CHAPTER II

STEADY STATE ANALYSIS OF A BULK ARRIVAL GENERAL BULK SERVICE QUEUEING SYSTEM WITH MULTIPLE VACATIONS AND CLOSEDOWN TIMES*

A computational analysis of steady state probabilities of bulk service queues and related non bulk queues was studied by Chaudhry et al. [17]. They illustrated a technique for evaluating steady state probabilities and moments for the number of customers in system or queue at numerically different time epochs. Lee et al.[60] discussed a control policy for the $M^X/G/1$ queueing system. A single server $M/G/1$ queue with server's vacation and setup times was analysed by Bischof [11].

Queueing systems with arrival and services in batches of variable size were studied by Borthakur and Medhi [13], using supplementary variable technique. They also derived the queue length distribution for the $M^X/G(a, b)/1$ model. Chae and Lee [15] analysed a $M^X/G/1$ vacation model with N-policy and discussed heuristic interpretation of mean waiting time.

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This chapter deals with the analysis of a single server queueing system with Poisson bulk arrival, general bulk service, multiple vacations and closedown times. Server vacation models are useful for the system in which the server wants to utilise the idle time for different purposes. Application of vacation models can be found in production systems, designing local area networks and data communication systems.

In this chapter, a Steady State Analysis of a Bulk Arrival General Bulk Service Queueing System with Multiple Vacations and Closedown Times is considered. It is assumed that, after completing a service, if the queue length is less than ‘a’, the server performs closedown work (such as arranging the finished products). After completing the closedown work, the server leaves for a vacation of random length, irrespective of the queue length. After a vacation, if the queue length is still less than ‘a’, the server leaves for another vacation and so on, until he finds at least ‘a’ customers waiting for service. After a vacation, if the server finds at least ‘a’ customers waiting for service, say \( \xi \), then he serves a batch of \( \min(\xi, b) \) customers, where \( b \geq a \).
A practical situation exists in a Globe Valve manufacturing industry where the single server model is applied. In Globe Valve manufacturing industry, after turning operation, the components arrive from job shop in batches to CNC turning center, for facing and turning processes. The operator of CNC turning center starts the process only if the minimum batch quantity is available. After processing a batch, if the number of components is not sufficient to process, then the operator leaves for arranging the tooling, writing the coding etc., to utilise his idle time. Before leaving, the operator must perform various types of work like closing the door, checking the tooling etc. When the operator returns to CNC turner, if the number of available
components is less than a batch quantity, he does secondary work repeatedly until he finds enough quantity.

For the proposed model, the probability generating function of the number of customers in the queue at an arbitrary time epoch is obtained, using supplementary variable technique. Expressions for expected queue length, expected length of idle period, expected length of busy period and expected waiting times are derived. A cost model of the queueing system is discussed. Numerical solution for particular values of parameters is presented.

2.1 Mathematical Model.

Let \( X \) be the group size random variable of the arrival, \( \lambda \) be the Poisson arrival rate, \( g_k \) be the probability that \( k \) customers arrive in a batch and \( X(z) \) be its probability generating function. Let \( S(.), V(.) \) and \( C(.) \) be the cumulative distributions of the service time, vacation time and closedown time, respectively. Let \( s(x), v(x) \) and \( c(x) \) be the probability density functions of service time, vacation time and closedown time, respectively. \( S^0(t) \) denotes the remaining service time of a batch in bulk service at an arbitrary time \( t \). \( V^0(t) \) and \( C^0(t) \) denote the remaining vacation time of a server and closedown time of a server at an arbitrary time \( t \), respectively. Let \( \tilde{S}, \tilde{V} \) and \( \tilde{C} \) denote the Laplace-Stiltjes transforms of \( S, V \) and \( C \), respectively. \( N_q(t) \) and \( N_s(t) \) are the number of customers in the queue and under service, respectively, at time \( t \).
The different states of the server at time 't' are defined as follows:

\[ Y(t) = 0, \text{ if the server is busy with bulk service} \]

\[ = 1, \text{ if the server is doing closedown job} \]

\[ = 2, \text{ if the server is on vacation.} \]

and define \( Z(t) = j \), if the server is on \( j^{th} \) vacation starting from the idle period.

To obtain system equations, the following probabilities are defined.

\[ P_{ij}(x, t) dt = P\{N_s(t) = i, N_q(t) = j, x \leq S^0(t) \leq x + dt, Y(t) = 0\}, \ a \leq i \leq b, j \geq 0 \]

\[ C_n(x, t) dt = P\{N_q(t) = n, x \leq C^0(t) \leq x + dt, Y(t) = 1\}, \ n \geq 0 \quad \text{and} \]

\[ Q_{jn}(x, t) dt = P\{N_q(t) = n, x \leq V^0(t) \leq x + dt, Y(t) = 2, Z(t) = j\}, \ n \geq 0, \ j \geq 1. \]

Now, the following equations are obtained for the queueing system, using supplementary variable technique:

\[ P_{i0}(x-\Delta t, t + \Delta t) = P_{i0}(x, t)(1-\lambda \Delta t) + \sum_{m=a}^{b} P_{i0}(0, t) s(x) \Delta t \]

\[ + \sum_{l=1}^{\infty} Q_{li}(0, t) s(x) \Delta t, \ a \leq i \leq b \]

\[ P_{ij}(x-\Delta t, t + \Delta t) = P_{ij}(x, t)(1-\lambda \Delta t) + \sum_{k=1}^{j} P_{ij-k}(x, t) \lambda g_k \Delta t, \ a \leq i \leq b-1, j \geq 1 \]

\[ P_{bj}(x-\Delta t, t + \Delta t) = P_{bj}(x, t)(1-\lambda \Delta t) + \sum_{m=a}^{b} P_{bm}(0, t) s(x) \Delta t \]

