 0 \leq t \leq T,$$

calling $\beta$ the intrinsic discount process for $S$. It will be convenient to define a discounted process $Z = (Z^1, ..., Z^K)$ by setting

$$Z_t^k = \beta_t S_t^k, \quad 0 \leq t \leq T \quad \text{and} \quad k = 1, ..., K.$$

Let $\mathcal{P}$ be the set of probability measures $Q$ on $(\Omega, \mathcal{F})$ which are equivalent to $P$ and such that $Z$ is a martingale under $Q$, since $\beta S_0 = 1$ is a martingale under any measure equivalent to $P$. Elements of $\mathcal{P}$ are called martingale measures.

We have that $S^0$ is a variation finite process and thus a semi martingale, that $Z^k$ is a martingale under any $Q \in \mathcal{P}$, and that $S^k = Z^k / \beta = S^0 Z^k$, $S^k$ is semi martingale under $Q$ and thus also under $P$. Hence $S$ is a vector semi martingale.

A trading strategy is defined as $K+1$ dimensional process $\phi = \{ \phi_t, 0 \leq t \leq T \}$ whose components are locally bounded and predictable. With each such strategy $\phi$ we associated a value process $V(\phi)$ and a gains process $G(\phi)$ by

$$V_t(\phi) = \phi_t S_t = \sum_{k=0}^K \phi_t^k S_t^k, \quad 0 \leq t \leq T,$$

$$G_t(\phi) = \int_0^t \phi_u dS_u = \sum_{k=0}^K \int_0^t \phi_u^k dS_u^k, \quad 0 \leq t \leq T.$$

We interpret $V_t(\phi)$ as the market value of the portfolio $\phi_t$ and $G_t(\phi)$ as the net capital gains. We say that a trading strategy $\phi$ is self financing if

$$V_t(\phi) = V_0(\phi) + G_t(\phi), \quad 0 \leq t \leq T.$$
2.7. Formulation of the Model

Let us select and fix a reference measure $P^* \in \mathcal{Q}$, denoting $E^*(\cdot)$ the associated expected operator. We define $\mathcal{E}(z)$ as the set of all predictable process $H = (H^1, \ldots, H^K)$ such that the increasing process, $\left( \int_0^t (H^k_s)^2 d[Z^k, Z^k_s] \right)^{1/2}$, $0 \leq t \leq T$, is locally integrable under $P^*$ for each $k = 1, \ldots, K$. It can be verified that $\mathcal{E}(z)$ contains all locally bounded and predictable $H$ and moreover $\int HdZ$ is still a local martingale for these integrands.

We now expand our definition of a trading strategy to include all predictable $\phi = (\phi^0, \phi^1, \ldots, \phi^K)$ such that $(\phi^1, \phi^2, \ldots, \phi^K) \in \mathcal{E}(z)$. With $V^*(\phi) = \beta \phi S$ and $G^*(\phi) = \int \phi dZ$, a trading strategy $\phi$ is said to be admissible if $V^*(\phi) \geq 0$, $V^*(\phi) = V_0^*(\phi) + G^*(\phi)$ and $V^*(\phi)$ is a martingale (under $P^*$).

A contingent claim defined as a positive random variable $X$. Such a claim is said to be attainable if there exists $\phi \in \Phi^*$ such that $V_T^*(\phi) = \beta_T X$, in which case $\phi$ is said to generate $X$ and $\pi = V_0^*(\phi)$ is called the price associated with $X$.

We give the following propositions in connection with the study of continuous market.

2.8. Complete Market

Proposition 2.1: (Harrison and Pliska [38]). The unique price $\pi$ associated with an attainable claim $X$ is $\pi = E^*(\beta_T X)$.

Proposition 2.2: (Harrison and Pliska [38]). Let $X$ be an integrable contingent claim and let $V^*$ be the modification of $V^*_t = E_\tau \left( \beta \tau X \right)$, $0 \leq t \leq T$. Then $X$ is attainable if and only if $V^*$ can be represented in the form $V^* = V_0^* + \int HdZ$ for some $H \in \mathcal{E}(z)$, in which case $V^*(\phi) = V^*$ for any $\phi \in \Phi^*$ which generates $X$. Let
\( \mathcal{M}(Z) \) consist of all \( M \in \mathcal{M} \) can be represented in the form \( M = M_0 + \int H \, dZ \) for some \( H \in \mathcal{E}(z) \).

