CHAPTER - I
BASIC STOCHASTIC MODELS

A. INTRODUCTION

Probability theory is a branch of Mathematics dealing with chance phenomena and has clearly discernible lines with real world. The exact way in which probabilities are determined from experiment is a deep problem in the philosophy of science. A number of aximatizations for subjective probabilities have appeared since Keynes with no single approach dominating. In the evolution of any scientific discipline there is a period in which attempts are made to develop Mathematical theories in order to account for and explain the observations generated by the phenomena with which the discipline is concerned. When formulating a mathematical model, we can select one of the two approaches, which are termed deterministic and stochastic, reflecting the casual nature of the postulated mechanism or model which we express in mathematical form.

J.L.Doob has defined a stochastic process as the mathematical abstraction of an empirical process whose development is governed by probabilistic laws. A class of stochastic process termed Markov chains or processes has been investigated rather extensively; it is of great importance in many branches of science and engineering and in other fields.

In this chapter we present four types of basic stochastic models for portfolio management in corporate finance. First, we describe the assumption and Markov Chain model of share market. This model and fitness criteria for share trade of different institutions are illustrated through data obtained from Sakthi Sugars Ltd in Tamilnadu. The prediction out of this model is best suited for almost all seasons.
Next, the other standard basic probabilistic models namely Higher-order Hidden Markov Model (HHMM) is chosen to fit share trading. We give an elaborate account of the assumption and the analysis of the model used by Wai-Ki Ching, Tak-Kuen Siu and Li-min Li [83].

Another well studied technique in the area of probabilistic modeling is a simple Random walk. Here the model helps us in the expectations of the duration in the ruin of our share trading and option pricing game. Here we attempt to give the probabilistic analysis of the Random walk in studying higher order transition of the price level and ultimate gain or loss in that trading.

B. MARKOV CHAIN MODEL

Definition 1.1: The stochastic process \( \{X_n, n=0, 1, 2...\} \) is called a Markov chain, if, for \( j, k, j_1, ..., j_{n-1} \in \mathbb{N} \) (or any subset of \( I \)).

\[
Pr \{X_n=k \mid X_{n-1} = j, X_{n-2} = j_1, ..., X_0 = j_{n-1}\} = Pr \{X_{n-1}=k \mid X_{n-2}=j\} = p_{jk} \quad \text{(say)}
\]

whenever the first member is defined.

The outcomes are called the states on the Markov chain; if \( X_n \) has the outcome \( j \) (i.e. \( X_n = j \)), the process is said to be at state \( j \) at \( n^{th} \) trail. To a pair of states \( (j, k) \) at the two successive trails (say, \( n^{th} \) and \( (n+1)^{st} \) trails) there is an associated conditional probability \( p_{jk} \). It is the probability of transition from the state \( j \) at \( n^{th} \) trail to the state \( k \) at \( (n+1)^{st} \) trail. The transition probabilities \( p_{jk} \) are basic to the study of the structure of the Markov chain.

The transition probability may or may not be independent of \( n \). If the transition probability \( p_{jk} \) is independent of \( n \), the Markov chain is said to be homogeneous (or to have stationary transition probabilities). If it is dependent on \( n \), the chain is said to be non-homogeneous. Here we shall confine to
homogeneous chains. The transition probability \( p_{jk} \) refers to the states \((j,k)\) at two successive trails (say, \(n^{th}\) and \((n+1)^{th}\) trail); the transition is one-step and \( p_{jk} \) is called one-step (or unit step) transition probability. In the more general case, we are concerned with the pair of states \((j,k)\) at two non-successive trials, state \(j\) at the \(n^{th}\) trail and state \(k\) at the \((n+m)^{th}\) trail. The corresponding transition probability is then called \(m\)-step transition probability and is denote by \( p_{jk}^{(m)} \).

That is \[ p_{jk}^{(m)} = Pr\{X_{n+m} = k \mid X_n = j\} \]

1.1. Transition Matrix

The transition probabilities \( p_{jk} \) satisfy

\[
p_{jk} \geq 0, \quad \sum_k p_{jk} = 1 \text{ for all } j.
\]

These probabilities may be written in the matrix from

\[
P = \begin{bmatrix}
p_{11} & p_{12} & p_{13} & \ldots \\
p_{21} & p_{22} & p_{23} & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

This is called the Transition Probability Matrix (TPM) of the Markov chain. \( P \) is a stochastic matrix that is a square matrix with non-negative elements and unit row sums.

1.2. Probability Distribution

It may be seen that the probability distribution \(X_r, X_{r+1}, \ldots, X_{r+n}\) can be computed in terms of the transition probabilities \( p_{jk} \) and the initial distribution of \(X_r\). Suppose for simplicity, that \(r = 0\), then

\[
Pr\{X_0 = a, X_1 = b, \ldots, X_{n-2} = i, X_{n-1} = j, X_n = k\} = Pr\{X_n = k \mid X_{n-1} = j, \ldots, X_0 = a\} \Pr\{X_{n-1} = j, X_0 = a\}
\]

\[
= Pr\{X_n = k \mid X_{n-1} = j, \ldots, X_0 = a\} Pr\{X_{n-1} = j \mid X_{n-2} = i\} \Pr\{X_{n-2} = i, \ldots, X_0 = a\}
\]

\[
= Pr\{X_n = k \mid X_{n-1} = j\} Pr\{X_{n-1} = j \mid X_{n-2} = i\} \ldots Pr\{X_1 = b \mid X_0 = a\} Pr\{X_0 = a\}
\]
Thus, 
\[ \Pr\{X_r = a, X_{r+1} = b, X_{r+n-2} = i, X_{r+n-1} = j, X_{r+n} = k\} = \{\Pr(X_r = a)\} p_{ab} ... P_{ij} P_{jk}. \]

1.3. Order of a Markov Chain

**Definition 1.2:** A Markov chain \( \{X_n\} \) is said to be of order \( s \) \((s = 1, 2, 3, \ldots)\), if, for all 
\[ \Pr\{X_n = k | X_{n-i} = j, X_{n-i-1} = j_{i-1}, \ldots, X_{n-s} = j_{s-1}\} = \Pr\{X_n = k | X_{n-i} = j, \ldots, X_{n-s} = j_{s-1}\} \]
whenever the L.H.S. is defined.

A Markov chain \( \{X_n\} \) is said to be of order one (or simply a Markov chain) if 
\[ \Pr\{X_n = k | X_{n-1} = j, X_{n-2} = j_1, \ldots, X_{n-s} = j_{s-1}\} = \Pr\{X_n = k | X_{n-1} = j\} \]
whenever \( \Pr\{X_{n-1} = j, X_{n-2} = j_1, \ldots\} > 0. \)

Unless explicitly stated otherwise, by Markov chain, we mean a chain of order one, to which we shall mostly confine ourselves here. A chain is said to be of order zero if \( p_{jk} = p_k \) for all \( j \). This implies independence of \( X_n \) and \( X_{n-1} \).

