Chapter II

Fuzzy Uniform Structures
CHAPTER II

FUZZY UNIFORM STRUCTURES

Section 1 of this chapter deals with preliminary definitions and results. This includes the new concepts F-star function, F-star refinement, fuzzy uniformity in terms of F-coverings, base and subbase for a fuzzy uniformity. In section 2, it is proved that every uniform structure \( \mu \) on a given set induces a fuzzy uniform structure \( \Gamma = w_u(\mu) \) and conversely every fuzzy uniform structure \( \Gamma \) on a set \( X \) induces a uniform structure \( i_u(\Gamma) \) on \( X \). They are connected by the relations:

(i) \( i_u(w_u(\mu)) = \mu \)
(ii) \( w_u(i_u(\Gamma)) \subseteq \Gamma \)

The third section is devoted to the study of fuzzy topology associated with a fuzzy uniformity. Here it is proved that the topology \( T(\mu) \) induced by a uniformity \( \mu \) and the fuzzy topology \( t(\Gamma) \) induced by the fuzzy uniformity \( \Gamma \) are connected by the following relations:

(i) \( t(w_u(\mu)) = w(T(\mu)) \)
(ii) \( T(i_u(\Gamma)) = i(t(\Gamma)). \)

Further, it is proved here that a topology \( \tau \) on a set \( X \) is uniformizable iff the associated fuzzy topology \( w(\tau) \) is fuzzy uniformizable. It is also proved that

(i) every fuzzy uniform space is fuzzy completely regular
and (ii) the fuzzy topology associated with the Hausdorff fuzzy uniform structure satisfies the FT\(_2\) axiom.

The last section deals with the properties of fuzzy uniform spaces and construction of coarsest and finest fuzzy uniformities.
Section 1

Preliminary definitions and results

Definition 2.1.1

A collection \( \{ f_\lambda \}_{\lambda \in \Lambda} \) of fuzzy sets in a set \( X \) is called an \textit{F-covering} iff \( \bigvee_{\lambda \in \Lambda} f_\lambda = 1 \).

If \( U = \{ f_\lambda \}_{\lambda \in \Lambda} \) and \( V = \{ g_\alpha \}_{\alpha \in \Sigma} \) are two F-coverings of a set \( X \), then \( U \wedge V = \{ f_\lambda \wedge g_\alpha / \lambda \in \Lambda, \alpha \in \Sigma \} \) is an F-covering of \( X \). \( U \wedge V \) is referred to as the \textit{intersection} of \( U \) and \( V \).

Definition 2.1.2

If \( U \) and \( V \) are F-coverings of a set \( X \), then \( V \) is a \textit{refinement} of \( U \) written as \( V < U \), iff for every \( g_\alpha \in V \) there is an \( f_\lambda \in U \) such that \( g_\alpha(x) \leq f_\lambda(x) \) for every \( x \in X \).

Definition 2.1.3

An F-covering \( U \) of a set \( X \), \textit{F-separates} two points \( x \) and \( y \) in \( X \) iff \( f(x) \wedge f(y) = 0 \) for every \( f \) in \( U \).

Notation:

Given \( x \in X \), the set of all \( y \) in \( X \) such that \( x \) and \( y \) are not F-separated by an F-covering \( U \) of \( X \), is denoted by \( <U>(x) \). In the set notation, it is written as

\[
<U>(x) = \{ y \mid x \text{ and } y \text{ are not F-separated by } U \}.
\]

Obviously, \( x \in <U>(x) \) for every \( x \in X \) and for every F-covering \( U \) of \( X \).

Definition 2.1.4

If \( U \) is an F-covering of \( X \) and \( f \) any fuzzy set in \( X \), then \textit{F-star of} \( f \) \textit{with respect to} \( U \), denoted by \( FSt(f, U) \), is the fuzzy set in \( X \), defined by

\[
FSt(f, U)(x) = \sup_{y \in <U>(x)} f(y)
\]
Properties of F-star function: 2.1.5

If X is any set, f, g ∈ Ix and U, V are F-coverings of X, then

1. for any constant fuzzy set α in Ix, FSt (α, U) = α
2. f ≤ FSt(f, U)
3. f ≤ g ⇒ FSt(f, U) ≤ FSt(g, U)
4. if V refines U, FSt(f, V) ≤ FSt(f, U)
5. if H is a family of fuzzy sets in X, then
   \[ FSt(\bigvee_{h \in H} h, U) = \bigvee_{h \in H} FSt(h, U) \]
6. If U is a covering of X and if U = {χU | U ∈ U}, then
   a. FSt(χG, U) = χG(U), for any subset G of X
   b. For any f in Ix,
      \[ FSt(f, U)(x) = \sup \{f(y) | \text{there is an } U \in U \text{ containing both } x \text{ and } y\} \]

Proof: (1), (2), (3) and (4) follow directly from the definition of F-star

5. As h ≤ V h, using (3) one gets FSt(h, U) ≤ FSt(\bigvee_{h \in H} h, U)
   Hence \[ \bigvee_{h \in H} FSt(h, U) \leq FSt(\bigvee_{h \in H} h, U) \]

If for some x in X, \[ \bigvee_{h \in H} FSt(h, U)(x) < FSt(\bigvee_{h \in H} h, U)(x) \]
then there is a \( y_0 \in <U>(x) \) and a \( h_o \in H \) such that
\[ \bigvee_{h \in H} FSt(h, U)(x) < h_o(y_0) \]
(i)

But for every \( y_o \in <U>(x), \)
\[ h_o(y_o) \leq FSt(h_o, U)(x) \]
(ii)

(i) and (ii) are contradictory. Thus \[ \bigvee_{h \in H} FSt(h, U) = FSt(\bigvee_{h \in H} h, U) \]
(6a) Case (i) \( \chi_{s(\mathcal{G},\mathcal{U})}(x) = 1 \)

Then \( x \in \text{St}(\mathcal{G},\mathcal{U}) \). Hence there exists an \( U' \in \mathcal{U} \) such that \( x \in U' \) and \( U' \cap \mathcal{G} \neq \varnothing \)

i.e. there exists an \( U' \in \mathcal{U} \) and a \( y \in \mathcal{G} \) such that both \( x, y \in U' \)

i.e. \( \chi_{U'}(x) \wedge \chi_{U'}(y) \neq 0 \) and \( \chi_{\mathcal{G}}(y) = 1 \)

i.e. \( y \in \langle U \rangle(x) \) and \( \chi_{\mathcal{G}}(y) = 1 \)

Hence \( \sup_{y \in \langle U \rangle(x)} \chi_{\mathcal{G}}(y) = 1 \)

i.e. \( F\text{St} \left( \chi_{\mathcal{G}}, U \right)(x) = 1 \)

Case (ii) \( \chi_{s(\mathcal{G},\mathcal{U})}(x) = 0 \)

Then \( x \notin \text{St}(\mathcal{G},\mathcal{U}) \)

\[ \therefore \text{whenever} \ x \in U' \in \mathcal{U}, \ U' \cap \mathcal{G} = \varnothing \] ———(iii)

\( y \in \langle U \rangle(x) \)

\[ \Rightarrow \text{there exists an} \ U'' \in \mathcal{U} \ \text{such that} \ \chi_{U''}(x) \wedge \chi_{U''}(y) \neq 0, \text{which in term implies} \ x, \ y \in U'' \]

\[ \Rightarrow \text{As} \ x \in U'', \ U'' \cap \mathcal{G} = \varnothing \ \text{by (iii)} \]

\[ \Rightarrow y \notin \mathcal{G} \]

\[ \Rightarrow \chi_{\mathcal{G}}(y) = 0 \]

True for every \( y \in \langle U \rangle(x) \)

Thus \( F\text{St}(\chi_{\mathcal{G}}, U)(x) = 0 \)

Hence by cases (i) and (ii), \( F\text{St} \left( \chi_{\mathcal{G}}, U \right) = \chi_{s(\mathcal{G},\mathcal{U})} \)

(6b) For any \( f \) in \( I^X \), the result follows immediately by noting that \( y \in \langle U \rangle(x) \) iff there is an \( U \in \mathcal{U} \) such that \( \chi_{U}(x) \wedge \chi_{U}(y) \neq 0 \).

**Definition 2.1.6**

An F-covering \( \mathcal{U} \) is an **F-star refinement** of an F-covering \( \mathcal{G} \) of \( X \) iff

(FST1) For every \( g \in \mathcal{G} \), there is an \( f \in \mathcal{U} \) such that \( F\text{St} \left( g, \mathcal{U} \right) \leq f \)
(FST2) If $x$ and $y$ are not F-separated by $\mathcal{U}$, and $y$ and $z$ are not F-separated by $\mathcal{U}$, then $x$ and $z$ are not F-separated by $\mathcal{U}$.

ie. $x \notin \mathcal{U}(y)$ and $y \notin \mathcal{U}(z) \Rightarrow x \notin \mathcal{U}(z)$

Remark 2.1.7

If an F-covering $\mathcal{V}$ is an F-star refinement of an F-covering $\mathcal{U}$, then $\text{FSt}(\text{FSt}(f,\mathcal{V}),\mathcal{V}) \leq \text{FSt}(f,\mathcal{U})$ for every fuzzy set $f$ in $X$.

Proof: Consider $\text{FSt}(\text{FSt}(f,\mathcal{V}),\mathcal{V})(x) = \sup_{y \in \mathcal{V}(x)} \text{FSt}(f,\mathcal{V})(y)$

$= \sup_{y \in \mathcal{V}(x)} \sup_{z \in \mathcal{V}(y)} f(z)$

By (FST2), $z \in \mathcal{V}(y)$ and $y \in \mathcal{U}(x)$ implies that $z \in \mathcal{U}(x)$

Thus $\text{FSt}(\text{FSt}(f,\mathcal{V}),\mathcal{V})(x) \leq \sup_{z \in \mathcal{V}(x)} f(z)$

$= \text{FSt}(f,\mathcal{U})(x)$

Definition 2.1.8

Let $X$ be a nonempty set. A family $\Gamma$ of F-coverings of $X$ is called a fuzzy uniformity on $X$ iff

(FU1) For $\mathcal{U}$, $\mathcal{V}$ in $\Gamma$, $\mathcal{U} \land \mathcal{V}$ is in $\Gamma$

(FU2) If $\mathcal{V} \leq \mathcal{U}$ and if $\mathcal{V} \in \Gamma$, then $\mathcal{U} \in \Gamma$

(FU3) Every member of $\Gamma$ has an F-star refinement in $\Gamma$

A fuzzy uniformity $\Gamma$ is called a Hausdorff fuzzy uniformity iff the following condition is satisfied:

(FU4) For every pair $x, y \in X$, with $x \neq y$, there is an F-covering in $\Gamma$ which F-separates $x$ and $y$.