**Proposition 2.3:** The model is complete if and only if \( \mathcal{M} = \mathcal{M}(Z) \) Harrison and Pliska [38] have conjectured. If \( \mathcal{P} \) is a singleton, and then the model is complete. Their conjecture is settled by the following result. Therefore the following statements are equivalent.

(i) The model is complete under \( \mathcal{P}^* \)
(ii) Every martingale \( M \) can be represented in the form

\[
M = M_0 + \int H \, dZ \text{ is for some } H \in \mathcal{E}(z).
\]

(iii) \( \mathcal{P} \) is a singleton.

By a martingale we mean the real valued stochastic process

\[
M = \{ M_t : 0 \leq t \leq T \},
\]

satisfying the usual definition of a martingale under the filtration \( \mathcal{F} \) and reference measure \( \mathcal{P}^* \).

**2.9. Conclusion**

The data of three major industries namely auto industry, medical industry and textile industry [79] have been taken. The investor’s preference on these three is calculated and in a market in which investors are engaged in these three industries finds a compactable risk free investment.
### AUTO INDUSTRIES

**Table 2.1 ASHOK LEYLAND**

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*Courtesy – www.bseindia.com transactions by ASHOK LEYLAND*

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### Table 2.3 HERO HONDA MOTORS

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TOTAL 634.104

*Courtesy – www.bseindia.com transactions by HERO HONDA MOTORS*

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TOTAL 1284.403

*Courtesy – www.bseindia.com transactions by CIPLA LTD*
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*Courtesy – www.bseindia.com transactions by RANBAXY LAB*

### Table 2.6 DR. REDDY

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*Courtesy – www.bseindia.com transactions by DR. REDDY*
## TEXTILE INDUSTRIES

### Table 2.7 RAYMOND LIMITED

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**TOTAL** 348.461

*Courtesy – www.bseindia.com transactions by RAYMOND LIMITED*

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**TOTAL** 590.176

*Courtesy – www.bseindia.com transactions by BOMBAY DYEING*
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C. WIENER PROCESS

At epoch \( t \), let \( X(t) \) be the displacement along a fixed axis of a particle undergoing Brownian motion and let \( X(0) = x_0 \). Consider an interval \( (s, t) \) of time; let us regard this interval as the sum of a large number of small intervals. The total displacement \( \{X(t) - X(s)\} \) in this interval can be regarded as the limit of the sum of random displacements over the small intervals of time. Suppose the random displacements are independently distributed, then it can be seen that the central limit theorem applies, whence it follows that the total displacement \( \{X(t) - X(s)\} \) is normally distributed. Further, suppose that the displacement \( \{X(t) - X(s)\} \) depends on the length of the interval \( (s, t) \) and not on the time-point \( s \) and that \( \{X(t) - X(s)\} \) has the same distribution as \( \{X(t + h) - X(s + h)\} \) for all \( h > 0 \).

We assume that the process \( \{X(t), t \geq 0\} \) is Markovian. Let the cumulative transition probability be

\[
P(x_0, s; x, t) = Pr\{X(t) \leq x \mid X(s) = x_0\}, s < t, \quad ...2.1
\]

and let the transition probability density \( p \) be given by

\[
p(x_0, s; x, t)dx = Pr\{x \leq X(t) < x + dx \mid X(s) = x_0\}. \quad ...2.2
\]
For a homogeneous process the transition probability depends only on the length of the interval \((t - s)\) and then the transition probability may be denoted in terms of the three parameters \(x_0, x, t - s\). We denote
\[
Pr \{x \leq X(t + t_0) < x + dx \mid X(t_0) = x\} \text{ by } p(x_0, x; t) \, dx \text{ for any } t_0.
\]

The Chapman-Kolmogorov equation can be written as follows:
\[
P \{x_0, s; x, t\} = \int d_z P(x_0, s; z, v) \, P(z, v; x, t).
\]

In terms of transition probabilities \(p(x_0, s; x, t)\), we have
\[
p \{x_0, s; x, t\} = \int p(x_0, s; z, v) \, p(z, v; x, t) \, dz.
\]

Consider that a (Brownian) particle performs a random walk such that in a small interval of time of duration \(\Delta t\), the displacement of the particle to the right or to the left is also of small magnitude \(\Delta x\), the total displacement \(X(t)\) of the particle in time \(t\) being \(x\). Suppose that the random variable \(Z_i\) denotes the length of the \(i^{th}\) step taken by the particle in a small interval of time \(\Delta t\) and that
\[
Pr \{Z_i = \Delta x\} = p \quad \text{and} \quad Pr \{Z_i = -\Delta x\} = q, \quad p + q = 1, \quad 0 < p < 1,
\]
where \(p\) is independent of \(x\) and \(t\).