1.4. Higher Transition Probabilities

**Chapman-Kolmogorov Equation:** We have so far considered unit-step or one-step transition probabilities, the probability of \( X_n \) given \( X_{n-1} \) i.e., the probability of the outcome at the \( n^{th} \) step or trail given the outcome at the previous step; \( p_{jk} \) gives the probability of unit-step transition from the state \( j \) at a trial to the state \( k \) at the next following trail. The \( m \)-step transition probability is denoted by 
\[ \Pr\{X_{m+n} = k | X_n = j\} = p_{jk}^{(m)}; \]
\( p_{jk}^{(m)} \) gives the probability that from the state \( j \) at \( n^{th} \) trail, the state \( k \) is reached at \((m+n)^{th}\) trail in \( m \) steps, i.e., the probability of transition from the state \( j \) to the state \( k \) in exactly \( m \) steps. The number \( n \) does not occur in the R.H.S. of the equation 1.7 and the chain is homogeneous. The one step transition probabilities \( p_{jk}^{(1)} \) are denoted by \( p_{jk} \) for simplicity.
1.5. Classification of States and Chains

The states \( j, j = 0, 1, 2, \ldots \) of a Markov chain \( \{X_n, n \geq 0\} \) can often be classified in a distinctive manner according to some fundamental properties of the system. By means of such classification it is possible to identify certain types of chains.

**Definition 1.3:**

Define \( F_{jk} = \sum_{n=1}^{\infty} f_{jk}^{(n)} \) and \( \mu_{jk} = \sum_{n=1}^{\infty} n f_{jk}^{(n)} \). A state \( j \) is said to be persistent (the word recurrent is also used by some authors [52]; we shall however use the word persistent), if \( F_{jj} = 1 \) (that is return to state \( j \) is certain) and transient if \( F_{jj} < 1 \) (that is return to state \( j \) is uncertain). A persistent state \( j \) is said to be null persistent if \( \mu_{jj} = \infty \), that is, if the mean recurrence time is infinite, and is said to be non-full (or positive) persistent if \( \mu_{jj} < \infty \).

Thus the states of a Markov chain can be classified as transient and persistent and persistent states can be subdivided as non-null and null persistent.

A persistent non-null and aperiodic state of a Markov chain is said to be ergodic.

1.6. Data and Analysis

Stock prices of the script have been obtained from the company Sakthi Sugars Ltd from 1991 to 2008. This is put to further analysis in getting the forward and backward moments. After getting the data, a frequency table is drawn and based on that the transient from one state to another is calculated by the standard result Markov chain [78]. This will be giving the Transition Probability Matrix of the model to be fitted for the data. The states are divided into 0 to 90 with the interval of 10 each. Using a program for matrix multiplication the higher powers of \( P \) are obtained.
For large values of $P^n$ is approximately equal to

\[
\begin{bmatrix}
0.8333 & 0.1296 & 0.0278 & 0.0000 & 0.0000 & 0.0093 & 0.0000 & 0.0000 & 0.0000 \\
0.3200 & 0.3400 & 0.2000 & 0.0600 & 0.0800 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0417 & 0.5000 & 0.3333 & 0.0833 & 0.0000 & 0.0000 & 0.0417 & 0.0000 & 0.0000 \\
0.0000 & 0.5714 & 0.2857 & 0.0000 & 0.0000 & 0.0000 & 0.1429 & 0.0000 & 0.0000 \\
0.0000 & 0.2000 & 0.4000 & 0.0000 & 0.0000 & 0.2000 & 0.0000 & 0.2000 & 0.0000 \\
0.0000 & 0.1667 & 0.1667 & 0.0000 & 0.1667 & 0.5000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.3333 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.3333 & 0.0000 & 0.3334 \\
0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
\end{bmatrix}
\]

**C. HIDDEN MARKOV CHAIN MODEL**

**Definition 1.4:** A Hidden Markov Model (HMM) is a finite set of states, each of which is associated with a probability distribution. Transitions among the states are governed by a set of probabilities called transition probabilities. In a particular state an outcome or observation can be generated, according to the associated symbol observation probability distribution. It is only the outcome, not the state that is visible to an external observer and therefore states are “hidden” to the outside; hence the name Hidden Markov Model.
1.7. Asset Price Dynamic by Higher-order Hidden Markov Model (HHMM)

In this section, we present a Markov modulated process driven by a Higher-order Hidden Markov Model (HHMM) [83] for modeling the asset price dynamic of an underlying risky asset. First, we consider a discrete time economy with two primary traded assets, namely, a bank account and a share. Let \( T \) be the time index set \( \{0, 1, \ldots \} \) of the economy. Fix a complete probability space \((\Omega, \mathcal{F}, P)\), where \( P \) is a real world probability. We suppose that the uncertainties due to the fluctuations of market prices and the hidden economic states are described by the probability space \((\Omega, \mathcal{F}, P)\). In the sequel, we shall define a HHMM for describing the hidden states of an economy.

Let \( X := \{X_t\}_{t \in T} \) be an \( l^\text{th} \)-order discrete-time homogeneous HHMM, which takes values in the state space \( X := \{x_1, x_2, \ldots, x_M\} \).

Write \( i(t, l) := (i_0, i_{t-1}, \ldots, i_{t-l}) \),

where \( t \geq l, l = 1, 2, \ldots \) and \( i_0, i_{t-1}, \ldots, i_{t-l} \in \{1, 2, \ldots, M\} \). The state transition probabilities of \( X \) are specified as follows:

\[
P(i_{t+1} | i(t, l)) := P[X_{t+1} = x_{i_{t+1}} | X_t = x_i, \ldots, X_{t-l} = x_{i_{t-l}}], i_{t+1} = 1, 2, \ldots, M.
\] ...1.8

The order \( l \) represents the degree of long-range dependence of the hidden states of the economy. When \( l = 1 \), \( X \) becomes a short-memory or just the first-order hidden Markov model.

To determine the HHMM completely, we need to define the following initial distributions:

\[
P(i_{t+1} | i(t, l)) := \pi_{i_{t+1} | i(t, l), i_{t+1} = 1, 2, \ldots, M}.
\] ...1.9
We shall then describe the Markov modulated process for the price dynamic of the underlying risky asset. We assume that the market interest rate of the bank account, the drift and the volatility of the risky asset switch over time according to the states of the economy modeled by $X$.