Members of $\Gamma$ are called uniform F-coverings.
**Definition 2.1.9**

Let $X$ be a nonempty set. A collection $E$ of $F$-coverings is called a **base for a fuzzy uniformity** iff the following conditions are satisfied.

(FUB1) For $U, V$ in $E$, $U \land V$ is refined by a member of $E$.

(FUB2) Every $U$ in $E$ has an $F$-star refinement $V$ in $E$.

$E$ is said to form a **base for a Hausdorff fuzzy uniformity**, if, in addition it satisfies the following condition:

(FUB3) Given $x, y \in X$ with $x \neq y$, there is an $F$-covering $U$ in $E$ which $F$-separates $x$ and $y$.

**Definition 2.1.10**

A **subbase** for a fuzzy uniformity (Hausdorff fuzzy uniformity) is a collection of $F$-coverings whose finite intersections satisfy the conditions for a base for a fuzzy uniformity (Hausdorff fuzzy uniformity).

**Definition 2.1.11**

If $E$ is a base for a Hausdorff fuzzy uniformity (fuzzy uniformity) then the **Hausdorff fuzzy uniformity** $\Gamma_E$ (fuzzy uniformity $\Gamma_E$) generated by $E$ is defined to be the collection of all $F$-coverings, which are refined by members of $E$. In symbols

$\Gamma_E = \{ U | U$ is an $F$-covering of $X$ and $U$ is refined by a member of $E \}$.

**Definition 2.1.12**

If $(X, \Gamma)$ is a fuzzy uniform space, then a subcollection $E \subset \Gamma$ is a **base for $\Gamma$** iff every member of $\Gamma$ is refined by a member of $E$. 
Section 2

Relation between uniformity and fuzzy uniformity

In this section, the following problem is analysed:

"How to associate a fuzzy uniformity \( w_\mu(\mu) \) with a uniformity \( \mu \) and conversely, a uniformity \( i_\Gamma(\Gamma) \) with a fuzzy uniformity \( \Gamma \)?"

For the discussion of the above problem the definition of uniform structure on a set \( X \) is in terms of coverings. For all the results on uniform spaces, one can refer to “Uniform spaces” by Isbell [30].

**Definition:** [Isbell, 30] 2.2.1

A preuniformity \( \mu \) on a set \( X \) is a collection of coverings satisfying the following conditions:

(U1) For \( U, V \) in \( \mu \), \( U \wedge V = \{ U_\alpha \cap V_\beta \mid U_\alpha \in U, V_\beta \in V \} \) belongs to \( \mu \).

(U2) If \( V \) refines \( U \) and \( V \in \mu \) then \( U \in \mu \).

(U3) Every member of \( \mu \) has a star refinement in \( \mu \).

A preuniformity \( \mu \) is called a uniformity if

(U4) For \( x, y \in X \) with \( x \neq y \), there is a covering \( U \) in \( \mu \), no element of which contains both \( x \) and \( y \).

The pair \( (X, \mu) \) is called a uniform space and the members of \( \mu \) are called uniform coverings.

**Definition:** [Isbell, 30] 2.2.2

A base \( \mathcal{B} \) for a preuniformity on a set \( X \) is a collection of coverings which satisfies the following conditions:

(B1) For \( U, V \in \mathcal{B} \), \( U \wedge V \) is refined by a member of \( \mathcal{B} \).
(B2) Every member of $\mathcal{E}$ has a star-refinement in $\mathcal{E}$

$\mathcal{E}$ is a base for a uniformity if $\mathcal{E}$ also satisfies:

(B3) For $x, y \in X$ with $x \neq y$, there is a covering $U$ in $\mathcal{E}$, no element of which contains both $x$ and $y$.

**Definition** [Isbell, 30] 2.2.3

If $\mathcal{E}$ is a base for a uniformity (preuniformity) on a set $X$ then the uniformity $\mu_\mathcal{E}$ (preuniformity $\mu_\mathcal{E}$) generated by $\mathcal{E}$ is defined to be the collection of all coverings which are refined by member of $\mathcal{E}$. In symbols,

$$\mu_\mathcal{E} = \{U \mid U \text{ is a covering of } X \text{ and } U \text{ is refined by a member of } \mathcal{E}\}$$

**Theorem** 2.2.4

Given a uniform space $(X, \mu)$, there is a Hausdorff fuzzy uniform space $(X, w_u(\mu))$ and given a Hausdorff fuzzy uniform space $(X, \Gamma)$, there is a uniform space $(X, i_u(\Gamma))$ having the following relations.

(i) $i_u(w_u(\mu)) = \mu$

(ii) $w_u(i_u(\Gamma)) \subseteq \Gamma$

**Proof:**

Define, for every $U$ in $\mu$ the collection $U$ by

$$U = \{\chi_U \mid U \in U\}$$

Since $X = \bigcup_{U \in U} U$, for every $x \in X$ there is an $U \in U$ such that $x \in U$ i.e. there exists $U \in U$ such that $\chi_U(x) = 1$. Thus $\bigvee_{U \in U} \chi_U(x) = 1$

i.e. $U$ is an F-covering of $X$. Thus for every uniform covering $U$ in $\mu$, an F-covering $U$ is associated.

Let $\mathcal{B}_{w_u(\mu)} = \{U \mid U = \{\chi_U \mid U \in U\}, U \in \mu\}$
To prove this collection forms a base for a Hausdorff fuzzy uniformity

(FUB1) Consider $U$, $V$ in $\mathcal{B}_{w_\mu}$

Let $U$, $V$ be the uniform coverings in $\mu$ associated with $U$, $V$ respectively.

Consider $U \wedge V = \{ \chi_U \wedge \chi_V \mid U \in U, V \in V \}
= \{ \chi_{U \cap V} \mid U \cap V \in U \wedge V \}$

Thus $U \wedge V$ is an F-covering associated with the uniform covering $U \wedge V$
and hence must be in $\mathcal{B}_{w_\mu}$.

(FUB2) Consider any $U$ in $\mathcal{B}_{w_\mu}$

To prove $U$ has an F-star refinement $V$ in $\mathcal{B}_{w_\mu}$. If $U$ is the uniform
covering associated with $U$, then, as $\mu$ is a uniformity, $U$ has a star
refinement $V$ in $\mu$.

Let $V = \{ \chi_V \mid V \in \mathcal{V} \}$

Claim: $V$ is an F-star refinement of $U$

(FST1) Let $\chi_V \in V$.

Then $\chi_V \in V$ and hence there is an element $U$ of $U$ such that $St(V, V) \subseteq U$.

Hence $\chi_{St(V, V)} \leq \chi_U$

:. By property (6) of F-star function as $FSt(\chi_V, U) = \chi_{St(V, V)}$

$FSt(\chi_V, U) \leq \chi_U$

(FST2) Assume that $x \in \langle U \rangle(y)$ and $y \in \langle U \rangle(z)$

To prove $x \in \langle U \rangle(z)$

As $U = \{ \chi_V \mid V \in \mathcal{V} \}$, there exists $V'$, $V'' \in \mathcal{V}$ such that $x$, $y \in V'$
and $y$, $z \in V''$. The three points $x$, $y$, $z$ belong to $St(V', V)$ and hence
to a member $U$ of $U$. Hence $\chi_U(x) \wedge \chi_U(z) \neq 0$ and thus $x \in \langle U \rangle(z)$.

Thus $V$ is an F-star refinement of $U$.

(FUB3) Consider $x$, $y \in X$ with $x \neq y$. Since $\mu$ is a uniformity on $X$, there
is a covering \( U \) in \( \mu \) such that no member of \( U \) contains both \( x \) and \( y \).

Thus \( \chi_U(x) \wedge \chi_U(y) = 0 \) for every \( U \) in \( \mathcal{U} \) i.e. \( x \) and \( y \) are \( F \)-separated by \( \mathcal{U} \). Thus \( \mathcal{B}_{\mathcal{w}_u(\mu)} \) is a base for a Hausdorff fuzzy uniformity on \( X \).

The Hausdorff fuzzy uniformity generated by \( \mathcal{B}_{\mathcal{w}_u(\mu)} \) is denoted by \( \mathcal{w}_u(\mu) \).

Conversely

Let \( \Gamma \) be a Hausdorff fuzzy uniformity on \( X \). For every \( \mathcal{U} \) in \( \Gamma \), define

\[ \mathcal{U} = \{ S(f) \mid f \in \mathcal{U} \} \], where \( S(f) = \text{support of } f = \{ x \in X \mid f(x) > 0 \} \)

Since, for every \( x \) in \( X \), there is an \( f \in \mathcal{U} \) such that \( f(x) > 0 \), \( \mathcal{U} \) is clearly a covering of \( X \).

It is useful to note here that \( y \in \langle \mathcal{U} \rangle(x) \) iff there exists \( f \) in \( \mathcal{U} \) such that \( x \) and \( y \in S(f) \).

Let \( \mathcal{B}_{\mathcal{w}_u(\Gamma)} = \{ \mathcal{U} \mid \mathcal{U} = \{ S(f) \mid f \in \mathcal{U} \}, \mathcal{U} \in \Gamma \} \)

To prove this collection forms a base for a uniformity on \( X \).

(B1) Consider \( \mathcal{U} \), \( \mathcal{V} \) in \( \mathcal{B}_{\mathcal{w}_u(\Gamma)} \) and let them correspond to \( \mathcal{U} \), \( \mathcal{V} \) in \( \Gamma \)

\[ \mathcal{U} \wedge \mathcal{V} = \{ S(f) \cap S(g) \mid f \in \mathcal{U}, g \in \mathcal{V} \} \]

\[ S(f) \cap S(g) = S(f \wedge g) \]

Thus \( \mathcal{U} \wedge \mathcal{V} \) corresponds to \( \mathcal{U} \wedge \mathcal{V} \) in \( \Gamma \) and hence belongs to \( \mathcal{B}_{\mathcal{w}_u(\Gamma)} \)

(B2) Consider \( \mathcal{U} \) in \( \mathcal{B}_{\mathcal{w}_u(\Gamma)} \) associated with the uniform \( F \)-covering \( \mathcal{U} \) in \( \Gamma \).