\[ + \sum_{k=1}^{j} P_{bj-k}(x, t) \lambda g_k \Delta t + \sum_{l=1}^{\infty} Q_{blj}(0, t) s(x) \Delta t, \ j \geq 1 \]
\begin{align*}
C_0(x-\Delta t, t + \Delta t) &= C_0(x, t)(1 - \lambda \Delta t) + \sum_{m=a}^{b} P_{m0}(0, t) c(x) \Delta t \\
C_n(x-\Delta t, t + \Delta t) &= C_n(x, t)(1 - \lambda \Delta t) + \sum_{m=a}^{b} P_{mn}(0, t) c(x) \Delta t \\
&\quad + \sum_{k=1}^{n} C_{n-k}(x, t) \lambda g_k \Delta t, \quad 1 \leq n \leq a-1 \\
C_n(x-\Delta t, t + \Delta t) &= C_n(x, t)(1 - \lambda \Delta t) + \sum_{k=1}^{n} C_{n-k}(x, t) \lambda g_k \Delta t, \quad n \geq a \\
Q_{10}(x-\Delta t, t + \Delta t) &= Q_{10}(x, t)(1 - \lambda \Delta t) + C_0(0, t) v(x) \Delta t, \\
Q_{1n}(x-\Delta t, t + \Delta t) &= Q_{1n}(x, t)(1 - \lambda \Delta t) + C_n(0, t) v(x) \Delta t \\
&\quad + \sum_{k=1}^{n} Q_{1-n-k}(x, t) \lambda g_k \Delta t, \quad n \geq 1 \\
Q_{j0}(x-\Delta t, t + \Delta t) &= Q_{j0}(x, t)(1 - \lambda \Delta t) + Q_{j-10}(0, t) v(x) \Delta t, \quad j \geq 2 \\
Q_{jn}(x-\Delta t, t + \Delta t) &= Q_{jn}(x, t)(1 - \lambda \Delta t) + Q_{j-1n}(0, t) v(x) \Delta t \\
&\quad + \sum_{k=1}^{n} Q_{j-n-k}(x, t) \lambda g_k \Delta t, \quad j \geq 2, 1 \leq n \leq a-1 \\
Q_{jn}(x-\Delta t, t + \Delta t) &= Q_{jn}(x, t)(1 - \lambda \Delta t) + \sum_{k=1}^{n} Q_{j-n-k}(x, t) \lambda g_k \Delta t, \quad j \geq 2, n \geq a 
\end{align*}

From the above equations, the steady state queue size equations are obtained as follows:

- \( P'_{i0}(x) = -\lambda P_{i0}(x) + \sum_{m=a}^{b} P_{mi}(0) s(x) + \sum_{l=1}^{\infty} Q_{l0}(0) s(x), \quad a \leq i \leq b \) \hspace{1cm} (2.1)

- \( P'_{ij}(x) = -\sum_{k=1}^{i} P_{i-jk}(x) \lambda g_k, \quad a \leq i \leq b-1, \quad j \geq 1 \) \hspace{1cm} (2.2)
- \( P_{bjo}(x) = -\lambda \ P_{bjo}(x) + \sum_{m=a}^{b} P_{m \ bjo} \ s(x) + \sum_{k=1}^{j} P_{bjo-k}(x) \lambda \ g_k + \sum_{l=1}^{\infty} Q_{l \ bjo} \ s(x), \)
  \( j \geq 1 \) (2.3)

- \( C'_0(x) = -\lambda \ C_0(x) + \sum_{m=a}^{b} P_{m \ 0} \ c(x) \) (2.4)

- \( C'_n(x) = -\lambda \ C_n(x) + \sum_{m=a}^{b} P_{m \ n} \ c(x) + \sum_{k=1}^{n} C_{n-k}(x) \lambda \ g_k, \)
  \( 1 \leq n \leq a-1 \) (2.5)

- \( C'_n(x) = -\lambda \ C_n(x) + \sum_{k=1}^{n} C_{n-k}(x) \lambda \ g_k \)
  \( n \geq a \) (2.6)

- \( Q'_{1 \ 0}(x) = -\lambda \ Q_{1 \ 0}(x) + C_0(0) \ v(x) \) (2.7)

- \( Q'_{1 \ n}(x) = -\lambda \ Q_{1 \ n}(x) + C_n(0) \ v(x) + \sum_{k=1}^{n} Q_{1 \ n-k}(x) \lambda \ g_k, \)
  \( n \geq 1 \) (2.8)

- \( Q'_{j \ 0}(x) = -\lambda \ Q_{j \ 0}(x) + Q_{j-1 \ 0} \ v(x), \)
  \( j \geq 2 \) (2.9)

- \( Q'_{j \ n}(x) = -\lambda \ Q_{j \ n}(x) + Q_{j-1 \ n}(0) \ v(x) + \sum_{k=1}^{n} Q_{j \ n-k}(x) \lambda \ g_k, \)
  \( j \geq 2, 1 \leq n \leq a-1 \) (2.10)

- \( Q'_{j \ n}(x) = -\lambda \ Q_{j \ n}(x) + \sum_{k=1}^{n} Q_{j \ n-k}(x) \lambda \ g_k, \)
  \( j \geq 2, n \geq a \) (2.11)

The Laplace-Stieltjes transforms of \( P_{i \ n}(x) \), \( Q_{j \ n}(x) \) and \( C_n(x) \) are defined as

\[ \widetilde{P}_{i \ n}(\theta) = \int_0^{\infty} e^{-\theta x} \ P_{i \ n}(x) \ dx, \quad \widetilde{Q}_{j \ n}(\theta) = \int_0^{\infty} e^{-\theta x} \ Q_{j \ n}(x) \ dx \quad \text{and} \]
\[ \widetilde{C}_n(\theta) = \int_0^{\infty} e^{-\theta x} \ C_n(x) \ dx. \]
Now, taking Laplace-Stieltjes transforms on both sides of the equation