Let $r_{t,j}$, be the market interest rate of the bank account in the $t^{\text{th}}$ period. For each $j = 0, 1, \ldots, l$, we write $X_{t,j}$ for $(X_0, X_{t-1}, \ldots, X_{t-j})$, for each $t \geq l, j = 0, 1, \ldots, l$. We suppose that $r_t$ depends on the current value and the past values of the HHMM up to lag $j$, that is

$$r_{t,j} := r(X_{t,j}).$$

Then, the price dynamic $B := \{B_t\}_{t \in T}$ of the bank account is given by

$$B_t = B_{t-1} e^{r_{t,j}} , \quad B_0 = 1 , \quad P - \text{a.s} \quad \ldots 1.11$$

Let $S := \{S_t\}_{t \in T}$ be the price process of the risky stock. For each $t \in T$, let $Y_t := \ln (S_t/S_{t-1})$ be the logarithmic return in the $t^{\text{th}}$ period. We denote

$$\mu_{t,j} := \mu(X_{t,j}) \quad \text{and} \quad \sigma_{t,j} := \sigma(X_{t,j})$$

the drift and the volatility of the risky stock in the $t^{\text{th}}$ period respectively. In other words, the drift and the volatility depend on the current value and the past values of the HHMM up to lag $j$. In particular,

$$\mu(x_{i_0}, x_{i_1}, \ldots, x_{i_j}) = \mu_t(t, j), \quad \ldots 1.12$$

and

$$\sigma(x_{i_0}, x_{i_1}, \ldots, x_{i_j}) = \sigma_t(t, j), \quad \ldots 1.13$$

where $\mu_t(t, j) \in R$ and $\sigma_t(t, j) > 0$.

Let $\{\xi_t\}_{t=1,2,\ldots}$ be a sequence of independent and identically distributed (i.i.d) random variables with common distribution $N(0, 1)$, a standard normal distribution with zero mean and unit variance. We assume that $\xi_t$ and $X$ are
We suppose that the dynamic of \( Y \) is governed by the following Markov modulated model

\[
Y_t = \mu(X_{t,j}) - \frac{1}{2} \sigma^2(X_{t,j}) + \sigma(X_{t,j}) \xi_t, \quad t = 1, 2, \ldots,
\]

By convention, \( Y_0 = 0, P-a.s. \).

When \( j = 0 \), the Markov modulated model for \( Y \) becomes:

\[
Y_t = \mu(X_t) - \frac{1}{2} \sigma^2(X_t) + \sigma(X_t) \xi_t, \quad t = 1, 2, \ldots,
\]

where the drift and the volatility are driven by the current state of the Markov chain \( X \) only.

If we further assume that \( l = 1 \), the Markov modulated model for \( Y \) resembles the first-order HMM for logarithmic returns in Elliott [25].

### 1.8. Regime-Switching Esscher Transform

The Esscher transform is a well-known tool in actuarial science. The seminal work of Gerber and Shiu [31], pioneers in the use of the Esscher transform for option valuation. Their approach provides a convenient and flexible way for the valuation of options under a general asset price model. The use of the Esscher transform for option valuation can be justified by the maximization of the expected power utility. It also highlights the interplay between actuarial and financial pricing, which is an important topic for contemporary actuarial research as pointed out by Buhlmann [13], Elliott [24] adopted the regime-switching version of the Esscher transform to determine an equivalent martingale measure for the valuation of options in an incomplete market described by a Markov modulated geometric Brownian motion. Here, we consider a discrete-time version of the regime-switching Esscher transform and apply it to determine an equivalent martingale measure for pricing options in an incomplete market described by our model. In the sequel, we shall introduce the discrete-time regime-switching Esscher transform.
First, for each \( t \in T \), let \( F_t^X \) and \( F_t^Y \) denote the \( \sigma \)-algebras generated by the values of the Markov chain \( X \) and the observable logarithmic returns \( Y \) up to and including time \( t \) respectively. We write \( G_t \) for \( F_t^Y \vee F_t^X \), for each \( t \in T \). Let \( \Theta_t \) be a \( F_t^X \)-measurable random variable, for each \( t = 1, 2, \ldots \). We interpret \( \Theta_t \) as the regime-switching Esscher parameter at time \( t \) conditional on \( F_t^X \). Let \( M_Y(t, \Theta_t) \) denote the moment generating function of \( Y_t \) given \( F_t^X \) evaluated at \( \Theta_t \) under \( P \), that is,

\[
M_Y(t, \Theta_t) := E(e^{\Theta_t Y_t} | F_t^X),
\]

where \( E(\cdot) \) is the expectation under \( P \).

Here we assume that there exists a \( \Theta_t \) such that \( M_Y(t, \Theta_t) < \infty \). Then, we define a process

\[
A := \{ A_t \}_{t \in T}
\]

with \( A_0 = 1, P \)-a.s., as follows:

\[
A_t := \prod_{k=1}^t e^{\Theta_{k} Y_k} \frac{M_Y(k, \Theta_k)}{M_Y(t, \Theta_t)}.
\]

**Lemma 1.1:** Assume that \( Y_{t+1} \) is conditionally independent of \( F_t^Y \) given \( F_t^X \). Then, \( A \) is a \((G, P)\) martingale.

**Proof:** We note that \( A_t \) is \( G_t \) measurable, for each \( t \in T \). Given that \( Y_{t+1} \) is conditionally independent of \( F_t^Y \) given \( F_t^X \),

\[
E\left( A_{t+1} | G_t \right) = E \left[ e^{\Theta_{t+1} Y_{t+1}} \left| F_t^X \right. \right] = 1, P - \text{a.s.}
\]

Hence, the result follows.

Now, we define a discrete-time version of the regime-switching Esscher transform in Elliott [24], \( P^\Theta \sim P \) on \( G_t \), associated with \( (\Theta_1, \Theta_2, \ldots, \Theta_T) \) as follows:

\[
P^\Theta(A) = E(A_T \cdot I_A), A \in G_T.
\]
Let $M_Y(t, z|\Theta)$ be the moment generating function of $Y_t$ given $F^X_t$ under $P^\Theta$ evaluated at $z$, that is

$$M_Y(t, z|\Theta) = E^\Theta(e^{zY_t} | F^X_t),$$

where $E^\Theta(\cdot)$ is an expectation under $P^\Theta$.

**Lemma 1.2:** We have

$$M_Y(t, z|\Theta) = M_Y(t, \Theta_t + z)$$

$$M_Y(t, \Theta_t).$$

**Proof:** By the Bayes' rule, Lemma 1.1 and the fact that $Y_t$ is independent of $F^Y_{t-1}$ given $F^X_t$.

$$M_Y(t, z|\Theta) = E^\Theta(e^{zY_t} | F^Y_{t-1} \cup F^X_t)$$

$$= E \left( \Lambda_t e^{zY_t} | G_{t-1} \right)$$

$$= E \left( e^{(e^Y_t + \Theta_t)Y_t} | F^Y_{t-1} \cup F^X_t \right)$$

$$M_Y(t, \Theta_t)$$

$$= M_Y(t, \Theta_t + z)$$

$$M_Y(t, \Theta_t).$$

The seminal works of Harrison and Pliska [37, 38] establish an important link between the absence of arbitrage and the existence of an equivalent martingale measure under which discounted price processes are martingales. This is known as the fundamental theorem of asset pricing and is then extended by several authors, including Dybvig and Ross [21], Back and Pliska [6] and Delbaen and Schachermayer [18]. In our case, we specify an equivalent martingale measure by the risk-neutral regime-switching Esscher transform and provide a necessary and sufficient condition on the regime-switching Esscher parameters $(\Theta_1, \Theta_2, \ldots, \Theta_T)$ for $P^\Theta$ to be a risk-neutral regime-switching Esscher transform.
Proposition 1.1: The discounted price process $\{S_t, B_t\}_{t \in T}$ is a $(G, P_0)$-martingale, if and only if

$$\theta_{t+1} := \theta(X_{t+1,j}) = r_{t+1,j} - \mu_{t+1,j}, \quad t = 0, 1, \ldots, T-1. \quad \ldots 1.23$$