Let \( \mathcal{V} \) be an \( F \)-star refinement of \( \mathcal{U} \) in \( \Gamma \).

Set \( \mathcal{V} = \{ S(g) \mid g \in \mathcal{V} \} \)

Claim: \( \mathcal{V} \) is a star refinement of \( \mathcal{U} \). Consider \( S(g) \in \mathcal{V} \)

Then there is an \( f \in \mathcal{U} \) such that \( FSt (g, \mathcal{U}) \leq f \quad \text{----(1)} \)
Now \( z \in \text{St}(S(g), \mathcal{V}) \)

\( \Rightarrow \) there exists \( g' \in \mathcal{V} \) such that \( z \in S(g') \) and \( S(g') \cap S(g) \neq \emptyset \).

\( \Rightarrow \) there exists \( g' \in \mathcal{V} \) and an \( x \in X \) such that \( g'(z) > 0 \), \( g'(x) > 0 \) and \( g(x) > 0 \).

Thus \( g'(x) \wedge g'(z) \neq 0 \) and therefore \( x \in \mathcal{V}(z) \).

Thus \( 0 < g(x) \leq \sup_{y \in \mathcal{V}(x)} g(y) = \text{FSt}(g, \mathcal{V})(z) \leq f(z) \), by (1).

ie \( z \in S(f) \).

Thus \( \text{St}(S(g), \mathcal{V}) \subseteq S(f) \). Hence \( \mathcal{V} \) is a star refinement of \( \mathcal{U} \).

(B3) Consider \( x, y \in X \) with \( x \neq y \). Then there is an \( \mathcal{U} \in \Gamma \) such that \( \mathcal{U} \), \( F \)-separates \( x \) and \( y \).

ie. \( f(x) \wedge f(y) = 0 \) for every \( f \in \mathcal{U} \). Hence \( x \) and \( y \) do not belong to the same set \( S(f) \) for every \( f \in \mathcal{U} \).

Thus \( \mathcal{B}_{i_u(\Gamma)} \) is a base for a uniformity on \( X \).

The uniformity generated by \( \mathcal{B}_{i_u(\Gamma)} \) is denoted by \( i_u(\Gamma) \).

To prove the relation between the operators \( i_u \) and \( w_u \).

(i) \( i_u(w_u(\mu)) = \mu \)

\( U \in \mu \)

\( \Rightarrow \) \( \{ \chi_U \mid U \in \mathcal{U} \} \in w_u(\mu) \)

\( \Rightarrow \) \( \{ S(\chi_U) \mid U \in \mathcal{U} \} \in i_u(w_u(\mu)) \)

\( \Rightarrow \) \( \{ U \mid U \in \mathcal{U} \} \in i_u(w_u(\mu)) \)

Hence \( \mu \subseteq i_u(w_u(\mu)) \)

To prove the other inclusion, consider a basis element \( \mathcal{U} \) in \( i_u(w_u(\mu)) \)

Then \( \mathcal{U} = \{ S(f) \mid f \in \mathcal{U}' \}, \mathcal{U} \in w_u(\mu) \)

Then there is a basis element \( \mathcal{U}' \) in \( w_u(\mu) \) such that \( \mathcal{U}' < \mathcal{U} \) with \( \mathcal{U'} = \{ \chi_{U'} \mid U' \in \mathcal{U}' \}, \mathcal{U}' \) in \( \mu \).
Claim: $U'$ refines $U$

$U' \in U' \Rightarrow \chi_{U'} \in U'$

$\Rightarrow$ there exists an $f \in U$ such that $\chi_{U'} \leq f$

$\Rightarrow S(\chi_{U'}) \subseteq S(f)$

$\Rightarrow U' \subseteq S(f)$

$\Rightarrow U'$ refines $U$

Since $U' \in \mu$, $U \in \mu$. Thus $i_u(w_u(\mu)) \subseteq \mu$. Combining both the parts $i_u(w_u(\mu)) = \mu$

(ii) $w_u(i_u(\Gamma)) \subseteq \Gamma$

It is enough to show that the F-covering $U = \{\chi_U | U \in U\}, U$ in $i_u(\Gamma)$, is in $\Gamma$.

$\Rightarrow$ there exists $U' \in i_u(\Gamma)$ such that

$U' = \{S(g) | g \in U' \text{ in } \Gamma\}$ and $U' < U$

Claim: $U'$ refines $U$

Consider $g \in U'$. Then $S(g) \in U'$. Hence there exists $U$ in $U$ such that $S(g) \subseteq U$

$\therefore g \leq \chi_{S(g)} \leq \chi_U$

Thus $U'$ refines $U$. Hence $U \in \Gamma$

Section 3

Fuzzy topology associated with a fuzzy uniformity

With every fuzzy uniformity $\Gamma$, a fuzzy closure operator satisfying the Lowen's closure axioms (Definition 1.1.9) can be defined as follows:

Definition 2.3.1

For any $f$ in $I^X$,

$$\overline{f} = \bigwedge_{u \in r} \text{FSt} (f, U)$$

(FC11) $\overline{\alpha} = \alpha$ for any constant fuzzy set $\alpha$ in $X$.

This follows immediately since FSt ($\alpha, U$) = $\alpha$ for every $U$ in $\Gamma$. 

66
\[(\text{FC12}) \quad f \leq \overline{f} \text{ for any fuzzy set } f \text{ in } X.\]

As \( f \leq \text{FSt}(f,\mathcal{U}) \) for every \( \mathcal{U} \) in \( \Gamma \), \( f \leq \overline{f} \)

\[(\text{FC13}) \quad f \vee g = \overline{f} \vee \overline{g} \text{ for every } f, g \text{ in } I^X.\]

Since \( h_1 \leq h_2 \Rightarrow \text{FSt}(h_1,\mathcal{U}) \leq \text{FSt}(h_2,\mathcal{U}) \), one must have \( h_1 \leq h_2 \Rightarrow \overline{h_1} \leq \overline{h_2} \).

Thus \( \overline{f} \vee \overline{g} \leq \overline{f \vee g} \)

Conversely,

\[
y \in \mathcal{U}(x) \Rightarrow f(y) \leq \text{FSt}(f,\mathcal{U})(x)
\]

and \( g(y) \leq \text{FSt}(g,\mathcal{U})(x) \)

\[
\Rightarrow (f \vee g)(y) \leq (\text{FSt}(f,\mathcal{U}) \vee \text{FSt}(g,\mathcal{U}))(x)
\]

\[
\Rightarrow \text{FSt}(f \vee g,\mathcal{U})(x) \leq (\text{FSt}(f,\mathcal{U}) \vee \text{FSt}(g,\mathcal{U}))(x)
\]

This is true for every \( \mathcal{U} \) in \( \Gamma \).

Thus \( \wedge_{\mathcal{U} \in \Gamma} \text{FSt}(f \vee g,\mathcal{U}) \leq \wedge_{\mathcal{U} \in \Gamma} (\text{FSt}(f,\mathcal{U}) \vee \text{FSt}(g,\mathcal{U})) \)

\[
= (\wedge_{\mathcal{U} \in \Gamma} (\text{FSt}(f,\mathcal{U}))) \vee (\wedge_{\mathcal{U} \in \Gamma} \text{FSt}(g,\mathcal{U}))
\]

i.e. \( \overline{f \vee g} \leq \overline{f} \vee \overline{g} \)

Combining the two inequalities, one must have \( \overline{f} \vee \overline{g} = \overline{f \vee g} \)

\[(\text{FC14}) \quad \overline{f} = \overline{f} \text{ for every } f \text{ in } I^X.\]

Consider \( \overline{\overline{f}} = \wedge_{\mathcal{U} \in \Gamma} \text{FSt}(f,\mathcal{U}) \)

\[
\leq \wedge_{\mathcal{U} \in \Gamma} \text{FSt}(\text{FSt}(f,\mathcal{U}),\mathcal{U}) \quad \text{as } \overline{f} \leq \text{FSt}(f,\mathcal{U}) \]

Since for every \( \mathcal{U} \) in \( \Gamma \) there is a \( \mathcal{V} \) in \( \Gamma \) such that

\[
\text{FSt}(\text{FSt}(f,\mathcal{U}),\mathcal{V}) \leq \text{FSt}(f,\mathcal{U}) \quad \text{(by Remark 2.1.7)}
\]

\[
\overline{f} \leq \wedge_{\mathcal{U} \in \Gamma} \text{FSt}(f,\mathcal{U})
\]

i.e. \( \overline{f} \leq \overline{f} \)

By \( (\text{FC12}) \), \( \overline{f} \leq \overline{f} \). Hence \( \overline{f} = \overline{f} \)

Thus all the four conditions for a fuzzy closure operator are satisfied.

67
Remark: 2.3.2

It is to be noted that $\overline{f}$ can also be expressed as $\overline{f} = \bigwedge_{\mathcal{B} \in \mathcal{B}} \text{FSt}(f, \mathcal{U})$, where $\mathcal{B}$ is a base for the structure $\Gamma$.

Definition: 2.3.3

A fuzzy set $f$ in $X$ is said to be fuzzy closed iff $f = \overline{f}$ and is said to be fuzzy open iff $1 - f = (1 - \overline{f})$.

Theorem: 2.3.4

If $(X, \Gamma)$ is a fuzzy uniform space and if $\mathcal{U} \in \Gamma$, $f, g \in I^X$ with $\text{FSt}(f, \mathcal{U}) \leq g$, then $f \leq \text{int} g$.

Proof: Consider $\mathcal{U} \in \Gamma$ and $f, g \in I^X$ satisfying $\text{FSt}(f, \mathcal{U}) \leq g$.

Since $\text{FSt}(f, \mathcal{U})(y) \leq g(y)$,

$f(x) \leq g(y)$ for every $x \in \langle \mathcal{U} \rangle$, (1)

$\text{FSt}(1-g, \mathcal{U})(x) = \sup_{y \in \langle \mathcal{U} \rangle} (1-g)(y)$

Since $y \in \langle \mathcal{U} \rangle(x) \iff x \in \langle \mathcal{U} \rangle(y)$

$(1-g)(y) \leq (1-f)(x)$, by (1)

Thus $\text{FSt}(1-g, \mathcal{U})(x) \leq (1-f)(x)$.