(2.1) through (2.11), we get

\[ \theta \tilde{P}_{i,0}(\theta) - P_{i,0}(\theta) = \lambda \tilde{P}_{i,0}(\theta) - \sum_{m=a}^{b} P_{m,0}(\theta) \tilde{S}(\theta) - \sum_{l=1}^{\infty} Q_{i,0}(\theta) \tilde{S}(\theta), \]

\[ \text{a} \leq i \leq b \quad (2.12) \]

\[ \theta \tilde{P}_{i,j}(\theta) - P_{i,j}(\theta) = \lambda \tilde{P}_{i,j}(\theta) - \lambda \sum_{k=l}^{j} \tilde{P}_{i,j-k}(\theta) g_k, \quad \text{a} \leq i \leq b-1, \ j \geq 1 \quad (2.13) \]

\[ \theta \tilde{P}_{b,j}(\theta) - P_{b,j}(\theta) = \lambda \tilde{P}_{b,j}(\theta) - \left[ \sum_{m=a}^{b} P_{m,b+j}(\theta) + \sum_{l=1}^{\infty} Q_{1,b+j}(\theta) \right] \tilde{S}(\theta) \]

\[ - \lambda \sum_{k=1}^{j} \tilde{P}_{i,j-k}(\theta) g_k, \quad j \geq 1 \quad (2.14) \]

\[ \theta \tilde{C}_0(\theta) - C_0(\theta) = \lambda \tilde{C}_0(\theta) - \sum_{m=a}^{b} P_{m,0}(\theta) \tilde{C}(\theta) \]

\[ (2.15) \]

\[ \theta \tilde{C}_n(\theta) - C_n(\theta) = \lambda \tilde{C}_n(\theta) - \sum_{m=a}^{b} P_{m,n}(\theta) \tilde{C}(\theta) - \lambda \sum_{k=1}^{n} \tilde{C}_{n-k}(\theta) g_k, \quad 1 \leq n \leq a-1 \quad (2.16) \]

\[ \theta \tilde{C}_n(\theta) = C_n(\theta) = \lambda \tilde{C}_n(\theta) - \lambda \sum_{k=1}^{n} \tilde{C}_{n-k}(\theta) g_k, \quad n \geq a \quad (2.17) \]

\[ \theta \tilde{Q}_{1,0}(\theta) - Q_{1,0}(\theta) = \lambda \tilde{Q}_{1,0}(\theta) - C_0(\theta) \tilde{V}(\theta) \]

\[ (2.18) \]

\[ \theta \tilde{Q}_{1,n}(\theta) - Q_{1,n}(\theta) = \lambda \tilde{Q}_{1,n}(\theta) - C_n(\theta) \tilde{V}(\theta) - \lambda \sum_{k=1}^{n} \tilde{Q}_{1,n-k}(\theta) g_k, \quad n \geq 1 \quad (2.19) \]

\[ \theta \tilde{Q}_{j,0}(\theta) - Q_{j,0}(\theta) = \lambda \tilde{Q}_{j,0}(\theta) - Q_{j-1,0}(\theta) \tilde{V}(\theta), \quad j \geq 2 \quad (2.20) \]
\[ \theta \tilde{Q}_{n}(\theta) - Q_{n}(0) = \lambda \tilde{Q}_{n}(\theta) - \sum_{k=1}^{n} \tilde{Q}_{n-k}(\theta) g_{k}, \quad j \geq 2, 1 \leq n \leq a-1 \tag{2.21} \]

\[ \theta \tilde{Q}_{n}(\theta) - Q_{n}(0) = \lambda \tilde{Q}_{n}(\theta) - \sum_{k=1}^{n} \tilde{Q}_{n-k}(\theta) g_{k}, \quad j \geq 2, n \geq a \tag{2.22} \]

### 2.2 Queue Size Distribution

Lee [58] developed a new technique to find the steady state probability generating function of the number of customers in the system at an arbitrary time epoch. To apply the technique, the following probability generating functions are defined

\[ \tilde{P}_{i}(z, \theta) = \sum_{n=0}^{\infty} \tilde{P}_{i,n}(\theta) z^{n} \quad \text{and} \quad P_{i}(z, 0) = \sum_{n=0}^{\infty} P_{i,n}(0) z^{n}; \quad a \leq i \leq b \]

\[ \tilde{Q}_{j}(z, \theta) = \sum_{n=0}^{\infty} \tilde{Q}_{j,n}(\theta) z^{n} \quad \text{and} \quad Q_{j}(z, 0) = \sum_{n=0}^{\infty} Q_{j,n}(0) z^{n}; \quad j \geq 2 \]

\[ \tilde{C}(z, \theta) = \sum_{n=0}^{\infty} \tilde{C}_{n}(\theta) z^{n} \quad \text{and} \quad C(z, 0) = \sum_{n=0}^{\infty} C_{n}(0) z^{n}; \tag{2.23} \]

Multiplying (2.18) by \( z^{0} \) and (2.19) by \( z^{n} (n \geq 1) \), summing up from \( n = 0 \) to \( \infty \) and using (2.23), we get

\[ (\theta - \lambda + \lambda X(z)) \tilde{Q}_{1}(z, \theta) = Q_{1}(z, 0) - C(z, 0) \tilde{V}(\theta) \tag{2.24} \]
Multiplying (2.20) by $z^0$, (2.21) by $z^n$ ($1 \leq n \leq a-1$) and (2.22) by $z^n$ ($n \geq a$), summing up from $n = 0$ to $\infty$ and using (2.23), we get

$$(\theta - \lambda + \lambda X(z)) \tilde{Q}_j(z, \theta) = Q_j(z, 0) - \tilde{V}(\theta) \sum_{n=0}^{a-1} Q_{j-1,n}(0) z^n, \quad j \geq 2 \quad (2.25)$$

Multiplying (2.15) by $z^0$, (2.16) by $z^n$ ($1 \leq n \leq a-1$) and (2.17) by $z^n$ ($n \geq a$), summing up from $n = 0$ to $\infty$ and using (2.23), we get

$$(\theta - \lambda + \lambda X(z)) \tilde{C}(z, \theta) = C(z, 0) - \tilde{C}(\theta) \sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{m,n}(0) z^n \quad (2.26)$$