Proof: By Lemma 1.2,

$$E^\theta\left(\frac{S_{t+1}}{B_{t+1}} \bigg| G_t\right) = \frac{S_t}{B_t} e^{-r_{t+1}} E^\theta\left(e^{r_{t+1}} \big| G_i\right)$$

$$= \frac{S_t}{B_t} e^{-r_{t+1}} M_Y(t+1, \theta)$$

$$= \frac{S_t}{B_t} e^{-r_{t+1}} M_Y(t+1, \theta_{t+1} + 1)$$

$$= \frac{S_t}{B_t} , \quad P-a.s \quad \ldots 1.24$$

if and only if

$$M_Y(t+1, \theta_{t+1} + 1) = e^{\sigma_{t+1}} \quad \ldots 1.25$$

Since $Y_{t+1} \mid F_r \sim N(\mu_{t+1,j} - \frac{1}{2} \sigma^2_{t+1,j}, \sigma^2_{t+1,j})$,

$$M_Y(t+1, \theta_{t+1}) = \exp \left[ \Theta_{t+1} \left( \mu_{t+1,j} - \frac{1}{2} \sigma^2_{t+1,j} \right) + \frac{1}{2} \Theta^2_{t+1} \sigma^2_{t+1,j} \right]. \quad \ldots 1.26$$

Then we have

$$M_Y(t+1, \theta_{t+1} + 1) = \exp(\mu_{t+1,j} + \Theta_{t+1} \sigma^2_{t+1,j}). \quad \ldots 1.27$$

Hence, we have the result that

$$E^\theta\left(\frac{S_{t+1}}{B_{t+1}} \bigg| G_t\right) = \frac{S_t}{B_t} , \quad P-a.s. \quad \ldots 1.28$$
if and only if
\[ \Theta_{t+1} = r_{t+1,j} - \mu_{t+1,j} \]
\[ \sigma^2_{t+1,j}. \]  

...1.29

The risk-neutral dynamic of \( Y \) under \( P^\Theta \) is presented in the following corollary.

**Corollary 1.1:** Suppose \( \nu := \{ \nu_t \}_{t=1,2,...,T} \) is a sequence of independent and identically distributed (i.i.d) random variables such that \( \nu_t \sim N(0, 1) \) under \( P^\Theta \). Then, under \( P^\Theta \),
\[ Y_{t+1} = r(X_{t+1,j}) - \frac{1}{2} \sigma^2(X_{t+1,j}) + \sigma(X_{t+1,j}) \nu_{t+1,j}, \quad t = 0, 1, \ldots, T - 1, \]  

...1.30

and the dynamic of \( X \) remains unchanged under the change of measures.

**Proof:** By Lemma 1.2,
\[ M_Y(t+1, z| \Theta) = \exp \left[ z \left( \mu_{t+1,j} - \frac{1}{2} \sigma^2_{t+1,j} \right) + \frac{1}{2} z(2\Theta_{t+1} + z)\sigma^2_{t+1,j} \right]. \]  

...1.31

By Proposition 1.1,
\[ \Theta_{t+1} = r_{t+1,j} - \mu_{t+1,j} \]
\[ \sigma^2_{t+1,j}. \]  

...1.32

This implies that
\[ M_Y(t+1, z| \Theta) = \exp \left[ z \left( r_{t+1,j} - \frac{1}{2} \sigma^2_{t+1,j} \right) + \frac{1}{2} z^2 \sigma^2_{t+1,j} \right]. \]  

...1.33

Hence,
\[ Y_{t+1} = r(X_{t+1,j}) - \frac{1}{2} \sigma^2(X_{t+1,j}) + \sigma(X_{t+1,j}) \nu_{t+1,j}, \quad t = 0, 1, \ldots, T - 1. \]  

...1.34

Since the processes \( X \) and \( \xi \) are independent, the dynamic of \( X \) remains the same under the change of measures from \( P \) to \( P^\Theta \).

We shall consider the pricing of three different types of exotic options, namely, Asian options, look back options and barrier options. First, we deal with
an arithmetic average floating-strike Asian call option with maturity $T$. The payoff of the Asian option at the maturity $T$ is given by

$$P_{AA}(T) = \max(S_T - J_T, 0),$$

where the arithmetic average $J_T$ of the underlying stock price is

$$J_T = \frac{1}{T} \sum_{t=0}^{T} S_t.$$  

Then, we consider the pricing of a down-and-out European call option with barrier level $L$, strike price $K$ and maturity at time $T$. The payoff of the barrier option at time $T$ is

$$P_B(T) = \max(S_T - K, 0) I_{\{\min_{0 \leq t \leq T} S_t > L\}},$$

where $I_E$ is the indicator of an event $E$. Finally, we deal with a European-style look back floating-strike call option with maturity at time $T$. The payoff of the look back option is

$$P_{LB}(T) = \max(S_T - M_{0,T}, 0),$$

where $M_{0,T} := \min_{0 \leq t \leq T} S_t$.

1.9. Data and Analysis

In all cases, we assume that the current price of the underlying stock $S_0 = 100$ and that the time to maturity ranges from 21 trading days (one month) to 126 trading days (six months), with an increment of 21 trading days. Table 1.1 displays the prices of the Asian options implied by the three models Model I, Model II and Model III for various maturities.
Table 1.1 Prices of arithmetic average floating-strike Asian call options

<table>
<thead>
<tr>
<th>Maturity (Days)</th>
<th>Model I</th>
<th>Model II</th>
<th>Model III</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>0.355948</td>
<td>0.185336</td>
<td>0.00134111</td>
</tr>
<tr>
<td>42</td>
<td>1.62487</td>
<td>1.19555</td>
<td>0.248206</td>
</tr>
<tr>
<td>63</td>
<td>2.59119</td>
<td>2.05436</td>
<td>0.738202</td>
</tr>
<tr>
<td>84</td>
<td>3.27272</td>
<td>2.68161</td>
<td>1.19598</td>
</tr>
<tr>
<td>105</td>
<td>3.98376</td>
<td>3.21456</td>
<td>1.58142</td>
</tr>
<tr>
<td>126</td>
<td>4.49409</td>
<td>3.77677</td>
<td>1.94598</td>
</tr>
</tbody>
</table>

Assume the barrier level $L = 80$ and the strike price $K = 100$. Table 1.2 displays the prices of the barrier options implied by the three models for various maturities.