Hence $(\bigwedge_{\mathcal{U}} \text{FSt}(1-g, \mathcal{U}))(x) \leq (1-f)(x)$

ie. $(1-g) \leq (1-f)$

ie. $f \leq 1 - (1-g) = \text{int} g$.

Hence the result.

Corollary: 2.3.5

$f \leq \text{int} \text{FSt}(f, \mathcal{U})$

Follows immediately from the above theorem by taking $g = \text{FSt}(f, \mathcal{U})$. 
**Lemma :** 2.3.6

Let \((X, \Gamma)\) be a fuzzy uniform space. Given \(U\) in \(\Gamma\), there is a \(V\) in \(\Gamma\) such that \(\text{FSt}(f, V) \leq \text{int FSt}(f, U)\), for every \(f\) in \(I^X\).

**Proof :** By taking \(V'\) as an F-star refinement of \(U\), by remark 2.1.7 one must have, for any \(f\) in \(I^X\)

\[
\text{FSt}(\text{FSt}(f, V'), V') \leq \text{FSt}(f, U) \quad ------(1)
\]

For this \(V'\) there is a \(V\) in \(\Gamma\) such that

\[
\text{FSt}(\text{FSt}(f, V), V) < \text{FSt}(f, V) \quad ------(2)
\]

Consider

\[
\text{FSt}(f, V) \leq \text{FSt}(\text{FSt}(f, V), V) \text{ by the definition of closure} \\
\quad \leq \text{FSt}(f, V') \quad ------(3)
\]

From (1) using the above theorem, one gets that

\[
\text{FSt}(f, V') \leq \text{int FSt}(f, U)
\]

Using (3), one gets,

\[
\text{FSt}(f, V) \leq \text{int FSt}(f, U)
\]

**Theorem :** 2.3.7

\[
f = \bigwedge_{\forall \Gamma} \text{FSt}(f, U) = \bigwedge_{\forall \Gamma} \text{FSt}(f, U) = \bigwedge_{\forall \Gamma} \text{int FSt}(f, U)
\]

**Proof :** By the above lemma for every \(U\) in \(\Gamma\) there is a \(V_u\) in \(\Gamma\) such that \(\text{FSt}(f, V_u) \leq \text{int FSt}(f, U) \leq \text{FSt}(f, U)\)

Now

\[
\bigwedge_{\forall \Gamma} \text{FSt}(f, U) \leq \bigwedge_{\forall \Gamma} \text{FSt}(f, U) = \bigwedge_{\forall U} \text{FSt}(f, V_u) \\
\quad \leq \bigwedge_{\forall} \text{int FSt}(f, U) \\
\quad \leq \bigwedge_{\forall} \text{FSt}(f, U)
\]

Since the first and the last terms are the same, one gets the required equation.
Theorem : 2.3.8

If \((X,\Gamma)\) is a fuzzy uniform space, then every uniform \(F\)-covering in \(\Gamma\) has a refinement in \(\Gamma\) consisting of open fuzzy sets.

Proof : Consider any \(\mathcal{U}\) in \(\Gamma\).

Let \(\mathcal{V}\) be an \(F\)-star refinement of \(\mathcal{U}\) in \(\Gamma\).

Then for every \(g\) in \(\mathcal{V}\) there is an \(f\) in \(\mathcal{U}\) such that

\[FSt(g,\mathcal{U}) \leq f \quad \text{------(1)}\]

Let \(\mathcal{V}' = \{\text{int}(FSt(g,\mathcal{U})) \mid g \in \mathcal{V}\}\). As \(g \leq \text{int}(FSt(g,\mathcal{U}))\) by corollary 2.3.5, \(\mathcal{V}'\) is an \(F\)-covering of \(X\) and \(\mathcal{U}\) refines \(\mathcal{V}'\).

As \(\mathcal{V} \in \Gamma, \mathcal{V}' \in \Gamma \quad \text{----- (2)}\)

Also, as \(\text{int}(FSt(g,\mathcal{U})) \leq FSt(g,\mathcal{U}) \leq f\) (by (1)),

\(\mathcal{V}'\) refines \(\mathcal{U} \quad \text{----- (3)}\)

Thus from (2) and (3) one gets the result.

Definition [Isbell, 30] 2.3.9

Given a uniformity \(\mu\) on \(X\), the topology induced by \(\mu\) is defined as follows:

A subset \(N\) of \(X\) is a neighbourhood of a point \(x\) of \(N\) if for some uniform covering \(\mathcal{U}\) in \(\mu\), \(N\) contains \(St(x,\mathcal{U})\). \(N\) is open iff it is a neighbourhood of each of its points.

Theorem 2.3.10

If the topology, induced by a uniformity \(\mu\) is denoted by \(T(\mu)\) and the fuzzy topology induced by a fuzzy uniformity \(\Gamma\) is denoted by \(t(\Gamma)\), then

(i) \(t(w_u(\mu)) = w(T(\mu))\)

(ii) \(T(i_u(\Gamma)) = i(t(\Gamma))\)
**Proof:**

(1) Consider a closed fuzzy set \( f \) with respect to \( t(w_u(\mu)) \)

To prove \( f \) is closed with respect to \( w(T(\mu)) \) it is enough to prove

\[ 1 - f \in w(T(\mu)). \]

This is equivalent to proving \( (1-f)^{-1} ((\varepsilon, 1]) \in T(\mu) \) for every \( \varepsilon \geq 0 \).

Consider \( x \in (1-f)^{-1} ((\varepsilon, 1]) \)

Then \( (1-f)(x) > \varepsilon \)

i.e. \( f(x) < 1-\varepsilon \)

As \( f \) is closed w.r.t. \( t(w_u(\mu)) \)

\[
\begin{align*}
f(x) &= \bigwedge_{u \in w_u(\mu)} \text{FSt}(f,U)(x) \\
&= \bigwedge_{u} \sup_{y \in (\text{U}^*)(x)} f(y)
\end{align*}
\]

Hence \( f(x) < 1-\varepsilon \Rightarrow \) there is an \( U \) in \( w_u(\mu) \) such that

\[
\sup_{y \in (\text{U}^*)(x)} f(y) < 1-\varepsilon \quad \text{--------(1)}
\]

**Claim :** \( \text{St} (x,U) \subseteq (1-f)^{-1} ((\varepsilon, 1]) \) where \( U \) is the uniform covering associated with \( U \).

i.e., \( U = \{ S(g) \mid g \in \text{U} \} \)

\( y \in \text{St} (x, U) \)

\[ \Rightarrow \] there exists \( S(g) \in U \) containing both \( x \) and \( y \).

\[ \Rightarrow \] \( y \in (\text{U}^*)(x) \)

\[ \Rightarrow \] \( f(y) < 1-\varepsilon \) by (1)

\[ \Rightarrow \] \( 1 - f(y) > \varepsilon \)

\[ \Rightarrow \] \( y \in (1-f)^{-1} ((\varepsilon, 1]) \)

Hence the claim.

Thus \( (1-f)^{-1} ((\varepsilon, 1]) \) belongs to \( T(\mu) \)

Thus \( t(w_u(\mu)) \subseteq w(T(\mu)) \)

Conversely, consider a closed fuzzy set \( f \) w.r.t. \( w(T(\mu)) \).
To prove that \( f \) is closed w.r.t. \( t(w_u(\mu)) \), it is enough to show that

\[
f = \bigwedge_{\mu \in w_u(\mu)} \text{FSt} (f, U)
\]

If \( f(x) = 1 \), then \( \bigwedge_{y} \sup_{y \in \{x \in U(x)\}} f(y) = 1 \), as \( x \in \{x \in U(x)\} \) for every \( U \) in \( w_u(\mu) \)

\[
\therefore \bigwedge_{y} \text{FSt} (f, U)(x) = f(x) \quad \text{--------(2)}
\]

If \( f(x) \neq 1 \), let \( f(x) = t < 1 \)

Choose \( t' \) such that, \( t < t' < 1 \)

Then \( t' = 1 - \varepsilon \) for some \( \varepsilon > 0 \).

\[
\therefore f(x) < 1 - \varepsilon \text{ i.e. } 1 - f(x) > \varepsilon.
\]

ie. \( x \in (1-f)^{-1} ((\varepsilon, 1]) \)

By assumption, \( (1-f)^{-1} ((\varepsilon, 1]) \in T(\mu) \)

Hence there is an \( U \) in \( \mu \) such that \( St (x, U) \subset (1-f)^{-1} ((\varepsilon, 1]) \)

This \( \Rightarrow \) there is an \( U \in \mu \) such that whenever \( x, y \) belong to the same \( U \in \mu \), \( y \in (1-f)^{-1} ((\varepsilon, 1]) \)

\( \Rightarrow \) there is an \( U = \{x_U \mid U \in \mu\} \) in \( w_u(\mu) \) such that whenever \( y \in \{x \in \{x \in U(x)\}\} \) in \( w_u(\mu) \), \( f(y) < 1 - \varepsilon \).