Multiplying (2.12) by $z^0$ and (2.13) by $z^i$ ($j \geq 1$), summing up from $j = 0$ to $\infty$ and using (2.23), we get

$$(\theta - \lambda + \lambda X(z)) \tilde{P}_i(z, \theta) = P_i(z, 0) - \tilde{S}(\theta) \left[ \sum_{m=a}^{b} P_{m,i}(0) + \sum_{l=1}^{\infty} Q_{l,i}(0) \right], \quad a \leq i \leq b-1 \quad (2.27)$$

Multiplying (2.12) by $z^0$ and (2.14) by $z^i$ ($j \geq 1$), summing up from $j = 0$ to $\infty$ and using (2.23), we get

$$z^b(\theta - \lambda + \lambda X(z)) \tilde{P}_b(z, \theta) = z^b P_b(z, 0) - \tilde{S}(\theta) \left[ \sum_{m=a}^{b} (P_{m}(z, 0) - \sum_{j=0}^{b-1} P_{m,j}(0) z^j) \right]$$

$$+ \tilde{S}(\theta) \left[ \sum_{l=1}^{\infty} (Q_l(z, 0) - \sum_{j=0}^{b-1} Q_{l,j}(0) z^j) \right] \quad (2.28)$$
By substituting $\theta = \lambda - \lambda X(z)$ in the equations (2.24) through (2.28), we get

$$Q_1(z, 0) = \widetilde{V}(\lambda - \lambda X(z)) C(z, 0)$$

(2.29)

$$Q_j(z, 0) = \widetilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} Q_{j-1}(0) z^n, \quad j \geq 2$$

(2.30)

$$C(z, 0) = \widetilde{C}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{mn}(0) z^n$$

(2.31)

$$P_i(z, 0) = \tilde{S}(\lambda - \lambda X(z)) \left[ \sum_{m=a}^{b} P_{mi}(0) + \sum_{i=1}^{\infty} Q_i(0) \right], \quad a \leq i \leq b-1$$

(2.32)

and

$$z^b P_b(z, 0) = \tilde{S}(\lambda - \lambda X(z)) \left\{ \left[ \sum_{m=a}^{b-1} P_m(z, 0) + P_b(z, 0) - \sum_{j=0}^{b-1} \sum_{m=a}^{b} P_{mj}(0) z^j \right] ight.$$ \left. + \left[ \sum_{i=1}^{\infty} (Q_i(z, 0) - \sum_{j=0}^{b-1} Q_{ij}(0) z^j) \right] \right\}$$

(2.33)

Now, solving for $P_b(z, 0)$ in (2.33), we have

$$(z^b - \tilde{S}(\lambda - \lambda X(z))) P_b(z, 0) = \tilde{S}(\lambda - \lambda X(z)) \left\{ \left[ \sum_{m=a}^{b-1} P_m(z, 0) - \sum_{j=0}^{b-1} \sum_{m=a}^{b} P_{mj}(0) z^j \right] ight.$$ \left. + \left[ \sum_{i=1}^{\infty} Q_i(z, 0) - \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} Q_{ij}(0) z^j \right] \right\}$$

(2.34)

From the equation (2.34), we get

$$P_b(z, 0) = \frac{\tilde{S}(\lambda - \lambda X(z)) f(z)}{(z^b - \tilde{S}(\lambda - \lambda X(z)))}$$

(2.35)

where $f(z) = \sum_{m=a}^{b-1} P_m(z, 0) - \sum_{j=0}^{b-1} \sum_{m=a}^{b} P_{mj}(0) z^j + \sum_{i=1}^{\infty} (Q_i(z, 0) - \sum_{j=0}^{b-1} Q_{ij}(0) z^j).$
Substituting the expressions for $P_m(z, 0)$, $(a < m < b-1)$ from (2.32), $Q_i(z, 0)$ from (2.29) and $Q_j(z, 0)$, $(j > 2)$ from (2.30) in $f(z)$, we get

\[
f(z) = \sum_{i=a}^{b-1} \left[ \sum_{m=a}^{b} P_{mi}(0) + \sum_{l=1}^{\infty} Q_{li}(0) \right] - \sum_{j=0}^{b-1} \sum_{m=a}^{b} P_{mj}(0) z^j
\]
\[
+ \left[ \tilde{V} (\lambda - \lambda X(z)) \tilde{C} (\lambda - \lambda X(z)) \right] \sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{mn}(0) z^n
\]
\[
+ \tilde{V} (\lambda - \lambda X(z)) \sum_{n=0}^{\infty} \sum_{j=2}^{\infty} Q_{j-1 n}(0) z^n.
\]

From the equations (2.24) and (2.29), we get

\[
\tilde{Q}_1(z, \theta) = \frac{\left( \tilde{V} (\lambda - \lambda X(z)) - \tilde{V}(\theta) \right) \tilde{C} (\lambda - \lambda X(z)) \sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{mn}(0) z^n}{(\theta - \lambda + \lambda X(z))} \tag{2.36}
\]

Similarly from the equations (2.25) and (2.30), we get

\[
\tilde{Q}_j(z, \theta) = \frac{\left( \tilde{V} (\lambda - \lambda X(z)) - \tilde{V}(\theta) \right) \sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{mn}(0) z^n}{(\theta - \lambda + \lambda X(z))}, \quad j \geq 2 \tag{2.37}
\]

Equations (2.26) and (2.31) give us

\[
\tilde{C}(z, \theta) = \frac{\left( \tilde{C} (\lambda - \lambda X(z)) - \tilde{C}(\theta) \right) \sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{mn}(0) z^n}{(\theta - \lambda + \lambda X(z))} \tag{2.38}
\]