Suppose that the interval of length \(t\) is divided into \(n\) equal subintervals of length \(\Delta t\) and that the displacements \(Z_i, i = 1 \ldots n\) in the \(n\) steps are mutually independent random variables. Then \(n(\Delta t) = t\) and the total displacement \(X(t)\) is the sum of \(n\) i.i.d. random variables \(Z_i\) that is,
\[
X(t) = \sum_{i=1}^{n(t)} Z_i, \quad n(t) = t / \Delta t.
\]

We have
\[
E \{Z_i\} = (p - q) \Delta x \quad \text{and} \quad \text{var} \{Z_i\} = 4pq(\Delta x)^2.
\]

Hence
\[
E\{X(t)\} = nE(Z_i) = t(p - q) \Delta x / \Delta t, \quad \ldots 2.3
\]
and
\[
\text{var} \{X(t)\} = n \text{var} \{Z_i\} = 4pq t(\Delta x)^2 / \Delta t.
\]
To get a meaningful result, as $\Delta x \to 0$, $\Delta t \to 0$, we must have

$$\frac{(\Delta x)^2}{\Delta t} \to \text{a limit}, \quad (p - q) \to \text{a multiple of } (\Delta x). \quad \ldots 2.4$$

We may suppose, in particular, that in an interval of length $t$, $X(t)$ has mean-value function equal to $\mu t$ and variance function equal to $\sigma^2 t$. In other words, we suppose that as $\Delta x \to 0$, $\Delta t \to 0$, in such a way that equation 2.4 is satisfied, and per unit time.

$$E \{X(t)\} \to \mu \quad \text{and} \quad \text{var} \{X(t)\} \to \sigma^2. \quad \ldots 2.5$$

From the equation 2.3 for $t = 1$ and the equation 2.5 we have

$$\frac{(p - q) \Delta x}{\Delta t} \to \mu; \quad \frac{4pq(\Delta x)^2}{\Delta t} \to \sigma^2. \quad \ldots 2.6$$

The equations 2.4 and 2.6 will be satisfied when

$$\Delta x = \sigma (\Delta t)^{1/2}, \quad \ldots 2.7a$$

$$p = \frac{1}{2} \left(1 + \mu(\Delta t)^{1/2}/\sigma\right), \quad q = \frac{1}{2} \left(1 - \mu(\Delta t)^{1/2}/\sigma\right). \quad \ldots 2.7b$$

Now since $Z_i$ are i.i.d. random variables, the sum $\sum_{i=1}^{n(t)} Z_i = X(t)$ for large $n(t) (= n)$, is asymptotically normal with mean $\mu t$ and variance $\sigma^2 t$ (by virtue of the central limit theorem for equal components). Note that here also $t$ represents the length of the interval of time during which the displacement, that takes place is equal to the increment $X(t) - X(0)$. We thus find that for $0 < s < t$, $\{X(t) - X(s)\}$ is normally distributed with mean $\mu (t - s)$ and variance $\sigma^2 (t - s)$. Further, the increments $\{X(s) - X(0)\}$ and $\{X(t) - X(s)\}$ are mutually independent; this implies that $\{X(t)\}$ is a Markov process.

We may now define a Wiener process or a Brownian motion process as follows:
The stochastic process \( \{X(t), \; t \geq 0\} \) is called a Wiener process (or a Wiener-Einstein process or a Brownian motion process) with drift \( \mu \) and variance parameter \( \sigma^2 \), if

\( i \) \( X(t) \) has independent increments, i.e., for every pair of disjoint intervals of time \((s, t)\) and \((u, v)\), where \( s \leq t \leq u \leq v \), the random variables \( \{X(t) - X(s)\} \) and \( \{X(v) - X(u)\} \) are independent.

\( ii \) Every increment \( \{X(t) - X(s)\} \) is normally distributed with mean \( \mu (t - s) \) and variance \( \sigma^2 (t - s) \).

Note that (i) implies that Wiener process is a Markov process with independent increments and (ii) implies that a Wiener process is Gaussian.