\( \Rightarrow \) there is an \( U \) in \( w_u(\mu) \) such that \( \sup_{y \in \{x \in \{x \in U(x)\}\}} f(y) \leq 1 - \varepsilon \)

Thus for every \( t' > f(x) \), there is an \( U \) in \( w_u(\mu) \) such that

\( f(x) \leq \text{FSt} (f, U)(x) \leq t' \)

Thus \( f(x) \leq \bigwedge_{y} \text{FSt}(f, U) (x) \leq t' \) for every \( t' > f(x) \)

\[
\therefore \bigwedge_{y} \text{FSt}(f, U) (x) = f(x) \quad \text{--------(3)}
\]

Thus from (2) and (3)

\[
f = \bigwedge_{\mu \in w_u(\mu)} \text{FSt} (f, U)
\]

ie \( f \) is closed w.r.t. \( t(w_u(\mu)) \)

Thus \( w(T(\mu)) \subset t(w_u(\mu)) \)

Hence both the inclusions are true.

\[
\therefore t(w_u(\mu)) = w(T(\mu))
\]
Consider for \( f \) in \( \mathcal{T}(\Gamma) \) the element

\[ N = f^\dagger((\epsilon, 1]) \text{ in } i(\mathcal{T}(\Gamma)) \]

Now, \( x \in N \)

\[ \Rightarrow f(x) > \epsilon \]
\[ \Rightarrow 1-f(x) < 1-\epsilon \]
\[ \Rightarrow \bigwedge_{u \in \Gamma} \text{FS} (1-f, u)(x) < 1-\epsilon, \text{ as } 1-f \text{ is closed in } t(\Gamma) \]
\[ \Rightarrow \text{there exists an } u \in \Gamma \text{ such that} \]
\[ \text{FS} (1-f, u)(x) < 1-\epsilon \]

Let \( U \) be the covering associated with \( U \)

ie. \( U = \{S(g) | g \in U\} \)

**Claim:** \( \text{St}(x, U) \subseteq N \)

\( y \in \text{St}(x, U) \)

\[ \Rightarrow \text{there is an } S(g) \in U \text{ containing both } x \text{ and } y \]
\[ \Rightarrow \text{there is } g \in U \text{ such that } g(x) > 0 \text{ and } g(y) > 0 \]
\[ \Rightarrow y \in (U)(x) \]
\[ \Rightarrow \text{from (4), } (1-f)(y) < 1 - \epsilon \]
\[ \Rightarrow f(y) > \epsilon \]
\[ \Rightarrow y \in f^\dagger((\epsilon, 1]) = N \]

Thus \( \text{St}(x, U) \subseteq N \)

Thus \( N \in T(i_u(\Gamma)) \)

\[ : \text{i}(t(\Gamma)) \subseteq T(i_u(\Gamma)) \]

Conversely, consider a member \( N \) of \( T(i_u(\Gamma)) \).

Let \( x \in N \). Then there exists \( U \in \Gamma \) such that

\[ U = \{S(f) | f \in U\} \text{ and } \text{St}(x, U) \subseteq N \]

Since, every uniform F-covering has a refinement in terms of open fuzzy sets (by theorem 2.3.8) all \( f \) in \( U \) can be assumed to be open fuzzy
sets. Hence $f \in U \Rightarrow f \in t(\Gamma)$, which in turn implies $f^{-1}(\varepsilon, 1]) \in i(t(\Gamma))$ for every $\varepsilon \geq 0$.

Since $U$ is an $\mathcal{F}$-covering of $X$, there is an $f \in U$ such that $f(x)>0$. Let $f(x) = \eta$. Choose $\varepsilon < \eta$

Then $x \in f^{-1}(\varepsilon, 1])$

Claim: $f^{-1}(\varepsilon, 1]) \subseteq N$

If $y \in f^{-1}(\varepsilon, 1])$, then $f(y)>0$

Thus $f(x)>0$ and $f(y)>0$. Hence $x, y$ belong to the same $S(f)$ in $U$

$\therefore y \in St(x, U) \subseteq N$

Hence the claim.

Thus for every $x \in N$, there is a basis element $f^{-1}(\varepsilon, 1])$ in $i(t(\Gamma))$ such that $x \in f^{-1}(\varepsilon, 1]) \subseteq N$. Therefore $N \in i(t(\Gamma))$

Hence $T(i_u(\Gamma)) \subseteq i(t(\Gamma))$

Thus both the inclusions hold good ie. $i(t(\Gamma)) = T(i_u(\Gamma))$

**Definition 2.3.11**

A fuzzy topological space $(X, \delta)$ is said to be **fuzzy uniformizable** iff there is a fuzzy uniformity $\Gamma$ on $X$ such that $\delta = t(\Gamma)$.

**Theorem 2.3.12**

A topological space $(X, \tau)$ is uniformizable iff $(X, w(\tau))$ is fuzzy uniformizable.

**Proof:** Assume $(X, \tau)$ is uniformizable. Then there exists an uniformity $\mu$ on $X$ such that $\tau = T(\mu)$

$\therefore w(\tau) = w(T(\mu))$

$= t(w_u(\mu))$ by (i) in theorem 2.3.10

Hence $(X, w(\tau))$ is fuzzy uniformizable.
Conversely,

Assume \((X, w(\tau))\) is fuzzy uniformizable. Then there is a fuzzy uniformity \(\Gamma\) on \(X\) such that \(w(\tau) = t(\Gamma)\)

Therefore \(i(w(\tau)) = i(t(\Gamma))\)

ie. \(\tau = i(t(\Gamma))\)

\[\therefore \tau = T(i_u(\Gamma)) \text{ by (ii) of theorem 2.3.10}\]

Thus \((X, \tau)\) is uniformizable.

**Theorem : 2.3.13**

If \((X, \Gamma)\) is a Hausdorff fuzzy uniform space, then the fuzzy points and crisp points are fuzzy closed.

**Proof :** Consider \(x_t\) with \(t \in (0,1]\)

By the definition of closure

\[\overline{x}_t = \bigwedge_{\forall \epsilon \Gamma} FSt (x_t, \mathcal{U})\]

\[FSt (x_t, \mathcal{U}) (y) = \sup_{\forall z \in p(y)} x_t(z)\]

Hence

\[FSt (x_t, \mathcal{U})(y) = t, \text{ if } x \in \mathcal{U}(y)\]

\[= 0, \text{ otherwise}\]

If \(y \neq x\), then by (FU4), there is an \(F\) - covering \(\mathcal{U}_o\) in \(\Gamma\) which \(F\)-separates \(x\) and \(y\).

Thus \(FSt (x_t, \mathcal{U}_o)(y) = 0\)

\[\therefore \bigwedge_{\forall \epsilon \Gamma} FSt (x_t, \mathcal{U}) (y) = 0\]

Hence \(\overline{x}_t (y) = 0, \text{ if } y \neq x\)

and \(\overline{x}_t(x) = \bigwedge_{\forall \epsilon \Gamma} FSt (x_t, \mathcal{U})(x)\)

\[= t \text{ as } x \in \mathcal{U}(x) \text{ for every } \mathcal{U} \text{ in } \Gamma\]

Thus \(\overline{x}_t = x_t\)
**Corollary 2.3.14** Every Hausdorff fuzzy uniform space is FT$_1$

**Proof:** Follows immediately using the above theorem and theorem 1.2.8

**Theorem :** 2.3.15

If $(X,\Gamma)$ is a Hausdorff fuzzy uniform space, then it is FT$_2$

**Proof :** Let $\alpha$ and $\beta$ be nonzero fuzzy sets with $\alpha \land \beta = 0$

Since $\alpha > 0$, $\beta > 0$, there exist $x, y \in X$ such that $\alpha(x) > 0$, $\beta(y) > 0$

As $\alpha \land \beta = 0$, $x \neq y$

Let $\alpha(x) = t_1$ and $\beta(y) = t_2$

By the previous theorem, $x_{t_1}$ and $y_{t_2}$ are fuzzy closed. Hence using theorem 2.3.7.

\[ x_{t_1} = x_{t_1} = \bigwedge_{\forall \epsilon FSt(x_{t_1}, \mathcal{U}) = \bigwedge_{\forall \epsilon \text{int } FSt(x_{t_1}, \mathcal{U})} \]

Similarly $y_{t_2} = \bigwedge_{\forall \epsilon \text{int } FSt(y_{t_2}, \mathcal{U})}$

As $x \neq y$, by (FU4), there is an F-covering $\mathcal{U}_o$ which F-separates $x$ and $y$. Let $\mathcal{U}_o$ be an F-star refinement of $\mathcal{U}_o$.

Then $\mathcal{U}_o$ also F-separates $x$ and $y$.

\[ \therefore \quad \text{FSt}(x_{t_1}, \mathcal{U}_o)(y) = 0 \]

and $\text{FSt}(y_{t_2}, \mathcal{U}_o)(x) = 0$

Thus $\text{int } \text{FSt}(x_{t_1}, \mathcal{U}_o)(y) = 0$

and $\text{int } \text{FSt}(y_{t_2}, \mathcal{U}_o)(x) = 0$

Let $f = \text{int } \text{FSt}(x_{t_1}, \mathcal{U}_o)$

and $g = \text{int } \text{FSt}(y_{t_2}, \mathcal{U}_o)$

Then $f(x) \land g(x) = f(x) \land 0 = 0 \quad ---(1)$

$f(y) \land g(y) = 0 \land g(y) = 0 \quad ---(2)$

Consider any $z \neq x$ and $\neq y$. 76
If $x$ and $z$ are F-separated by $V_0$ then
\[ f(z) \land g(z) = 0 \land g(z) = 0 \quad \text{----}(3) \]

If $x$ and $z$ are not F-separated by $V_0$ then $y$ and $z$ are F-separated by $V_0$. This follows from the properties of F-star refinement.

Thus $f(z) \land g(z) = f(z) \land 0 = 0 \quad \text{-------}(4)$

From (1), (2), (3) and (4), one must have $f \land g = 0$

\[ \alpha(x) \land f(x) = \alpha(x) \land \text{int } \text{FSt}(x, U)(x) \geq \alpha(x) \land t_1 >0 \]

Similarly $\beta(y) \land g(y) > 0$

Hence $\alpha \land f > 0$, $\beta \land g > 0$ and $f \land g = 0$

**Definition:** 2.3.16

Let $(X, \Gamma_1)$ and $(Y, \Gamma_2)$ be two fuzzy uniform spaces. A function $F: (X, \Gamma_1) \rightarrow (Y, \Gamma_2)$ is **fuzzy uniformly continuous** iff $F^{-1}(U_2)$ is in $\Gamma_1$ for every $U_2$ in $\Gamma_2$. Here $F^{-1}(U)$ stands for the F-covering $\{ F^{-1}(g) \mid g \in U \}$.

**Definition:** [Chang, 12] 2.3.17

Let $(X, \delta_1)$ and $(Y, \delta_2)$ be two fuzzy topological spaces. A function $F: (X, \delta_1) \rightarrow (Y, \delta_2)$ is **fuzzy continuous** iff $F^{-1}(g) \in \delta_1$ for every $g \in \delta_2$.

**Theorem:** 2.3.18

Every fuzzy uniformly continuous function is fuzzy continuous.