From the equations (2.27) and (2.32), we get

\[
\tilde{P}_i(z, \theta) = \frac{\left( \tilde{S} (\lambda - \lambda X(z)) - \tilde{S}(\theta) \right) \sum_{m=a}^{b} P_{mi}(0) + \sum_{l=0}^{\infty} Q_{li}(0)}{(\theta - \lambda + \lambda X(z))}, \quad a \leq i \leq b-1 \tag{2.39}
\]
From the equations (2.28) and (2.35), we get
\[ \tilde{P}_b(z, \theta) = \frac{\left(\tilde{S}(\lambda - \lambda X(z)) - \tilde{S}(\theta)\right)f(z)}{(\theta - \lambda + \lambda X(z))(z^b - \tilde{S}(\lambda - \lambda X(z)))} \]  
(2.40)

Let \( P(z) \) be the probability generating function of the queue size at an arbitrary time epoch. Then,
\[ P(z) = \sum_{m=a}^{b-l} \tilde{P}_m(z, 0) + \tilde{P}_b(z, 0) + \tilde{C}(z, 0) + \sum_{l=1}^{\infty} \tilde{Q}_1(z, 0) \]  
(2.41)

Using the equations (2.36) through (2.40) in \( P(z) \) with \( \theta = 0 \), we get
\[ P(z) = \frac{\left(\tilde{S}(\lambda - \lambda X(z)) - 1\right)\sum_{i=a}^{b-l} \sum_{m=a}^{b} P_{ni}(0) + \sum_{l=1}^{\infty} Q_{1i}(0)}{(-\lambda + \lambda X(z))} \]
\[ + \frac{[\tilde{S}(\lambda - \lambda X(z)) - 1] f(z)}{(-\lambda + \lambda X(z))(z^b - \tilde{S}(\lambda - \lambda X(z)))} \]
\[ + \frac{[\tilde{C}(\lambda - \lambda X(z)) - 1] \sum_{n=0}^{a-l} \sum_{m=a}^{b} P_{mn}(0) z^n}{(-\lambda + \lambda X(z))} \]
\[ + \frac{\left(\tilde{V}(\lambda - \lambda X(z)) - 1\right)\tilde{C}(\lambda - \lambda X(z)) \sum_{n=0}^{a-l} \sum_{m=a}^{b} P_{mn}(0) z^n}{(-\lambda + \lambda X(z))} \]
\[ + \frac{\left(\tilde{V}(\lambda - \lambda X(z)) - 1\right) \sum_{n=0}^{a-l} \sum_{j=1}^{\infty} Q_{jn}(0) z^n}{(-\lambda + \lambda X(z))} \]  
(2.42)

Let \( p_i = \sum_{m=a}^{b} P_{mi}(0) \), \( q_i = \sum_{l=1}^{\infty} Q_{li}(0) \) and \( c_i = p_i + q_i \)  
(2.43)
Simplifying the equation (2.42) by using (2.43), we obtain

\[
\left(\tilde{S}(\lambda - \lambda X(z)) - 1\right)\sum_{i=a}^{b-1} (z^b - z^i) c_i + (z^b - 1)(\tilde{V}(\lambda - \lambda X(z)) - 1)\sum_{i=0}^{a-1} q_i z^i +
\]

\[
\left(\tilde{S}(\lambda - \lambda X(z)) - 1\right)(1 - \tilde{C}(\lambda - \lambda X(z)))\tilde{V}(\lambda - \lambda X(z))\sum_{i=0}^{a-1} p_i z^i +
\]

\[
P(z) = \frac{(z^b - 1)(\tilde{V}(\lambda - \lambda X(z))\tilde{C}(\lambda - \lambda X(z)) - 1)\sum_{i=0}^{a-1} p_i z^i}{(-\lambda + \lambda X(z))(z^b - \tilde{S}(\lambda - \lambda X(z)))}
\]

(2.44)

The probability generating function \(P(z)\) has to satisfy the condition \(P(1) = 1\). In order to satisfy the condition, applying L’Hospital’s rule and evaluating \(\lim_{z \to 1} P(z)\) and equating the expression to 1, we have to satisfy

\[
E(S) \sum_{m=a}^{b-1} (b - i) p_i + b(E(C) + E(V)) \sum_{i=0}^{a-1} p_i + b E(V) \sum_{i=0}^{a-1} q_i - \lambda (E(C)E(X)E(S)) \sum_{i=0}^{a-1} p_i = b - \lambda E(X)E(S)
\]

(2.45)

Since \(p_i\) and \(q_i\) are probabilities of ‘i’ customers being in the queue at service completion epoch and vacation completion epoch, respectively, it follows that the left hand side of the above expression must be positive. Thus \(P(1) = 1\) is satisfied if and only if \(b - \lambda E(X)E(S) > 0\). Define \(\rho\) as
Thus $\frac{\lambda E(X)E(S)}{b}$ is the condition to be satisfied for the existence of steady state for the model under consideration.

We express $q_0, q_1, q_2, \ldots, q_{a-1}$, in terms of $p_0, p_1, p_2, \ldots, p_{a-1}$ in the following theorem.

**Lemma 2.1**

Let $\alpha_i$ and $\gamma_i$ be the probabilities that ‘i’ customers arrive during a vacation and closedown time, respectively, then the probability generating functions of $\alpha_i$ and $\beta_i$ are given by

$$\sum_{i=0}^{\infty} \alpha_i z^i = \tilde{V}(\lambda - \lambda X(z))$$ and $$\sum_{i=0}^{\infty} \beta_i z^i = \tilde{C}(\lambda - \lambda X(z))$$

**Proof:**

Conditioning on the actual vacation length, number of arrivals and the group size, we get

$$\alpha_i = \int_0^\infty \left[ \sum_{m=0}^{\infty} \frac{(e^{-\lambda t})(\lambda t)^m g_i^{(m)}}{m!} \right] dV(t),$$

Where $g_i^{(m)}$ is the $m$ - fold convolution of $g_i$ with itself (i.e., total of $m$ arrivals make ‘i’ customers).
Multiplying the above equation by $z^i$ and taking the summation from $i = 0$ to $\infty$, we get

$$
\sum_{i=0}^{\infty} \alpha_i z^i = \int_0^\infty e^{-\lambda t} \left[ \sum_{m=0}^{\infty} \frac{(\lambda t)^m}{m!} \sum_{i=m}^{\infty} g_i^{(m)} z^i \right] dV(t)
$$

$$
= \int_0^\infty e^{-\lambda t} \left[ \sum_{m=0}^{\infty} \frac{(\lambda t)^m}{m!} [X(z)]^m \right] dV(t)
$$

$$
= \widetilde{V}(\lambda - \lambda X(z))
$$

Similarly the other result can be proved.