Since \( \{X(t) - X(0)\} \) is normally distributed with mean \( \mu t \) and variance \( \sigma^2 t \), the transition probability density function \( p \) of a Wiener process is given by

\[
p(x_0, x; t) \; dx = Pr \{x \leq X(t) < x + dx \mid X(0) = x_0\}
= \frac{1}{\sigma \sqrt{2\pi}} \exp\left\{-\frac{(x - x_0 - \mu t)^2}{2\sigma^2 t}\right\} dx \quad \ldots 2.8
\]

A Wiener process \( \{X(t), \; t \geq 0\} \) with \( X(0) = 0 \), \( \mu = 0 \), \( \sigma = 1 \) is called a standard Wiener process [52].

2.10. Differential Equations for a Wiener Process

Let \( \{X(t), \; t \geq 0\} \) be a Wiener process. We can consider the displacement in such a process as being caused by the motion of a particle undergoing displacements of small magnitude in a small interval of time. Suppose that \((t - \Delta t, t)\) is an infinitesimal interval of length \( \Delta t \) and that the particle makes in this interval a shift equal to \( \Delta x \) with probability \( p \) or a shift equal to \( -\Delta x \) with probability \( q = 1 - p \). Suppose that \( p \) and \( q \) are independent of \( x \) and \( t \). Let the
transition probability that the particle has a displacement from $x$ to $x + \Delta x$ at epoch $t$, given that it started from $x_0$ at time $0$, be $p(x_0, x; t) \Delta x$. Further suppose that $p(x_0, x; t)$ admits of an expansion in Taylor’s series, i.e.

$$p(x_0, x + \Delta x; t - \Delta t) = p(x_0, x; t) - \Delta t \frac{\partial p}{\partial t} \pm \Delta x \frac{\partial p}{\partial x} + \frac{1}{2} (\pm \Delta x)^2 \frac{\partial^2 p}{\partial x^2} + o(\Delta t). \quad ...2.9$$

From simple probability arguments we have

$$p(x_0, x; t) \Delta x = p.p (x_0, x - \Delta x; t - \Delta t) \Delta x$$

$$+ q.p (x_0, x + \Delta x; t - \Delta t) \Delta x. \quad ... 2.10$$

Making use of equation 2.9, and cancelling out the factor $\Delta x$ from both sides of equation 2.10 we get

$$\frac{\partial p}{\partial t} p(x_0, x; t) = p(x_0, x; t) - \Delta t \frac{\partial p}{\partial t} - \Delta x(p - q) \frac{\partial p}{\partial x}$$

$$+ \frac{1}{2} (\Delta x)^2 \frac{\partial^2 p}{\partial x^2} + o(\Delta t).$$

Divide both sides by $\Delta t$, using equations 2.6 and 2.7 and taking limits as $\Delta t \to 0$, $\Delta x \to 0$, we get

$$\frac{\partial}{\partial t} p(x_0, x; t) = -\mu \frac{\partial}{\partial x} p(x_0, x; t) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} p(x_0, x; t). \quad ...2.11$$

This is a partial differential equation in the variables $x$ and $t$, being of the first order in $t$ and of the second order in $x$. The equation is known as the forward diffusion equation of the Wiener process. One can likewise obtain the backward diffusion equation of the process in the form

$$\frac{\partial}{\partial t} p(x_0, x; t) = \mu \frac{\partial}{\partial x_0} p(x_0, x; t) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x_0^2} p(x_0, x; t). \quad ...2.12$$
Definition 2.2:
Let $X$ be a measurable function whose integral exists. Let $\mathcal{B} \subseteq \mathcal{A}$ and be a $\sigma$-field. Then,
$$\phi(B) = \int_B X \, dP, \quad B \in \mathcal{B}$$
is the restriction of $\phi$ to sets of $\mathcal{B}$ denoted by $\phi_{\mathcal{B}}$. It is a $\mathcal{B}$ measurable, $P$-continuous and $\sigma$-additive set function. Hence by Radon-Nikodym theorem, there exists a $\mathcal{B}$-measurable function $d\phi/dP$, called the conditional expectation of $X$ given $\mathcal{B}$, denoted by $E^\mathcal{B}X$. It is defined by,
$$\int_B (E^\mathcal{B}X) \, dP = \int_B \left( \frac{d\phi}{dP} \right) \, dP = \int_B X \, dP, \quad \forall B \in \mathcal{B}$$