**Proof:** Let $(X, \Gamma_1)$ and $(Y, \Gamma_2)$ be two fuzzy uniform spaces and let $F: (X, \Gamma_1) \rightarrow (Y, \Gamma_2)$ be fuzzy uniformly continuous. Let $g$ be any closed fuzzy set in $Y$.

Consider $F^{-1}(g)(x) = \bigwedge_{U_1 \in F^{-1}(U_2)} FSt(F^{-1}(g), U_1)(x) \sup_{y \in F^{-1}(x)} g(F(y))$
Since $F$ is fuzzy uniformly continuous,

$$F'(U_2) \in \Gamma_1 \text{ for every } U_2 \in \Gamma_2$$

Thus $F^{-1}(g)(x) \leq \text{sup}_{y \in F(U_2)} F(g)(y)$

$$y \in <F'(U_2)(x) \Rightarrow \text{there is an } h \in \mathcal{U}_2 \text{ such that}$$

$$h(F(x)) \land h(F(y)) \neq 0$$

$$\Rightarrow F(y) \in <U_2>(F(x))$$

Hence $F^{-1}(g)(x) \leq \text{sup}_{F(y) \in F(U_2)} g(F(y))$

$$\leq \text{sup}_{z \in F(U_2)} g(z)$$

$$= \text{St}(F, U_2)(F(x))$$

$$= g(F(x)), \text{ as } g \text{ is fuzzy closed.}$$

$$= F^{-1}(g)(x).$$

Hence $F^{-1}(g) \leq F^{-1}(g)$

Thus $F^{-1}(g) = F^{-1}(g)$. Hence $F$ is fuzzy continuous.

**Theorem : 2.3.19**

Let $(X, \mu)$ and $(Y, \mu')$ be two uniform spaces. A mapping $F: (X, \mu) \rightarrow (Y, \mu')$ is uniformly continuous iff $F: (X, w_u(\mu)) \rightarrow (Y, w_u(\mu'))$ is fuzzy uniformly continuous.

**Proof :** The result follows by noting that for any $U' \subset Y$, $F^{-1}(\chi_{U'}) = \chi_{F^{-1}(U')}$ and $i_u(w_u(\mu)) = \mu$.

**Theorem : 2.3.20**

Let $(X, \Gamma)$ and $(Y, \Gamma')$ be fuzzy uniform spaces. If a mapping $F: (X, \Gamma) \rightarrow (Y, \Gamma')$ is fuzzy uniformly continuous then $F: (X, i_u(\Gamma)) \rightarrow (Y, i_u(\Gamma'))$ is uniformly continuous.

**Proof :** The result follows by noting that $S(F^{-1}(f')) = F^{-1}(S(f'))$ for any fuzzy set $f'$ in $Y$. 78
In the remaining part of this section the relation between the fuzzy uniformity and the fuzzy uniformity defined by Hutton [28] is discussed.

**Definition: [Hutton, 28] 2.3.21**

Let X be any set and S be the set of maps D: I^X → I^X which satisfy the following conditions:

(A1) f ≤ D(f) for every f ∈ I^X

(A2) D(∇_{λ ∈ H} f_λ) = ∇_{λ ∈ H} D(f_λ) for f_λ ∈ I^X

A Hutton fuzzy quasi-uniformity on a set X is a subset HΓ of S such that

(Q1) HΓ ≠ ∅

(Q2) D ∈ HΓ and D ≤ E and E ∈ S implies E ∈ HΓ.

(Q3) D ∈ HΓ and E ∈ HΓ implies D ∩ E ∈ HΓ

(Q4) D ∈ HΓ implies there exists E ∈ HΓ such that E ∩ E ≤ D, where E ∩ E denotes the composition of mappings E and E.

A Hutton fuzzy quasi-uniformity HΓ is a Hutton fuzzy uniformity if it also satisfies

(Q5) D ∈ HΓ implies D^{-1} ∈ HΓ, where D^{-1}(g) = \{ f \mid D(f) ≤ g \}.

**Properties of D^{-1} [Hutton, 28]: 2.3.22**

If D: I^X → I^X is any mapping satisfying (A1) and (A2) then the mapping D^{-1}: I^X → I^X defined by

D^{-1}(g) = \{ f \mid D(f) ≤ g \} satisfies the following properties:

(i) D^{-1} satisfies (A1) and (A2)

(ii) (D^{-1})^{-1} = D

(iii) D ≤ E iff D^{-1} ≤ E^{-1}
To define a base for a Hutton fuzzy quasi-uniformity consider the following condition:

(Q3') \( D_1 \in \mathcal{H}_\Gamma \) and \( D_2 \in \mathcal{H}_\Gamma \) imply that there exists \( D \in \mathcal{H}_\Gamma \) such that \( D \leq D_1 \) and \( D \leq D_2 \).

**Definition:** [Hutton, 28] 2.3.23

Any subset \( \mathcal{B} \) of \( \mathcal{B} \) which satisfies (Q4) and (Q3') is called a base for a Hutton fuzzy quasi-uniformity.

**Definition:** [Hutton, 28] 2.3.24

Let \( X \) be a nonempty set and \( \mathcal{H}_\Gamma \) a Hutton fuzzy quasi-uniformity on \( X \). Define

\[
\text{int}_H : I^X \to I^X \text{ by}
\]

\[
\text{int}_H(f) = \bigvee \{ g \in I^X / D(g) \leq f \text{ for some } D \in \mathcal{H}_\Gamma \}
\]

Then this satisfies the axioms for an interior operator. The **fuzzy topology induced by \( \mathcal{H}_\Gamma \)** is the fuzzy topology associated with the above interior operator, \( \text{int}_H \).

**Theorem :** 2.3.25

Every fuzzy uniformity induces a Hutton fuzzy uniformity \( \mathcal{H}_\Gamma \) such that the associated fuzzy topologies coincide.

**Proof :** For every \( \mathcal{U} \in \Gamma \), define

\[
D_\mathcal{U} : I^X \to I^X \text{ by}
\]

\[
D_\mathcal{U}(f) = FSt(f, \mathcal{U})
\]

(A1) Since \( f \leq FSt(f, \mathcal{U}) \), \( f \leq D_\mathcal{U}(f) \)

(A2) \( D_\mathcal{U}(\vee f_\lambda) = FSt(\vee f_\lambda, \mathcal{U}) \)
Claim: $\mathcal{B}$ is a base for Hutton fuzzy quasi-uniformity

(Q3') Let $D_u, D_v \in \mathcal{B}$

Then $D_{u \wedge v}(f) = \text{FSt}(f, \mathcal{U} \wedge \mathcal{V})$

$\leq \text{FSt}(f, \mathcal{U}) = D_u(f)$

Similarly $D_{u \wedge v}(f) \leq D_v(f)$

Thus condition (Q3') is satisfied

(Q4) Consider $D_u \in \mathcal{B}$

Let $\mathcal{U}$ be an F-star refinement of $\mathcal{U}$.

Let $E = D_v$

Then $E \circ E(f) = D_v(D_v(f))$

$= D_v(\text{FSt}(f, \mathcal{U}))$

$= \text{FSt}(\text{FSt}(f, \mathcal{U}), \mathcal{U})$

$\leq \text{FSt}(f, \mathcal{U}) = D_u(f)$

Thus $\mathcal{B}$ forms a base for a Hutton fuzzy quasi-uniformity $(H\Gamma)_\mathcal{B}$

(Q5) Consider $D_u \in \mathcal{B}$

$D_u^{-1}(f) = \wedge \{ g \mid D_u(g^c) \leq f^c \}$

$= \wedge \{ g \mid \text{FSt}(g^c, \mathcal{U}) \leq f^c \}$

$\text{FSt}(g^c, \mathcal{U}) \leq f^c$

$\Rightarrow \text{FSt}(1-g, \mathcal{U}) \leq 1-f$

$\Rightarrow \sup_{y \in \mathcal{U}(x)}(1-g)(y) \leq 1-f(x)$

$\Rightarrow 1-g(y) \leq 1-f(x)$ for every $y \in \mathcal{U}(x)$

$\Rightarrow f(x) \leq g(y)$ for every $y \in \mathcal{U}(x)$

Now, consider $x \in \mathcal{U}(y)$
Then \( y \in \mathcal{U}(x) \) and thus \( f(x) \leq g(y) \)

\[ \therefore \sup_{x \in \mathcal{U}(y)} f(x) \leq g(y) \]

\[ \Rightarrow \quad \text{FSt}(f, \mathcal{U})(y) \leq g(y) \]

\[ \Rightarrow \quad D_y(f) \leq g \]

True for every \( g \) with \( \text{FSt}(g^\circ \mathcal{U}) \leq f^\circ \)

Thus \( D_y(f) \leq D_y^{-1}(f) \)

ie. \( D_y \leq D_y^{-1} \)

Hence by the properties of \( D^{-1} \), \( D_y^{-1} \leq (D_y^{-1})^{-1} = D_y \)

Hence \( D_y = D_y^{-1} \)

Consider \( D \in (\Gamma \mathcal{G})_\mathcal{G} \). Then there is a \( D_y \) in \( \mathcal{B} \) such that \( D_y \leq D \)

\[ \therefore \quad D_y^{-1} \leq D^{-1} \]

ie. \( D_y \leq D^{-1} \). Thus \( D^{-1} \in (\Gamma \mathcal{G})_\mathcal{G} \)

Hence \( (\Gamma \mathcal{G})_\mathcal{G} \) is a Hutton fuzzy uniformity on \( X \). Denote \( (\Gamma \mathcal{G})_\mathcal{G} \) by \( \Gamma \mathcal{G} \).

To Prove: The associated fuzzy topologies coincide.

\[ \text{int}_H(f) = \vee \{ g \in I^X \mid D(g) \leq f \text{ for some } D \in \Gamma \mathcal{G} \} \]

\( D \in \Gamma \mathcal{G} \Rightarrow \) there is an \( \mathcal{U} \in \Gamma \) such that \( D_y \leq D \)

Thus \( D(g) \leq f \Rightarrow D_y(g) \leq f \)

\[ \Rightarrow \quad \text{FSt}(g, \mathcal{U}) \leq f \]

\[ \Rightarrow \quad g \leq \text{int } f, \text{ where int is the interior with respect to } t(\Gamma) \]

True for every \( g \) with \( D(g) \leq f \).