**Theorem 2.1**

$$
q_n = \sum_{i=0}^{n} K_i p_{n-i}, \quad n = 0, 1, 2, 3, \ldots, a-1
$$

(2.46)

where

$$
K_n = \frac{h_n + \sum_{i=1}^{n} \alpha_i K_{n-i}}{1 - \alpha_0}, \quad n = 1, 2, 3, \ldots, a-1 \text{ with } K_0 = \frac{\alpha_0 \beta_0}{1 - \alpha_0}, \quad h_n = \sum_{i=0}^{n} \alpha_i \beta_{n-i}
$$

and $\alpha_i$ and $\beta_i$ are the probabilities that ‘$i$’ customers arrive during vacation time and closedown time, respectively.

**Proof:**

Using the equations (2.29) and (2.30), we get

$$
\sum_{i=1}^{a} Q_i(z, 0) = \widetilde{V}(\lambda - \lambda X(z)) [C(z, 0) + \sum_{n=0}^{a-1} q_n z^n]
$$
\[
\sum_{n=0}^{\infty} q_n z^n = \tilde{V}(\lambda - \lambda X(z)) \left[ \tilde{C}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} p_n z^n + \sum_{n=0}^{a-1} q_n z^n \right]
\]

which, using Lemma 2.1, gives
\[
\sum_{n=0}^{\infty} q_n z^n = \left( \sum_{n=0}^{\infty} \alpha_n z^n \right) \left[ \sum_{j=0}^{\infty} \sum_{n=0}^{a-1} \beta_j z^j \sum_{n=0}^{a-1} p_n z^n + \sum_{n=0}^{a-1} q_n z^n \right]
\]

Equating the coefficients of \( z^n \) on both sides of the above equation for \( n = 0, 1, 2, \ldots, a-1 \), we have
\[
q_n = \sum_{j=0}^{n} \sum_{i=0}^{n-j} \alpha_n \beta_{n-i-j} p_n + \sum_{i=0}^{n} \alpha_{n-i} q_i
\]

Solving for \( q_n \), we get
\[
q_n = \frac{\sum_{j=0}^{n} \left( \sum_{i=0}^{n-j} \alpha_i \beta_{n-i-j} \right) p_j + \sum_{i=0}^{n} \alpha_{n-i} q_i}{1 - \alpha_0}
\]  \hspace{1cm} (2.47)

Coefficient of \( p_n \) in \( q_n \) is \( \frac{\alpha_0 \beta_{0}}{1 - \alpha_0} = K_0 \)

Coefficient of \( p_{n-1} \) in \( q_n \) is \( \left( \sum_{i=0}^{1} \alpha_i \beta_{n-i} + \alpha_1 \text{ coefficient of } p_{n-1} \text{ in } q_{n-1} \right) / (1 - \alpha_0) \)

\[
= \frac{h_1 + \alpha_0 K_0}{1 - \alpha_0} = K_1, \text{ where } h_1 = \sum_{i=0}^{1} \alpha_i \beta_{n-i}
\]

Coefficient of \( p_{n-2} \) in \( q_n \) is \( \left[ \sum_{i=0}^{2} \alpha_i \beta_{n-i} + \alpha_1 \text{ Coefficient of } p_{n-2} \text{ in } q_{n-1} \right] / ((1 - \alpha_0))
\]
\[ h_2 + \alpha_1 K_1 + \alpha_2 K_0 = K_2, \text{ where } h_2 = \sum_{i=0}^{2} \alpha_i \beta_{n-i} \]

Proceeding like this, we get

\[ h_n + \alpha_1 K_{n-i} \]

Coefficient of \( p_0 \) in \( q_n \) is

\[ \frac{h_n + \sum_{i=1}^{n} \alpha_i K_{n-i}}{1 - \alpha_0} = K_n \]

Therefore \( q_n = \sum_{i=0}^{n} K_i p_{n-i}, n = 0, 1, 2, 3, ..., a-1 \)

Hence \( \sum_{n=0}^{a-1} q_n z^n = \sum_{n=0}^{a-1} \left( \sum_{i=0}^{n} K_i p_{n-i} \right) z^n \)

\[ = \sum_{i=0}^{a-1} \left( \sum_{j=0}^{a-1-i} K_j z^{i+j} \right) p_i \]

Hence the theorem \( \square \)

Since \( q_i \), \( i = 0 \) to \( a-1 \) are expressed in terms of \( p_i, i = 0 \) to \( a-1 \). Now, the equation (2.44) gives that the probability generating function \( P(z) \) involving only ‘\( b \)’ unknowns. To find these constants, Rouche’s theorem of complex variables is used. By Rouche’s theorem, the expression \( z^b - \tilde{S}(\lambda - \lambda X(z)) \) has \( b - 1 \) zeros inside and one on the unit circle \( |z| = 1 \). Since \( P(z) \) is analytic within and on the unit circle, the numerator must vanish at these points, which gives \( b \) equations in \( b \) unknowns. These equations can be solved by any suitable numerical technique. Thus, the equation (2.44) gives the probability generating function of the number of customers in the queue at an arbitrary time.