2.11. Expectation Properties

Conditional expectation possesses linearity property, etc., which are similar to those of an expectation of a random variable

i)  
(a) $E^\mathcal{B}(X+Y) = E^\mathcal{B}X + E^\mathcal{B}Y, \quad \text{a.s}$
(b) $E^\mathcal{B}(aX) = aE^\mathcal{B}X \quad \text{a.s}$

From the definition of conditioning and linearity property of expectations, we have
$$\int_B E^\mathcal{B}(X+Y) \, dP = \int_B (X+Y) \, dP = \int_B X \, dP + \int_B Y \, dP,$$
$$= \int_B E^\mathcal{B}X \, dP + \int_B E^\mathcal{B}Y \, dP,$$
$$= \int_B (E^\mathcal{B}X + E^\mathcal{B}Y) \, dP,$$
for all $B \in \mathcal{B}$. Hence we have (i) (a). Similarly, we can prove (i) (b).

(ii) $X \leq Y \Rightarrow E^\mathcal{B}X \leq E^\mathcal{B}Y, \quad \text{a.s.}$
This is immediate from the definition. This implies that $X \geq 0 \Rightarrow E^{\sigma}X \geq 0 \ a.s.$ Since $X \leq |X|$, we have $E^{\sigma}X \leq E^{\sigma}|X|$. Similarly, $|E^{\sigma}X| \leq E^{\sigma}|X|$.

In addition, result analogues to monotone convergence theorem and dominated convergence theorem also hold good for conditional expectations. In fact

(iii) If $|X_n| \leq Y$, $X_n \overset{a.s.}{\rightarrow} X \Rightarrow E^{\sigma}X_n \rightarrow E^{\sigma}X \ a.s.$

This follows, because

$$X_n \overset{a.s.}{\rightarrow} X \Rightarrow XnI(B) \rightarrow XI(B), \ a.s$$

$$\Rightarrow \int_B X_n \rightarrow \int_B X, \text{(by dominated convergence theorem)},$$

$$\Rightarrow \int_B E^{\sigma}X_n \rightarrow \int_B E^{\sigma}X, \ (B \in \mathcal{B}),$$

$$\Rightarrow E^{\sigma}X_n \rightarrow E^{\sigma}X \ a.s.$$ 

Similarly, if $0 \leq X_n \uparrow X$, $0 \leq E^{\sigma}X_n \uparrow E^{\sigma}X$.

If $\mathcal{B} = \{\emptyset, \Omega\}$ then $E^{\sigma}X$, which is a $\mathcal{B}$-measurable function, is a constant on $\Omega$, such that

$$E^{\sigma}X = \int_{\Omega} E^{\sigma}X \ dP = \int_{\Omega} X \ dP = EX$$

If $\mathcal{B} = \mathcal{A}$, then $B \in \mathcal{B} \iff B \in \mathcal{A}$, Hence

$$\int_{\mathcal{B}} E^{\sigma}X \ dP = \int_{\mathcal{B}} X \ dP, \ B \in \mathcal{A}$$

implies that $E^{\sigma}X = X $ a.s.

We may also note that whatever be $B$,

$$E(E^{\sigma}X) = \int_{\Omega} E^{\sigma}X \ dP = \int_{\Omega} X \ dP = EX.$$
2.12. Other Smoothing Properties

We have already established the basic smoothing property of conditioning viz.; $E^X$ is a constant on every non-null atom of $X$. Now we shall establish other smoothing properties [81]. Let $X$ be the $\sigma$-field induced by $X$.

If $X$ and $\mathcal{B}$ are independent, $E^X = EX$ a.s.

For, $\int X I(B) dP = \int X dP = \int E^X dP.$

$$= EX \cdot EI(B) \quad (\therefore X \text{ and } I(B) \text{ are independent})$$

$$= \int (EX) dP, B \in \mathcal{B}$$

In particular, if $\mathcal{B}$ and $\mathcal{B}_Y$ are independent,

$$E^Y \cdot X = EX \text{ a.s.}, \text{ and } E^Y \cdot X = EY \text{ a.s}$$

$E^Y(X)$ is sometimes written as $E(X/Y)$ and $E^Y(Y)$ as $E(Y/X)$. Since $[Y=y]$ is an atom of $\mathcal{B}_Y$, the value of $E^Y(X)$ of this atom will be

$$\frac{1}{P[Y = y]} = \int X dP, \text{ if } P[Y = y] > 0.$$ 

This may be interpreted as the expectation of $X$ with respect to the probability measure restricted to subsets of $[Y=y]$. Such a probability measure is called the conditional probability measure, given $[Y=y]$ and expectation w.r.t. that measure is denoted as $E[X/Y = y]$. If $X$ and $Y$ are independent $E[X/Y = y] = EX$ and a.s and $E[Y/X = x] = EY$ a.s. Taking $X = I(A)$ and $Y = I(B)$, we see that the conditional probability of $A$ given $B$ is equal to the unconditional probability of $A$, if $A$ and $B$ are independent.