Thus \( \text{int}_H(f) \leq \text{int } f \) \quad (1) \)

Consider,

\[ \text{int } f = 1-(1-f) \]

\[ = 1- \bigwedge_{\mathcal{U} \in \Gamma} \text{FSt}(1-f, \mathcal{U}) \]

\[ = \bigvee_{\mathcal{U} \in \Gamma}(1-\text{FSt}(1-f, \mathcal{U})) \]

\[ = \bigvee_{\mathcal{U} \in \Gamma} h_{\mathcal{U}}, \text{ where } h_{\mathcal{U}} = 1-\text{FSt}(1-f, \mathcal{U}) \]

82
\[ h(x) = (1 - F_{\text{St}}(1 - f, U))(x) \]
\[ = 1 - \sup_{y \in U(x)} (1 - f)(y) \]
\[ = \bigwedge_{y \in U(x)} f(y) \]

Thus \( h(x) \leq f(y) \) for every \( y \in U(x) \) ———(2)

Consider \( D_y(h_y)(z) = F_{\text{St}}(h_y, U)(z) \)
\[ = \sup_{x \in U(z)} h_y(x) \]

As \( x \in U(z) \Rightarrow z \in U(x) \), by (2) \( h_y(x) \leq f(z) \)
\begin{align*}
& \text{ie. } D_y(h_y)(z) \leq f(z) \\
& \text{ie. } D_y(h_y) \leq f
\end{align*}

Thus \( \forall h_y \leq \lor \{ g | D(g) \leq f \text{ for some } D \in \Gamma \} \)
\begin{align*}
& \text{ie. } \text{int } f \leq \text{int}_H f \quad ----(3)
\end{align*}

Hence from (1) and (3), one must have
\[ \text{int } f = \text{int}_H f \]

Thus the fuzzy topology \( t(\Gamma) \) associated with \( \Gamma \) and the fuzzy topology \( t(\Gamma H) \) associated with \( \Gamma H \) coincide.

**Definition** [Hutton, 28]: 2.3.26

A fuzzy topology space \( (X, \delta) \) is **fuzzy completely regular** iff for every open fuzzy set \( f \), there exists a collection \( \{ g_\alpha \} \) of fuzzy sets and a family \( \{ F_\alpha \} \) of fuzzy continuous functions from \( X \) into \([0, 1](L)\) ([0, 1](L) is the fuzzy unit interval [Hutton, 28]) such that
\[ \lor g_\alpha = f \text{ and } g_\alpha(x) \leq F_\alpha(x)(1-) \leq F_\alpha(x)(0+) \leq f(x) \text{ for } x \in X \]

**Theorem** [Hutton, 28]: 2.3.27

Every Hutton fuzzy uniform space is fuzzy completely regular.
Theorem 2.3.28

Every fuzzy uniform space is fuzzy completely regular.

Proof: Let \((X, \Gamma)\) be a fuzzy uniform space and let \(H_\Gamma\) be the induced Hutton fuzzy uniformity. Then as \(t(\Gamma) = t(H_\Gamma)\), the result follows from the above theorem.

Example 2.3.29

A Hausdorff fuzzy uniformity on the fuzzy real line

Consider the fuzzy real line \(R(I)\). Elements of \(R(I)\) are left continuous monotonically decreasing functions \(\lambda\) on \(R\) with \(\sup \lambda = 1\) and \(\inf \lambda = 0\) [50]

Given \(\varepsilon > 0\) and the crisp point \(\lambda\),

Define \(S(\lambda, \varepsilon) = \{\mu \mid |\mu(t) - \lambda(t)| < \varepsilon, \text{ for all } t, \mu \in R(I)\}\)

Given \(\varepsilon > 0\) and \(f \in R(I)\), define

\[U_\varepsilon(f) : R(I) \to [0,1] \text{ by} \]

\[U_\varepsilon(f)(\mu) = \vee_{\gamma \in S(\mu, \varepsilon)} f(\gamma)\]

For the crisp point \(\lambda\) in \(R(I)\)

\[U_\varepsilon(\lambda)(\mu) = \vee_{\gamma \in S(\mu, \varepsilon)} \lambda(\gamma)\]

Thus \(U_\varepsilon(\lambda)(\mu) = 1\) if \(\lambda \in S(\mu, \varepsilon)\)

\[= 0, \text{ otherwise}\]

To define a fuzzy uniform structure on \(R(I)\)

Let \(U_\varepsilon = \{U_\varepsilon(\lambda) \mid \lambda, \text{ a crisp point in } R(I)\}\)

Since \(\lambda \in S(\lambda, \varepsilon)\), \(\vee_{\varepsilon \in R(I)} U_\varepsilon(\lambda) = 1\)

Thus \(U_\varepsilon\) is an \(F\)-covering of \(X\)

Let \(\delta_\Gamma = \{U_\varepsilon \mid \varepsilon \in (0,1]\}\)
Claim: $\delta_\Gamma$ is a subbase for a fuzzy uniformity on $X$

(FST1) Consider $U_\varepsilon \in \delta_\Gamma$

To prove $U_{\varepsilon/4}$ is an F-star refinement of $U_\varepsilon$
Let $H = \text{FSt}(U_{\varepsilon/4}(\lambda), U_{\varepsilon/4})$
If $H(\gamma) = 0$, then $H(\gamma) \leq U_\varepsilon(\lambda)(\gamma)$
If $H(\gamma) = 1$, then there is a $\mu \in <U_{\varepsilon/4}>(\gamma)$ such that $U_{\varepsilon/4}(\lambda)(\mu) = 1$
$\mu \in <U_{\varepsilon/4}>(\gamma)$
$\Rightarrow$ there is a $\lambda' \in R(I)$ such that $U_{\varepsilon/4}(\lambda')(\mu) \wedge U_{\varepsilon/4}(\lambda')(\gamma) \neq 0$
$\Rightarrow \lambda' \in S(\mu, \varepsilon/4)$ and $S(\gamma, \varepsilon/4)$
$\Rightarrow |\mu(t) - \gamma(t)| < \varepsilon/2$
$U_{\varepsilon/4}(\lambda)(\mu) = 1 \Rightarrow \lambda \in S(\mu, \varepsilon/4)$
$\Rightarrow |\lambda(t) - \mu(t)| < \varepsilon/4$
Thus $|\lambda(t) - \gamma(t)| < \varepsilon$
$\Rightarrow \lambda \in S(\gamma, \varepsilon) \Rightarrow U_\varepsilon(\lambda)(\gamma) = 1$
Thus $H \leq U_\varepsilon(\lambda)$

(FST2) Let $\mu_1 \in <U_{\varepsilon/4}>(\mu_2)$ and $\mu_2 \in <U_{\varepsilon/4}>(\mu_3)$.
Then $|\mu_1(t) - \mu_2(t)| < \varepsilon/2$ and $|\mu_2(t) - \mu_3(t)| < \varepsilon/2$.
$\Rightarrow \mu_2 \in S(\mu_1, \varepsilon)$ and $\mu_2 \in S(\mu_3, \varepsilon)$
$\Rightarrow U_\varepsilon(\mu_2)(\mu_1) = 1$ and $U_\varepsilon(\mu_2)(\mu_3) = 1$.
$\Rightarrow \mu_1 \in <U_\varepsilon>(\mu_3)$

Thus $\delta_\Gamma$ forms a subbase for a fuzzy uniformity $\Gamma$ on $X$. This uniformity $\Gamma$ is a Hausdorff fuzzy uniformity on $X$. Consider $\lambda, \mu$ in $R(I)$ with $\lambda \neq \mu$. There exists $t \in R$ such that $\lambda(t) \neq \mu(t)$. Suppose $\lambda(t) > \mu(t)$.

Choose $\varepsilon$ such that $0 < \varepsilon < \lambda(t) - \mu(t)$ --- (1)
Consider the F-covering $U_{\varepsilon/2}$
Claim: $\mathcal{U}_{e^2}$ F-separates $\lambda$ and $\mu$

Suppose this is not true, then there is a $\gamma \in \mathbb{R}(I)$ such that

$\mathcal{U}_{e^2}(\gamma)(\lambda) \wedge \mathcal{U}_{e^2}(\gamma)(\mu) \neq 0.$

i.e. $\gamma \in S(\lambda,e/2) \cap S(\mu,e/2)$.

$\therefore | \lambda(t) - \mu(t) | < \varepsilon$ for every $t$ in $\mathbb{R}$.

This contradicts (1). Hence $\mathcal{U}_{e^2}$ F-separates $\lambda$ and $\mu$.

Thus $\Gamma$ is a Hausdorff fuzzy uniformity an $X$.

The following topology $\delta$ on the fuzzy real line is studied by Lowen [50].

A subbase for $\delta$ is given by the collection $\{ L_t | t \in \mathbb{R} \} \cup \{ R_t | t \in \mathbb{R} \}$

where $L_t(\lambda) = 1-\lambda(t)$ and $R_t(\lambda) = \lambda(t+)$. 

Claim: The fuzzy topology $t(\Gamma)$ induced by $\Gamma$ is finer than the above topology $\delta$. To prove this it is enough to show that the subbase elements $R_t$ and $L_t$ are open in $t(\Gamma)$ or equivalently, the fuzzy sets $\alpha_t = 1-R_t$ and $\beta_t = 1-L_t$ are fuzzy closed in $t(\Gamma)$.

Let $\{ \varepsilon_n \}$ be a sequence of positive real numbers converging to 0.

$\bar{\alpha}_t(\lambda) = \wedge_{\mathbb{R}} \text{FSt}(\alpha_t, \mathcal{U})(\lambda) \leq \wedge_{\mathbb{R}} \text{FSt}(\alpha_t, \mathcal{U}_{e_n})(\lambda)$

$= \wedge_{\mathbb{R}} \sup_{\gamma < U_{e_n}(\lambda)} \alpha_t(\gamma)$

$\gamma < U_{e_n}(\lambda) \Rightarrow | \gamma(t) - \lambda(t) | < 2\varepsilon_n$

$\Rightarrow \lambda(t) - 2\varepsilon_n < \gamma(t) < \lambda(t) + 2\varepsilon_n$

$\Rightarrow \lambda(t+) - 2\varepsilon_n < \gamma(t+) < \lambda(t+) + 2\varepsilon_n$

$\Rightarrow 1-\gamma(t+) < 1-(\lambda(t+) - 2\varepsilon_n)$

$\bar{\alpha}_t(\lambda) \leq \wedge_{\mathbb{R}} 1-(\lambda(t+) - 2\varepsilon_n)$

$= 1 - \lambda(t+) = \alpha_t(\lambda)$

Thus $\alpha_t$ is fuzzy closed in $t(\Gamma)$. Similarly it can be shown that $\beta_t$ is fuzzy closed in $t(\Gamma)$. 