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2.3 Expected Length of Busy Period

Let $B$ be the busy period random variable. We define another random variable $J$ as

$$J = 0, \text{ if the server finds less than } 'a' \text{ customers after the first service.}$$

$$= 1, \text{ if the server finds at least } 'a' \text{ customers after the first service.}$$

Now, expected length of busy period $E(B)$ is given by

$$E(B) = E(B/J = 0)P(J = 0) + E(B/J = 1)P(J = 1)$$

$$= E(S)P(J = 0) + [E(S) + E(B)] P(J = 1),$$

where $E(S)$ is the expected service time.

Solving for $E(B)$ we get,

$$E(B) = \frac{E(S)}{\sum_{i=0}^{a-1} p_i}$$  \hspace{1cm} (2.48)

2.4 Expected Length of Idle Period

If $I$ be idle period the random variable, then the expected length of idle period is given by,

$$E(I) = E(I_1) + E(C),$$

where $I_1$ is the random variable denoting the ‘Idle period due to multiple vacation process’, $E(C)$ is the expected closedown time.

We define another random variable $U$ as,
U = 0, if the server finds at least ‘a’ customers after the first vacation.
= 1, if the server finds less than ‘a’ customers after the first vacation.

Now, the expected length of idle period due to multiple vacations $E(I_1)$ is given by

$$E(I_1) = E(I_1/U = 0) P(U = 0) + E(I_1/U = 1) P(U = 1)$$

$$= E(V) P(U = 0) + [E(V) + E(I_1)] P(U = 1).$$

Where $E(V)$ is the expected vacation time

Solving for $E(I_1)$ we have,

$$E(I_1) = \frac{E(V)}{P(U = 0)}$$

(2.49)

To find $P(U = 0)$, we do some algebra using the equations (2.23) and (2.29), then

$$Q_1(z, 0) = \sum_{n=0}^{\infty} Q_{1n}(0) z^n = \tilde{V}(\lambda - \lambda X(z)) C(z, 0)$$

$$= \tilde{V}(\lambda - \lambda X(z)) [\tilde{C}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{mn}(0) z^n ]$$

$$= (\sum_{n=0}^{\infty} \alpha_n z^n)[\sum_{j=0}^{\infty} \beta_j z^j \sum_{n=0}^{a-1} p_n z^n ]$$

Equating the coefficients of $z^n (n = 0, 1, 2, 3, \ldots, a-1)$ on both sides, we get

$$Q_{1n}(0) = \sum_{j=0}^{n} \left( \sum_{j=0}^{n-1} \alpha_j \beta_{n-j} \right)p_n$$
\[ P(U = 0) = 1 - \sum_{n=0}^{a-1} Q_{1n}(0) \]

\[ = 1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \left( \sum_{j=0}^{n-1} \alpha_j \beta_{n-i-j} \right) p_n \]  

(2.50)

where \( \alpha_i, \beta_i \) are the probabilities that 'i' customers arrive during vacation and closedown time. Using (2.49) and (2.50), the expected idle period \( E(I) \) is obtained as

\[ E(I) = \frac{E(V)}{1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \left( \sum_{j=0}^{n-1} \alpha_j \beta_{n-i-j} \right) p_n} + E(C). \]  

(2.51)

### 2.5 Expected Queue Length

The expected queue length \( E(Q) \) at an arbitrary time epoch is obtained by differentiating \( P(z) \) at \( z = 1 \) and is given by

\[ E(Q) = \sum_{n=0}^{\infty} n p_n = P'(1) \]

\[ = f_1(X, S) \sum_{i=a}^{b-1} (b - i) c_i + f_2(X, S) \sum_{i=a}^{a-1} (b(b - 1) - i(i - 1)) c_i + f_3(X, S, V, S) \sum_{i=0}^{a-1} p_i \]

\[ + f_4(X, S, C, V) \sum_{i=0}^{a-1} p_i + f_5(X, S, V) \sum_{i=0}^{a-1} q_i + f_6(X, S, V) \sum_{i=0}^{a-1} i q_i \]

\[ E(Q) = \frac{2\lambda(E(X)(b - S1))^2}{2\lambda(E(X)(b - S1))^2} \]

(2.52)

The functions \( f_1, f_2, f_3, f_4, f_5 \) and \( f_6 \) are given by

\( f_1(X, S) = T2 S1 - T1 S2, \quad f_2(X, S) = T1 S1 \)

\( f_3(X, S, C, V) = T2 (S1-b)C1 + T1(b-2S1) C2 + T1(b(b-1)-2(b-S1)V1-2S2)C1 \)
\[ f_4(X, S, C, V) = 2bT_1V_1 - 4T_1S_1C_1, \]
\[ f_5(X, V, S) = bT_1 (V_2 + (b-1)V_1 - bV_1T_2) \]
\[ f_6(X, V, S) = 2bT_1V_1 \]

where
\[ S_1 = \lambda X_1 E(S), \quad S_2 = \lambda X_2 E(S) + \lambda^2 X_1^2 E(S^2), \quad X_1 = E(X), \quad X_2 = X''(1) \]
\[ V_1 = \lambda X_1 E(V), \quad V_2 = \lambda X_2 E(V) + \lambda^2 X_1^2 E(V^2), \]
\[ C_1 = \lambda X_1 E(C), \quad C_2 = \lambda X_2 E(C) + \lambda^2 X_1^2 E(C^2), \]
\[ T_1 = \lambda X_1(b - S_1), \quad T_2 = \lambda X_1(b - 1 - S_2) + X_2(b - S_1). \]

### 2.6 Expected Waiting Time

The expected waiting time is obtained by using the Little's formula

\[
E(W) = \frac{E(Q)}{\lambda E(X)}, \quad (2.53)
\]

where \( E(Q) \) is expected queue length as in \( (2.52) \).