Table 1.2 Prices of down-and-out European call options

<table>
<thead>
<tr>
<th>Maturity (Days)</th>
<th>Model I</th>
<th>Model II</th>
<th>Model III</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>3.26916</td>
<td>2.60273</td>
<td>1.36266</td>
</tr>
<tr>
<td>42</td>
<td>4.75085</td>
<td>3.87316</td>
<td>2.05369</td>
</tr>
<tr>
<td>63</td>
<td>5.97215</td>
<td>4.96323</td>
<td>2.69594</td>
</tr>
<tr>
<td>84</td>
<td>6.97627</td>
<td>5.82885</td>
<td>3.30289</td>
</tr>
<tr>
<td>105</td>
<td>7.88732</td>
<td>6.64117</td>
<td>3.89378</td>
</tr>
<tr>
<td>126</td>
<td>8.57693</td>
<td>7.34083</td>
<td>4.42853</td>
</tr>
</tbody>
</table>

Table 1.3 presents the prices of the look back options implied by the three models for various maturities.
We can regard Model III (i.e., the no regime-switching case) as a zero order HHMM and Model I as a first order HHMM. Then, from tables 1.1, 1.2 and 1.3, we can see that the prices of the Asian options, the barrier options and the look back options, respectively, increase substantially as the order of the HHMM does. These prices are sensitive to the order of the HHMM. This is true for the options with various maturities. In other words, the long-range dependence in the states of economy has significant impact on the prices of these path-dependent exotic options. The differences between the prices implied by the first-order HHMM and those implied by the zero-order HHMM are more substantial than the difference between the prices obtained from the second-order HHMM and those obtained from the first-order HHMM.

### D. RANDOM WALK MODEL

#### 1.10. Closed-Form Valuation

Consider the standard Black-Scholes environment where there are two securities: a zero-coupon bond maturing at the expiration date $T$ of the look back option with a flat term-structure $r$, and a risky security whose price $S_t$ at time $t$ follows a geometric Brownian motion under the risk-neutral probability measure. Formally,
\[ \tilde{S}_t = S_0 e^{B_t}, \quad t \geq 0 \]

where \( B_t = (r - \sigma^2/2) + \sigma W_t \), \( \sigma \) represents the volatility of the return of the underlying security, and \( \{ W_t \} \) is a standard Brownian motion initialized at 0.

When the underlying security of the contract is monitored only at the \( m \) dates \( \Delta t, \ldots, m\Delta t \), with \( \Delta t = T/m \), the corresponding discrete-time price model [26] is

\[ S_n = S_0 e^{U_n}, \quad n = 0, 1, \ldots, m \]

where \( U_0 = 0, U_n = X_1 + X_2 + \ldots + X_n \) for \( n \geq 1 \)

and the \( X_t \) are independent \( N(\mu, \sigma^2) \) random variables, with \( \mu = (r - \sigma^2/2) \Delta t, \sigma = \sigma \sqrt{\Delta t} \).

1.11. Duality Theory and Extreme of Random Walks

Let \( X_1, X_2 \ldots \) be independent, identically distributed random variables. Consider the random walk \( U_n = X_1 + \ldots + X_n \). Let \( U_0 = 0 \) and

\[ M_m = \max \{ U_n : 0 \leq n \leq m \} \]

\[ \tau = \inf \{ n : U_n \leq 0 \}, \tau^+ = \inf \{ n : U_n > 0 \} \]

Since the \( X_i \) is independent and has the same distribution, it follows that for \( x > 0 \),

\[ P\{M_m \in dx\} = P\{U_1 \in dx\} P\{X_2 \leq 0, X_2 + X_3 \leq 0, \ldots, X_2 + \ldots + X_m \leq 0\} \]

\[ + \sum_{\nu=2}^{m} P\{U_\nu > U_i, i < \nu; U_\nu \in dx\} P\{X_{\nu+1} \leq 0, X_{\nu+1} + X_{\nu+2} \leq 0, \ldots, X_{\nu+1} + \ldots + X_m \leq 0\} \]

By the duality of random walks for \( x > 0 \),

\[ P\{U_\nu > U_i, i < \nu; U_\nu \in dx\} = P\{U_\nu - U_\nu - 1 > 0, \ldots, U_\nu - U_1 > 0; U_\nu \in dx\} \]

\[ = P\{U_1 > 0, \ldots, U_{\nu-1} > 0; U_\nu \in dx\} \]

\[ = P\{\tau^+ > \nu; U_\nu \in dx\} \]

On the other hand,

\[ P\{X_{\nu+1} \leq 0, X_{\nu+1} + X_{\nu+2} \leq 0, \ldots, X_{\nu+1} + \ldots + X_m \leq 0\} \]

\[ = P\{U_1 \leq 0, U_2 \leq 0, \ldots, U_{m-1} \leq 0\} = P\{\tau^+ > m-1\} \]

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Putting equations 1.42 and 1.43 into equation 1.41 yields

\[ P\{M_m \in dx\} = P\{U_1 \in dx\} \sum_{v=2}^{m} P\{\tau_- > v; U_v \in dx\} \]

Moreover,

\[ P\{M_m = 0\} = P\{\tau_+ > m\} \]

Thus the distribution of the maximum can be expressed in terms of the so-called 'ladder epochs' \( \tau_+ \) and \( \tau_- \). This is particularly useful for pricing discretely monitored options because the quantities in equations 1.44 and 1.45 can be computed recursively.

Replacing \( U_n \) by \( -U_n \) in the preceding argument leads to an analogous representation of the distribution of the minimum

\[ \Lambda_m = \min\{U_n; 0 \leq n \leq m\} = -\max\{-U_n; 0 \leq n \leq m\} \]

of the random walk \( \{U_n\} \). For \( x < 0 \),

\[ P\{\Lambda_m \in dx\} =
\sum_{v=2}^{m} P\{\tau_- > v; U_v \in dx\} \]

in analogy with equation 1.44.

For the particular case where \( X_i \) are independent \( N(\mu, \sigma^2) \) random variables so that \( U_n \) is the normal random walk in the discrete-time price model equation 1.39, the following result Siegmund [71] can be used to determine the density function \( f_v \) of the measure \( P\{\tau_+ > v; U_v \in dx\} \) by recursive numerical integration.

**Proposition 1.2:** Let \( J \) be either \((0, \infty)\) or \((-\infty, 0]\), and \( \tau = \inf\{n: U_n \notin J\} \).

For \( x \in J \), let \( f_n(x)dx = P\{\tau > n; U_n \in dx\} \). Let \( \phi \) denote the density function of the
standard normal distribution and let \( \psi(x) = \varphi^{-1}(\frac{x - \mu}{\sigma}) \). Then for \( x \in J \),

\[
\begin{align*}
    f_1(x) &= \psi(x) \\
    f_n(x) &= \int f_{n-1}(y) \psi(x - y) \, dy \quad \text{for } 2 \leq n \leq m.
\end{align*}
\]

...1.48

### 1.12. Fixed Strike Options

A fixed strike (hindsight) call gives its holder the right to buy the underlying security at a fixed strike price \( K \) and to sell it at the maximum price achieved during the life of the option. Correspondingly, hindsight put grants the right to purchase the underlying security at the minimum price and to sell it at a fixed strike. We shall concentrate on the hindsight call, since the arguments and results for the hindsight put are similar. Define \( M_m \) by equation 1.40 with the same \( U_n \) as in equation 1.39. Then the payoff is

\[
e^{-rT} E(S_0 e^{M_m} - K)^+
\]

where \( K \) is the strike price of the call. The main quantity to evaluate, \( E(S_0 e^{M_m} - K)^+ \), can be expressed in terms of the density functions \( f_n \) in Proposition 1.2.