86
Section 4

Properties of fuzzy uniform spaces

Definition: 2.4.1

A family $\mathcal{Z}$ of F-coverings of a set $X$ is called a normal family iff every $\mathcal{U}$ in $\mathcal{Z}$ has an F-star refinement in $\mathcal{Z}$.

Definition: 2.4.2

A sequence $\{\mathcal{U}^n\}$ of F-coverings of a set $X$ is called a normal sequence iff $\mathcal{U}^{n+1}$ is an F-star refinement of $\mathcal{U}^n$ for every $n$.

Definition: 2.4.3

Let $\mathcal{Z}$ be a family of F-coverings of a set $X$. An F-covering $\mathcal{U}$ is said to be normal with respect to $\mathcal{Z}$ iff $\mathcal{U}$ is the first term of some normal sequence of F-coverings each of which has a refinement in $\mathcal{Z}$.

Remark: 2.4.4

If $\mathcal{U}$ is an F-covering belonging to some fuzzy uniformity contained in $\mathcal{Z}$ then $\mathcal{U}$ must be normal with respect to $\mathcal{Z}$.

Theorem: 2.4.5

Let $X$ be a nonempty set. Every normal family of F-coverings of $X$ is a subbase for a fuzzy uniformity.

Proof: Let $\mathcal{S}$ be a normal family of F-coverings of $X$. Let $\mathcal{B}$ denote the collection of finite intersections of members of $\mathcal{S}$.

(FUB1) is easily seen to be satisfied

(FUB2) Consider $\mathcal{U}$ in $\mathcal{B}$. Then $\mathcal{U} = \mathcal{U}_1 \wedge \mathcal{U}_2 \wedge \ldots \wedge \mathcal{U}_n$ where $\mathcal{U}_i \in \mathcal{S}$. Each
\( U_i \) has an F-star refinement \( V_i \) in \( \mathcal{S} \)

Let \( V = V_1 \wedge V_2 \wedge \ldots \wedge V_n \)

Proof of condition (FST1) is straightforward.

(FST2) \( x \in \langle V \rangle(y) \) and \( y \in \langle V \rangle(z) \) implies that there exists
\[
\bigwedge_{i=1}^n f_i \text{ and } \bigwedge_{i=1}^n g_i \in V \text{ such that }
\bigwedge_{i=1}^n f_i(x) \wedge f_i(y) \neq 0
\bigwedge_{i=1}^n g_i(y) \wedge g_i(z) \neq 0
\]

Hence for every \( i, x \in \langle V_i \rangle(y) \) and \( y \in \langle V_i \rangle(z) \)
\[ \therefore x \in \langle U_i \rangle(z) \text{ for every } i. \]

Hence \( x \in \langle U \rangle(z) \)
\[ \therefore V \text{ is an F-star refinement of } U. \text{ Thus } \mathcal{S} \text{ forms a subbase for a fuzzy uniformity.} \]

**Definition** [Isbell, 30]: 2.4.6

A family \( \{f_a\} \) of functions with the same domain \( X \) but possibly different ranges separates points iff given two distinct points \( x,y \) there exists \( \alpha \) such that \( f_a(x) \neq f_a(y) \).

**Theorem** : 2.4.7

For any family \( \{f_a\} \) of functions on a set \( X \) into various Hausdorff fuzzy uniform spaces there is a coarsest fuzzy uniformity on \( X \) including all the inverse images of uniform F-coverings under these functions. If these functions separate points, then this fuzzy uniformity is a Hausdorff fuzzy uniformity.
The proof of this theorem, involves the proof of the following lemma:

**Lemma : 2.4.8**

Let \( X \) and \( Y \) be two nonempty sets and \( F : X \rightarrow Y \) be any mapping. If \( \mathcal{U} \) is an \( F \)-covering of \( Y \) and \( \mathcal{V} \) is an \( F \)-star refinement of \( \mathcal{U} \), then \( F^{-1}(\mathcal{V}) \) is an \( F \)-star refinement of \( F^{-1}(\mathcal{U}) \).

**Proof :** Consider \( g \in \mathcal{V} \)

\[
(FST1) \quad \text{Let } H = \text{FSt}(F^{-1}(g), F^{-1}(\mathcal{V})) (x) \sup_{y \in F^{-1}(\mathcal{V})(x)}\ F^{-1}(g)(y).
\]

\( y \in F^{-1}(\mathcal{V})(x) \Leftrightarrow F(y) \in \mathcal{V}(F(x)) \)

Thus \( H(x) = \sup_{F(y) \in \mathcal{V}(F(x))} g(F(y)) \leq \sup_{z \in \mathcal{U}(F(x))} g(z) = \text{FSt}(g, \mathcal{U})(F(x)) \leq f(F(x)) \) for some \( f \) in \( \mathcal{U} \).

\( = F^{-1}(f)(x) \)

\( (FST2) \) is straightforward.

**Proof of the theorem :**

By the above lemma, one gets that the family of inverse images of uniform \( F \)-coverings form a normal family.

Hence by theorem 2.4.5, this family forms a subbase for a fuzzy uniformity \( \Gamma \) and obviously this is the coarsest fuzzy uniformity satisfying the required condition.

Assume \( \{f_\alpha\} \) separates points.

Consider \( x \neq y \). Then there exists \( \alpha \) such that \( f_\alpha(x) \neq f_\alpha(y) \). Let the codomain space corresponding to \( \alpha \) be \( (Y_\alpha, \Gamma_\alpha) \). Then there exists \( \mathcal{U}_\alpha \in \Gamma_\alpha \) such that \( g(f_\alpha(x)) \vee g(f_\alpha(y)) = 0 \) for every \( g \in \mathcal{U}_\alpha \).
As \( f^{-1}_a(g)(x) \land f^{-1}_a(g)(y) = g(f_a(x)) \land g(f_a(y)) = 0 \)
\( f^{-1}_a(U_a) \) F-separates \( x \) and \( y \). Thus \( \Gamma \) is a Hausdorff fuzzy uniformity on \( X \).

**Theorem :** 2.4.9

Let \( X \) be a nonempty set. If \( \Sigma \) is a family of F-coverings of \( X \) which forms a filter under refinement, then there is a finest fuzzy uniformity contained in \( \Sigma \) and it contains all F-coverings normal with respect to \( \Sigma \).

**Proof :**

Let \( \Gamma \) be the family of all F-coverings normal with respect to \( \Sigma \).

Since \( \Sigma \) is a filter, \( \Gamma \) is contained in \( \Sigma \).

(FU1) Consider \( U, V \) in \( \Gamma \). Then there are normal sequences \( \{U^n\} \) and \( \{V^n\} \) such that \( U = U^1, V = V^1 \) and for each \( n \) \( U^n \) and \( V^n \) have refinements in \( \Sigma \). Hence \( \{U^n \land V^n\} \) is a normal sequence, \( U \land V = U^1 \land V^1 \) and \( U^n \land V^n \) has a refinement in \( \Sigma \). Thus \( U \land V \) is in \( \Gamma \).

(FU2) and (FU3) are easily seen to be satisfied.

If \( \Gamma' \) is fuzzy uniformity contained in \( \Sigma \) then any F-covering in \( \Gamma' \) is normal with respect to \( \Sigma \) and therefore belongs to \( \Gamma \).

**Theorem 2.4.10**

Let \( (X, \Gamma) \) be a Hausdorff fuzzy uniform space, \( g \) a function from \( X \) into a set \( Y \) and \( \{f_a\} \), a family of functions, separating points, from \( Y \) into a collection \( \{Z_a\} \) of Hausdorff fuzzy uniform spaces. Let \( \Gamma' \) be the coarsest Hausdorff fuzzy uniformity on \( Y \) induced by the functions \( \{f_a\} \). Then \( g : (X, \Gamma) \rightarrow (Y, \Gamma') \) is fuzzy uniformly continuous iff the composition \( f_a \circ g : (X, \Gamma) \rightarrow Z_a \) is fuzzy uniformly continuous for every \( \alpha \).

Proof follows easily as in the classical theory.
Terminology

A fuzzy uniformity inducing a given fuzzy topology is said to be compatible with the fuzzy topology.

Theorem : 2.4.11

Let $(X, \delta)$ be a fuzzy uniformizable fuzzy topological space. Let $\Gamma$ be the family of all F-coverings which are normal with respect to the family of fuzzy open F-coverings. Then $\Gamma$ is a fuzzy uniformity on $X$ and $\Gamma$ contains all fuzzy uniformities compatible with $\delta$.

Proof:

Let $\Sigma$ be the family of F-coverings generated by the filter base consisting of all fuzzy open F-coverings. Then by theorem 2.4.9, $\Gamma$ is the finest fuzzy uniformity contained in $\Sigma$. If $\Gamma'$ is a fuzzy uniformity compatible with $\delta$, then by theorem 2.3.8, every uniform F-covering in $\Gamma'$ has a refinement consisting of fuzzy open sets. Thus $\Gamma' \subset \Sigma$. Hence $\Gamma' \subset \Gamma$.

Remark 2.4.12

It is not known that whether $\Gamma$ in the above theorem is compatible with $\delta$. In the classical topological case, it is well known that there is a finest uniformity compatible with every uniformizable topology.

Definition : 2.4.13

A space whose fuzzy uniformity is the finest compatible with its fuzzy topology is a fine space.
Theorem: 2.4.14

Every fuzzy continuous function on a fine space into any fuzzy uniform space is fuzzy uniformly continuous.

Proof: Straight forward.

Subspace of a Fuzzy Uniform Space:

Let \((X, \Gamma)\) be a fuzzy uniform space and \(M \subset X\). Define for every \(U \in \Gamma\), \(U_M = \{ f \mid_M \mid f \in U\}\). It can be easily verified that \(\Gamma_M = \{ U_M \mid U \in \Gamma\}\) is a fuzzy uniformity on \(M\) and \(t(\Gamma_M) = (t(\Gamma))_M\).