### 2.7 Cost Model

Cost analysis is an important phenomenon in any system. In this Section, the total average cost of the queueing system is derived with the following assumptions:

- \( C_s \): Start-up cost per cycle
- \( C_h \): Holding cost per customer per unit time
- \( C_o \): Operating cost per unit time
- \( C_r \): Reward due to vacation per unit time
- \( C_u \): Closedown cost per unit time.
The length of cycle is the sum of the idle period and busy period. From the equations, (2.48) and (2.51), the expected length of cycle, \( E(T_c) \) is obtained as

\[
E(T_c) = E(I) + E(B)
\]

\[
= \frac{E(V)}{1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \alpha_j \beta_{n-i-j} p_n} + E(C) + \frac{E(S)}{\sum_{i=0}^{a-1} p_i} \quad (2.54)
\]

The total average cost per unit is given by,

Total average cost = Start-up cost per cycle + Holding cost of number of customer in the queue + Operating cost\( \times \rho \)

+ Closedown time cost

- Reward due to vacation per unit time.

\[
\text{Total average cost} = C_s - C_t \frac{E(V)}{P(U = 0)} + C_u E(C) \frac{1}{E(T_c)} + C_h E(Q) + C_o \rho \quad (2.55)
\]

where \( \rho = \lambda E(X) E(S) / b \) and \( E(T_c), E(Q) \) are given in (2.54) and (2.52), respectively.

### 2.8 Numerical Example

A numerical model is analysed with the following assumptions:

(i) Service time distribution is k-Erlang with \( k = 2 \) and service rate \( \mu \)

(ii) Batch size distribution of the arrival is geometric with mean = 2.

(iii) Vacation time is exponential with parameter \( \alpha = 10 \)

(iv) Closedown time is exponential with parameter \( \beta = 7 \)
(v) Minimum service capacity \( a = 3 \)

(vi) Maximum service capacity \( b = 10 \)

(vii) Traffic intensity \( \rho = \frac{2\lambda k}{b \mu} \)

Since \( k = 2 \) and \( b = 10 \), \( z^b - \tilde{S}(\lambda - \lambda X(z)) \) will become a polynomial of degree twelve and it will have 9 roots inside, 2 roots outside and one on the unit circle \( |z| = 1 \). The zeros of the function \( z^b - \tilde{S}(\lambda - \lambda X(z)) \) are found by using MATLAB [41] and using the same, the simultaneous equations are solved.

The expected queue length \( E(Q) \), expected length of idle period \( E(I) \) and expected length of busy period \( E(B) \) and expected waiting times \( E(W) \) are computed and tabulated as detailed below:

Numerical results are presented in Tables 2.1 through 2.4

In Table 2.1, the results of performance measures of the queueing system are presented for the service rate 2.0 and the arrival rate ranging from 2.0 to 4.5. For the service rate 2.5 and arrival rate ranging from 2.0 to 6.0, results are given in Table 2.2. In Table 2.3, the service rate is taken as 3.0 and the arrival rate ranges from 2.0 to 7.0. Results are presented for the service rate 3.5 and the arrival rate ranges from 2.0 to 8.5 in Table 2.4.
From these tables, the following observations are made:

(i) Expected queue length increases, as arrival rate increases,

(ii) Expected queue length decreases, as service rate increases for a particular arrival rate (considering all the tables together).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \rho )</th>
<th>( \text{E(Q)} )</th>
<th>( \text{E(B)} )</th>
<th>( \text{E(I)} )</th>
<th>( \text{E(W)} )</th>
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<tr>
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Table 2.1 Arrival Rate versus Performance Measures for \( \mu = 2.0 \)

<table>
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<tr>
<th>( \lambda )</th>
<th>( \rho )</th>
<th>( \text{E(Q)} )</th>
<th>( \text{E(B)} )</th>
<th>( \text{E(I)} )</th>
<th>( \text{E(W)} )</th>
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<td>1.8425</td>
<td>0.1531</td>
<td>0.4352</td>
</tr>
<tr>
<td>2.5</td>
<td>0.40</td>
<td>2.4089</td>
<td>1.8869</td>
<td>0.1464</td>
<td>0.4818</td>
</tr>
<tr>
<td>3.0</td>
<td>0.48</td>
<td>3.2834</td>
<td>2.0399</td>
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</tr>
<tr>
<td>3.5</td>
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<td>4.4844</td>
<td>2.3118</td>
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<td>0.6406</td>
</tr>
<tr>
<td>4.0</td>
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Table 2.2 Arrival Rate versus Performance Measures for \( \mu = 2.5 \)
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<th>E(B)</th>
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<th>E(W)</th>
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Table 2.3 Arrival Rate versus Performance Measures for \( \mu = 3.0 \)
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<th>$\lambda$</th>
<th>$\rho$</th>
<th>E(Q)</th>
<th>E(B)</th>
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**Table 2.4 Arrival Rate versus Performance Measures for $\mu = 3.5$**

The numerical results are also presented Tables in between 2.5 - 2.9 for the following parameters:

- (i) Vacation rate $\alpha = 10$
- (ii) Arrival rate $\lambda = 2$
- (iii) Minimum service capacity $a = 3$
- (iv) Maximum service capacity $b = 10$
- (v) Closedown rate ranging from 2 to 10.
In Table 2.5, results are presented for the service rate 1.5. For the service rate 2.0, 2.5, 3.0 and 3.5 results are presented respectively, in Tables 2.6 through 2.9.

From these tables, the following observations are made:

(i) Expected queue length increases, as closedown rate increases

(ii) Expected queue length decreases, when the service rate increases for a particular closedown rate (considering all the tables together).

<table>
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<tr>
<th>β</th>
<th>E(Q)</th>
<th>E(B)</th>
<th>E(I)</th>
<th>E(W)</th>
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Table 2.5 Closedown Rate versus Performance Measures for μ = 1.5
### Table 2.6 Closedown Rate versus Performance Measures for $\mu = 2.0$

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### Table 2.7 Closedown Rate versus Performance Measures for $\mu = 2.5$

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### Table 2.8 Closedown Rate versus Performance Measures for $\mu = 3.0$

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### Table 2.9 Closedown Rate versus Performance Measures for $\mu = 3.5$

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