**Proposition 1.3:** The value of a hindsight look back call at inception is

\[
e^{-rT} E(S_0 e^{M_m} - K)^+ = e^{-rT} \alpha_m (S_0 - K)^+ + e^{-rT} \sum_{k=1}^{m} \alpha_{m-k} \int (S_0 e^x - K)^+ f_n(x) \, dx
\]

...1.49

where \( f_n(x) \) is defined recursively for \( x > 0 \) by equation 1.48 with \( J = (0, \infty) \),

\[
\alpha_0 = 1, \quad \alpha_k = \int_{-\infty}^{0} g_k(x) \, dx \quad \text{for } k \geq 1
\]

in which \( g_k(x) \) is the same as the \( f_k(x) \) defined for \( x \leq 0 \) by equation 1.48 with \( J = (-\infty, 0] \).

**Proof:** First note that

\[
E(S_0 e^{M_m} - K)^+ = \int (S_0 e^x - K)^+ P\{M_m < dx\}
\]
Define $\tau_+$ as in equation 1.40.

For $n \geq 1$ and $x < 0$, since $g_n(x) \text{dx} = P\{\tau_+ > n; U_n \in \text{d}x\}$, it follows that

$$
\alpha_n = \int_{-\infty}^{0} g_n(x) \text{dx} = P\{\tau_+ > n\} = \{U_1 \leq 0, \ldots, U_n \leq 0\}.
$$

By equations 1.44 and 1.51,

$$
P\{M_m \in \text{d}x\} = \alpha_{m-1} P\{U_1 \in \text{d}x\} + \sum_{v=2}^{m} \alpha_{m-v} f_v(x) \text{dx} \quad \text{for} \quad x > 0,
$$
yielding equation 1.49 in view of equations 1.50 and 1.45. With $\Lambda_m$ defined by equation 1.46, the price of a fixed strike put is $e^{-rT} E \left(K - S_0 e^{A_m}\right)^+$, which can be evaluated by an obvious modification of Proposition 1.3.

1.13. Floating Strike Options

The holder of a floating strike (or standard) look back put has the right to purchase the underlying security at its price on the exercise date, and to sell it at the maximum price it achieved during the life of the option. Correspondingly, a floating strike call is exercised by purchasing the underlying security at the minimum price it achieved during the life of the option, and selling it at the price on the exercise date. In the Black-Scholes environment, the discrete-time price of a floating strike option at time 0 is $e^{-rT} E (S_0 e^{M_m} - S_m)$, which can be evaluated by making use of the following proposition.

**Proposition 1.4:** The value of a standard (floating strike) look back put at inception is

$$
e^{-rT} E(S_0 e^{M_m} - S_m) = e^{-rT} S_0 \sum_{v=0}^{m-1} \beta_{m-v} I_v
$$

where

$$
\beta_{m-v} = \int_{-\infty}^{0} \left(1 - e^x\right) g_{m-v}(x) \text{dx} \quad \text{for} \quad 0 \leq v \leq m - 1
$$
where the first and second terms and the $v$th summand correspond to the cases $M_m = 0$, $M_m = U$, and $M_m = U_v$, respectively, noting that $P\{U_i = U_j\} = 0$ for $i \neq j$.

Define $\tau_+$ and $\tau_-$ as in equation 1.40. Since $P\{U_n = 0\} = 0$ for all $n$,

$$\tau_+ = \inf \{n: U_n \geq 0\}$$

with probability 1, and therefore

$$E(1 - e^{U_m}) |_{\{U_1 < 0, ..., U_m < 0\}} = E(1 - e^{U_m}) |_{\{\tau_+ > m\}} = \int_{-\infty}^{\infty} (1 - e^x) g_m(x) dx$$

The second term in sum of equation 1.53 can also be written as

$$E(e^{U_i} - e^{\sum_{i=1}^{m} X_i}) 1\{U_1 > 0, U_2 - U_1 < 0, U_m - U_1 < 0\}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^x - e^{xy}) dx dy$$

$$P \{U_1 \in dx, X_2 < 0, X_2 + X_3 < 0, ..., X_2 + X_3 + ... + X_m < 0, \sum_{i=2}^{m} X_i \in dy\}$$

$$= \int_0^\delta e^x P\{U_1 \in dx\}$$

$$\left[ \int_{-\infty}^{\infty} (1 - e^y) P\{X_2 < 0, X_2 + X_3 < 0, ..., X_2 + X_3 + ... + X_m < 0, \sum_{i=2}^{m} X_i \in dy\} \right]$$

where the last step follows from the independence of $U_1$ and $(X_2, ..., X_m)$.

Therefore the second term in sum in equation 1.53 reduces to

$$\int e^x \psi(x) dx \left[ \int_{-\infty}^{\delta} (1 - e^y) g_{m-1}(y) dy \right]$$
Similarly, the last term in sum of equation 1.53 can be written as

$$\sum_{v=2}^{n-1} E(e^{U_v} - e^{U_m}) \prod_{U_v > 0, U_{v-1} > 0, \ldots, U_v - U_{v-1} > 0, U_{v+1} > 0} \prod_{U_v - U_{v-1}} \prod_{U_v - U_{v+1} > 0} \prod_{U_v - U_m > 0}$$

$$= \sum_{v=2}^{n-1} \int_{0}^{\Theta} \int_{-\infty}^{\infty} (e^{x} - e^{-y}) \mathbb{P}(U_1 < U_v, \ldots, U_{v-1} < U_v; U_v \in dx).$$

$$P(X_{v+1} < 0, \ldots, X_{v+1} + \ldots + X_0 < 0; X_{v+1} + \ldots + X_m \in dy)$$

$$= \sum_{v=2}^{n-1} \int_{0}^{\Theta} e^{x}f(x)dx \int_{-\infty}^{\infty} (1 - e^{y})g_{m-v}(y)dy.$$

With $\Lambda_m$ defined as in equation 1.46, the price of a floating strike look back call is $e^{-rT} E(S_m - S_0 e^{\Lambda m})$, which can be evaluated by an obvious modification of Proposition 1.4.