Since Black and Scholes published their seminal paper [8], the pricing of financial derivatives has been an active (and perhaps lucrative) area of research. A financial derivative is a financial instrument whose value (at some time $t = T$ in the future) is completely determined by the price (or price history) of some other
set of instruments (called the underlying instruments). The general goal is to price the derivative (obtain its fair value) at time $t = 0$.

The general approach to pricing is to first assume a certain (stochastic) model for the dynamics of the market. One would then like to know the fair value of a given financial instrument/derivative given the current state of the market, and the (assumed) stochastic model. It is fair to assume that the possible future values of the instrument (and their associated probabilities) should play a central role in determining its present fair value.

2.13. Continuous State Economics

We have restricted our attention to a finite state economy. We will here present a very brief treatment of the continuous state (but finite instrument) economy [51].

The future state can be indexed by a random variable $s$ which we assume to have a probability density function $\pi(s)$. Then the price matrix at $t = T$ becomes $Z(s)$, a random vector. Given a portfolio $\Theta$ its value at $t = T$ is a random variable $V_\Theta(s) = \Theta^T Z(s)$. Then, type I arbitrage would require the existence of a portfolio that satisfies

$$\Theta^T S(0) \leq 0, \quad V_\Theta \geq 0, P[ V_\Theta > 0 ] > 0,$$

and type II arbitrage required the existence of portfolio that satisfies

$$\Theta^T S(0) < 0, \quad V_\Theta \geq 0.$$

The analog of positive supporting price is that under suitable regularity conditions, the absence of arbitrage opportunities is equivalent to the existence of a positive integrable function $\psi(s)$ such that

$$S(0) = \int ds Z(s)\psi(s), \quad \psi(s) > 0$$
Further, choosing a numeraire (instrument 1) and redefining

\[ \pi(s) = \psi(s)Z_i(s)/S_1(0), \]

we have that \( S_i(0) = \int \frac{S_1(0)}{Z_i(s)} Z_i(s) \pi(s) = E_p[D(T)Z_i]. \)

2.14. Some Illustrations

Let \( K \) be the number of possible future states for the economy, and let the number of independent instruments be \( N = K' + n \). Without loss of generality, we assume that the dynamics of the first \( K' \) (underlying) instruments are completely specified, including their current prices. The remaining \( n \) instruments are derivatives of the first \( K' \) instruments, in that their values in the possible future states are known, given the values of the underlying instruments. It is necessary to obtain the correct prices of the \( n \) derivatives at time \( t = 0 \).

1. Choose a numeraire (reference instrument), and call this \( S_i \).
2. Solve the set of \( K \) simultaneous linear equations in \( K \) unknowns \( \tilde{P}_i, \ i = 1, \ldots, K. \)
   \[ \frac{S_i}{S_1} = \sum_{j=1}^{K'} \frac{Z_j}{Z_{ij}} P_j, \text{ for } i = 1, \ldots, K', \] to obtain the martingale measure \( P \).
3. Obtain the risk neutral prices of all the instruments as follows.
   \[ S_i(0) = S_1(0) E_p \left[ \frac{S_i(T)}{S_1(T)} \right]^a = E_p D(T) E_p[S_i(T)\] \( = D(T)E_p[S_i(T)], \)
   where \( D(T) = S_i(0)/S_1(T) \) and (a) only follows if \( S_i \) is a risk free instrument.
4. These prices (if the measure exists) are unique if \( K' > K \). If such a measure cannot be found, then there exists arbitrage opportunities.

The connection with Monte Carlo Techniques should now be clear. It is often possible to obtain the risk neutral measure, but due to complexity of derivatives, it is often not possible to compute the desired expectation analytically. One still needs to price the derivatives efficiently and an ideal tool for obtaining
the necessary expectation is a Monte Carlo simulation. The following examples will illustrate the technique and how Monte Carlo simulations can be useful.