1.14. Data and Analysis

Tables 1.4 – 1.7 display numerical results on the above recursive integration algorithm giving an indication of their convergence and accuracy as compared to alternative methods. In tables 1.4 and 1.5, the results in the column labelled ‘Monte Carlo’ are taken from Kat [43]. Each Monte Carlo estimate is based on 100000 simulation runs, and the ‘width’ given in parentheses refers to half the length of the 95% confidence interval centered at the Monte Carlo estimate. Table 1.4 also includes Kat’s results on the binomial method of Cheuk and Vorst [14] with 5200 time steps, while table 1.5 also includes his results on Babbs’ [5] method. In tables 1.6 and 1.7, the results on continuity correction methods are taken from Broadie [11] and the column labelled ‘Trinomial method’ refers to their trinomial-tree adaptation of the methods of Babbs and of Cheuk and Vorst who used binomial trees instead.
### Table 1.4 Fixed strike (hindsight) look backs

<table>
<thead>
<tr>
<th>Option (K)</th>
<th>Cheuk and Vorst</th>
<th>Monote Carlo (width)</th>
<th>Recursive integration ($\delta = 0.02$)</th>
<th>Recursive integration ($\delta = 0.01$)</th>
<th>Recursive integration ($\delta = 0.005$)</th>
<th>Recursive integration ($\delta = 0.0025$)</th>
<th>Recursive integration ($\delta = 0.001$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Put (95)</td>
<td>4.04</td>
<td>4.25 (0.11)</td>
<td>4.2379</td>
<td>4.2227</td>
<td>4.2272</td>
<td>4.2267</td>
<td>4.2266</td>
</tr>
<tr>
<td>Put (100)</td>
<td>7.65</td>
<td>7.65 (0.14)</td>
<td>7.6588</td>
<td>7.6486</td>
<td>7.6480</td>
<td>7.6480</td>
<td>7.6480</td>
</tr>
<tr>
<td>Put (105)</td>
<td>12.52</td>
<td>12.52 (0.14)</td>
<td>12.5426</td>
<td>12.5255</td>
<td>12.5246</td>
<td>12.5246</td>
<td>12.5246</td>
</tr>
<tr>
<td>Call (95)</td>
<td>15.55</td>
<td>15.55 (0.20)</td>
<td>15.5739</td>
<td>15.5537</td>
<td>15.5526</td>
<td>15.5526</td>
<td>15.5526</td>
</tr>
<tr>
<td>Call (100)</td>
<td>10.67</td>
<td>10.67 (0.20)</td>
<td>10.6900</td>
<td>10.6768</td>
<td>10.6761</td>
<td>10.6760</td>
<td>10.6760</td>
</tr>
<tr>
<td>Call (105)</td>
<td>6.83</td>
<td>6.99 (0.18)</td>
<td>6.9946</td>
<td>6.9725</td>
<td>6.9772</td>
<td>6.9767</td>
<td>6.9765</td>
</tr>
</tbody>
</table>

$S_0 = 100, \sigma = 0.20, r = 0.05, T = 0.5$ (bi-weekly observations)

### Table 1.5 Floating strike look backs

<table>
<thead>
<tr>
<th>Option (m)</th>
<th>Babbs</th>
<th>Monote Carlo (width)</th>
<th>Recursive integration ($\delta = 0.02$)</th>
<th>Recursive integration ($\delta = 0.01$)</th>
<th>Recursive integration ($\delta = 0.005$)</th>
<th>Recursive integration ($\delta = 0.0025$)</th>
<th>Recursive integration ($\delta = 0.001$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Put (26)</td>
<td>8.81</td>
<td>8.82 (0.14)</td>
<td>8.8884</td>
<td>8.8197</td>
<td>8.8171</td>
<td>8.8170</td>
<td>8.8170</td>
</tr>
<tr>
<td>Put (13)</td>
<td>8.21</td>
<td>8.21 (0.14)</td>
<td>8.2181</td>
<td>8.2076</td>
<td>8.2071</td>
<td>8.2070</td>
<td>8.2070</td>
</tr>
<tr>
<td>Call (26)</td>
<td>10.62</td>
<td>10.61 (0.19)</td>
<td>10.7018</td>
<td>10.6209</td>
<td>10.6179</td>
<td>10.6177</td>
<td>10.6177</td>
</tr>
<tr>
<td>Call (13)</td>
<td>10.12</td>
<td>10.11 (0.19)</td>
<td>10.1300</td>
<td>10.1177</td>
<td>10.1171</td>
<td>10.1170</td>
<td>10.1170</td>
</tr>
</tbody>
</table>

$S_0 = 100, \sigma = 0.20, r = 0.05, T = 0.5$

### Table 1.6 Floating strike look back put price at inception

<table>
<thead>
<tr>
<th>$M$</th>
<th>1st-order correction</th>
<th>2nd-order correction</th>
<th>Recursive integration ($\delta = 0.01$)</th>
<th>Recursive integration ($\delta = 0.005$)</th>
<th>Recursive integration ($\delta = 0.0025$)</th>
<th>Recursive integration ($\delta = 0.001$)</th>
<th>Trinomial method</th>
</tr>
</thead>
</table>

$S_0 = 100, \sigma = 0.30, r = 0.10, T = 0.5$
Tables 1.4 and 1.5 give five choices for the value of $\delta$ in the recursive integration algorithm with $\delta$ ranging from 0.02 to 0.001. They show that this simple algorithm with $\delta = 0.01$ already gives results differing by no more than 0.004 from those with $\delta = 0.001$. These results all lie within the 95% confidence limits of the Monte Carlo estimates. The binomial/trinomial tree method also gives similar results, but its first entry in table 1.4 falls outside the Monte Carlo confidence interval while its last entry in table 1.4 is near the left end-point of the confidence interval. In tables 1.6 and 1.7, the results given by the recursive integration algorithm with $\delta \leq 0.005$ are in close agreement with each other and with those obtained by Broadie [11] using the trinomial tree method. The results based on continuity correction are close to those of the recursive integration or trinomial tree method for $m \geq 80$, but do not have comparable accuracy for $m \leq 20$.

Because its complexity increases linearly with $m$, the recursive integration method becomes less attractive for large $m$. However, the number $m$ of monitoring dates of a look back option is typically less than 80, for which the recursive integration method is relatively fast. Moreover, for large values of $m$, one can switch to the continuity correction formulas of Broadie [11] which are reasonably accurate for $m \geq 80$, as demonstrated by tables 1.6 and 1.7. Therefore,
we recommend a combined recursive integration / continuity correction approach in practice, using the former for \( m \leq 80 \) and switching to the latter for larger values of \( m \).

It follows that \( \int e^x P\{U_n \in dx\} = o(\delta) \) if we choose \( B \sim 2 \log \delta \). With this choice of \( B \) and noting that \((\mu + \sigma^2/2) n \leq rT \) and \( \sqrt{n} \leq \sigma \sqrt{T} \), it follows that, for fixed \( m \), the computational complexity of our recursive integration method is \( O(\delta^{-1} \log \delta^{1/5}) \) to give results accurate to within \( O(\delta) \). No comparable accuracy results have been established for the binomial/trinomial tree method, which is based on weak convergence of the tree to geometric Brownian motion as \( n \), the number of steps, approaches \( \infty \). The best one can expect is that the results are accurate to within \( O(n^{-1}) \), but the convergence rate results for Donsker's invariance principle established by some authors and yield the order \( O(n^{-1/2} \log n) \) for every \( \varepsilon > 0 \). As noted by Kat [43], the number of calculations required by a standard binomial tree with \( n \) steps is of the order \( n^2/2 \), which is a lower bound for the complexity of the Babbs/Cheuk-Vorst method. Hence, if one wants the binomial/trinomial method to yield results accurate to within \( O(\delta) \), then the number of time steps needed is at least some constant times \( \delta^{-1} \), resulting in a complexity of at least some constant times \( \delta^{-2} \) for the tree method, in contrast with the \( O(\delta^{-1} \log \delta^{1/5}) \) complexity for the recursive integration method that is accurate to within \( O(\delta) \). Although Breen [10] has made use of Richardson extrapolation to develop an accelerated binomial tree method for standard options such that the number of calculations increases linearly with \( n \), the convergence rate of this method may be slow and its extension to look back options is unresolved.

1.15. Expected Duration of the Game in the Ruin Problem

We next deal with the expected value of the duration of the random walk. The present discussion will demonstrate a simple method of very wide
applicability. Suppose the duration of the game has a finite expectation $d_k$ with the condition that the random walk be started from the point $x = k$. If the first trial leads to a win for the first gambler the conditional duration from that point on is $d_{k+1}$. The expected duration of the whole game is $1 + d_{k+1}$ if the first trial is a win. Similarly the expected duration of the whole game is $1 + d_{k-1}$ if the first trial is a loss. Therefore, we have

$$d_k = p(1 + d_{k+1}) + q(1 + d_{k-1}),$$

$$= 1 + pd_{k+1} + qd_{k-1} [1 < k < (a - 1)],$$

...1.54

with the boundary conditions

$$d_0 = 0 \quad \text{and} \quad d_a = 0.$$  

...1.55

The above equation is a non-homogeneous difference equation. One may rewrite equation 1.54 as a first order difference equation

$$pM_{k+1} = qM_k - l$$

...1.56

where

$$M_k = d_k - d_{k-1} \quad k = 1, 2, ..., a.$$  

...1.57

Solving equation 1.56 recursively, we get

$$M_{k+1} = (q/p)^k M_1 - l/p [1 + q/p + (q/p)^2 + ... + (q/p)^{k-1}]$$

$$= (q/p)^k M_1 - l/(p - q) [1 - (q/p)^k] \quad \text{if} \quad p \neq q$$

$$= M_1 - k/p \quad \text{if} \quad p = q.$$  

...1.58

Now for

$$j = 1, 2, ..., k - l$$

$$d_k = \sum_{j=1}^{k-1} M_{j+1} = d_k - d_0.$$  

...1.59

So we have

$$d_k = \{M_1 + l/(p - q)\} \frac{1 - (q/p)^k}{1 - (q/p)} - k/(p - q) \quad \text{if} \quad p \neq q$$

$$= kM_1 - k(k - 1)(1/2p) \quad \text{if} \quad p = 1.$$  

...1.60

Thus $\{d_k, k = 1, 2, ..., a\}$ is given by equation 1.60 up to an undetermined constant $M_1$. To determine $M_1$ we use the fact that $d_a = 0$. 

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Thus \( \{ M_i + 1/(p - q) \}^{1-(q/p)} \frac{1-(q/p)^p}{1-(q/p)} - a/(p-q) = 0 \) if \( p \neq q \) \(...1.61

or \( M_i = [a/(p-q)]^{1-(q/p)} \frac{1-(q/p)^p}{1-(q/p)} \frac{1}{(p-q)} \) if \( p \neq q \) \(...1.62

\[ = (a-1)/2p \quad \text{if} \quad p = q. \] \(...1.63

Substituting the values of equations 1.63 and 1.62 in equation 1.60, we have

\[ d_k = k/(q-p) -a (q-p) \frac{1-(q/p)^p}{1-(q/p)} \quad \text{if} \quad p \neq q, \] \(...1.64

\[ = k (a-k) / 2p \quad \text{if} \quad p = q. \]

If one lets \( a \) tend to \( \infty \) in equation 1.64, we get

\[ d_k = k/(q-p) \quad \text{if} \quad q > p, \]

\[ = \infty \quad \text{if} \quad q \leq p. \] \(...1.65

1.16. Random Walk with Absorbing Boundaries

**Theorem 1.1:** Let \( \{X_n\} \) be a random walk process on \( \{0, 1, 2... a\} \) with absorbing barriers 0 and \( a \), and initial position \( k \), \( X_n \) being specified by

\[ X_n = X_{n-1} + Z_n, X_{n-1} \neq 0, a \quad \text{where} \quad Pr\{Z_n = +1\} = p \]

\[ X_n = X_{n-1} \quad \text{or} \quad a. \]

Then \( G_k(s) \) being the generating function of the probability of the time to the termination of the random walk at position 0 is given by

\[ G_k(s) = \sum_{n=0}^{\infty} \pi_k(n)s^n \quad |s| < 1 \]

\[ = (q/p)^k \frac{\lambda_1^{a-k}(s) - \lambda_2^{a-k}(s)}{\lambda_1^a(s) - \lambda_2^a(s)} \] \(...1.66
where $\lambda_1(s)$ and $\lambda_2(s)$ are given by

$$\lambda_1(s) = \frac{1 + \sqrt{1 - 4pqs^2}}{2ps}, \quad \lambda_2(s) = \frac{1 - \sqrt{1 - 4pqs^2}}{2ps}$$ ...

**Proof:** We first note the probabilities $\pi_k(n)$ ($0 \leq k \leq a$) satisfy the equations

$$\pi_0(n) = \delta_{n0}, \quad \pi_a(n) = 0$$

and

$$\pi_k = p\pi_{k+1} + q\pi_{k-1}, \quad 1 \leq k \leq a - 1$$

$$\pi_0 = 1, \quad \pi_a = 0.$$ 

We multiply both sides of above equations by $s^n$ and sum overall values of $n$ to obtain

$$G_k(s) = psG_{k+1}(s) + qsG_{k-1}(s), \quad 1 \leq k \leq a - 1$$

$$G_0(s) = 1, \quad G_a(s) = 0$$ ...

We next define the polynomial $g(u, s)$ in $u$ by

$$g(u, s) = \sum_{k=0}^{a} G_k(s)u^k$$ ...

and obtain the following equation from equation 1.68

$$g(u, s) = 1 + \frac{ps}{u} [g(u, s) - l - G_1(s)u + qsu [g(u, s) = G_{a-1}(s) u^{a-1}]]$$ ...

or

$$g(u, s) = \frac{[u - ps - psG_1(s) u - qsG_{a-1}(s)u^{a+1}] / [u - u^2 qs - ps]}{u - psu^2 - qs}$$ ...

From this point, the chain of arguments proceeds exactly and we obtain finally equation 1.66. We immediately have the following corollary.

**Corollary 1.2:** In the random walk $\{X_n\}$ specified by theorem 1.1, when the absorbing barrier $a$ progresses to infinity, the probability generating function $G_k(s)$ is given by

$$G_k(s) = \lambda^k(s)$$ ...

where $\lambda(s)$ is the root of the equation

$$u - psu^2 - qs = 0, \quad \text{satisfying the condition } |\lambda(s)| < 1.$$
1.17. Conclusion

The above discussion helps to fair inference on the risk involved in the share trading and investment. The ruin nature of share holders can be explained through ruin probabilities. This result paves the way for multiparty involvement with share market.