**2.15. Stock and Bond Economy**

We would like to generalize our first example as follows. Suppose the economy contains a stock and a risk free asset whose value grows according to a compounding interest rate. This economy (and its dynamics) can be fully specified by giving $S(0)$, $Z(\Delta)$ and $P$.

i) Two State Stock Dynamics—Binary Lattice Models: In this dynamics the stock at time $t = \Delta$ can exists in an up state $S(0)\lambda_+$, or a down state $S(0)\lambda_-$. 

$$ S(0) = \begin{bmatrix} B(0) \\ S_1(0) \end{bmatrix} \quad Z(\Delta) = \begin{bmatrix} B(0)e^{r\Delta} & B(0)e^{r\Delta} \\ S(0)\lambda_+ & S(0)\lambda_- \end{bmatrix} $$

$$ P = \begin{bmatrix} P_u \\ 1 - P_u \end{bmatrix} \quad \lambda_\pm = 1 + \mu\Delta \pm \sigma\sqrt{\Delta} $$

The particular dependence on $\Delta$ for the two possible values of $S(\Delta)$ are chosen with a view toward taking limit $\Delta \to 0$. By summing up such changes for infinitesimal times, we can get the change for the finite time. In order to obtain acceptable behavior for finite times, one must choose this type of dependence for the infinitesimal changes.

ii) Three State Stock Dynamics: In this dynamics the stock can exist in an “unchanged” up or down state in the next time period.

$$ S(0) = \begin{bmatrix} B(0) \\ S_1(0) \end{bmatrix} \quad Z(\Delta) = \begin{bmatrix} B(0)e^{r\Delta} & B(0)e^{r\Delta} & B(0)e^{r\Delta} \\ S(0)\lambda_+ & S(0)(1 + \mu\Delta) & S(0)\lambda_- \end{bmatrix} $$
where $\lambda_2 = 1 + \mu \Delta + \sigma \sqrt{\Delta}$. In this case, there are more states than there are instruments and the risk neutral measure is not unique. There are infinitely many risk neutral probabilities that would be consistent with no arbitrage.

2.16. Interest Rate Derivatives

A spot roll over or money market account is an instrument whose value compounds continuously at the instantaneous interest rate. We allow the instantaneous interest rate to take on one of two possible values $r_1$ and $r_2$, with $r_1 > r_2$. Suppose that there also exists a one period zero coupon bond, that pays 51 at the end of the period and that it is priced at $B(0)$. The economy is represented by

$$ S'(0) = \begin{bmatrix} R \\ B(0) \end{bmatrix}, \quad Z(\Delta) = \begin{bmatrix} R e^{r_1 \Delta} & R e^{r_2 \Delta} \\ 1 & 1 \end{bmatrix} $$

Choosing the roll over account as numeraire, one can compute the risk neutral probability of being in state $r_1$ as

$$ p = P_r = \frac{B(0,1) - e^{-r_1 \Delta}}{e^{-r_1 \Delta} - e^{-r_2 \Delta}} \text{ provided that } e^{-r_1 \Delta} \geq B(0) \geq e^{-r_2 \Delta} $$

We now extend to $N$ time periods, and consider the instrument that yields $51 in every state at time $t = N \Delta$. The possible states of the roll over account are

$$ Re^{\Delta(kr_1 + (N-k)r_2)} \text{ where } k = 0 \ldots N. $$

Thus

$$ B(0, N) = E_p[e^{-\Delta^2(kr_1 + (N-k)r_2)}] = E_p[e^{-\Delta \Sigma r(i)}] $$

where $r(i)$ represents the instantaneous interest rate at time period $i$. Converting the sum into an integral (in the limit $\Delta \to 0$)

$$ B(0, N) = E_p \left[ e^{-\frac{1}{2} \int_{r(t)} dtr(t)} \right] $$

53
where the expectation is with respect to the risk neutral measure. This expression is generally true when one has chosen the roll-over account as numeraire. In our case, this expectation is over a binomial distribution and is given by

$$B(0, N) = \sum_{k=0}^{N} \binom{N}{k} p^k (1 - p)^{N-k} e^{-\Delta(kr_1 + (N-k)r_2)} = B(0)^N.$$  

A little insight would have led us directly to this result, since to guarantee $1 at time period $N$, one buys the one period bond for $B(0)$ at time $N-1$, which requires $B(0)$ one period bonds at time $N-2$, costing $B(0)^2$ and so on. The key is that our derivation was purely mechanical.