II. CONTROLLABILITY OF NONLINEAR VOLTERRA INTEGRODIFFERENTIAL SYSTEMS

2.1. INTRODUCTION

Several authors have studied the problem of controllability of linear and nonlinear systems in infinite dimensional spaces. Naito [51] established sufficient conditions for the controllability of nonlinear Volterra equations without any local restrictions on reachable sets in Banach spaces. Constrained controllability of nonlinear systems in Banach spaces has been discussed in [42,60]. Using the implicit function theorem, Chukwu and Lenhart [20] showed that the nonlinear system

\[ x'(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0 \]  

(2.1)

is locally approximate null controllable provided the linear approximation of (2.1) is locally null controllable. A special case of (2.1) in which the nonlinearity occurs as a perturbation was studied by Wong [77] and he considered the nonlinear evolution equations of the form

\[ x'(t) = Ax(t) + Bu(t) + h(x(t)) \]

and deduced the controllability results from the controllability assumptions on the linear part \( x'(t) = Ax(t) + Bu(t) \) with Lipschitz conditions on the function \( h \).

In this chapter, we study the controllability of nonlinear Volterra integrodifferential systems in Banach spaces by suitably adopting the technique of Ntouyas and Tsamatos [56]. The results generalize those of [6,8].

2.2. PRELIMINARIES

Consider the Volterra integrodifferential system of the form

\[ x'(t) = Ax(t) + (Bu)(t) + f(t, x(t), \int_0^t g(t, s, x(s))ds), \quad t \in J = [0, b], \]  

\[ x(0) = x_0, \]  

(2.2)

where the state \( x(\cdot) \) takes values in a Banach space \( X \) with the norm \( \| \cdot \| \) and the control function \( u(\cdot) \) is given in \( L^2(J, U) \), a Banach space of admissible control functions, with \( U \) as a Banach space. Here, \( A \) is the infinitesimal generator.
of a strongly continuous semigroup \(T(t), t \geq 0\) in the Banach space \(X\) and \(g : J \times J \times X \to X, f : J \times X \times X \to X\) are given functions and \(B\) is a bounded linear operator from \(U\) into \(X\).

We need the following fixed point theorem due to Schaefer [68]:

**Schaefer Theorem.** Let \(S\) be a convex subset of a normed linear space \(E\) and \(0 \in S\). Let \(F : S \to S\) be a completely continuous operator and let

\[\zeta(F) = \{x \in S; \ x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.\]

Then either \(\zeta(F)\) is unbounded or \(F\) has a fixed point.

We assume the following hypotheses:

(i) \(A\) is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators \(T(t), t \geq 0\) in \(X\) satisfying \(\|T(t)\| \leq M_1 e^{\omega t}, t \geq 0\) for some \(M_1 \geq 1\) and \(\omega \geq 0\).

(ii) The linear operator \(W : L^2(J, U) \to X\) defined by

\[Wu = \int_0^b T(b - s)Bu(s)ds\]

has an invertible operator \(W^{-1} : X \to L^2(J, U) \setminus \ker W\) and there exist positive constants \(M_2, M_3\) such that \(\|B\| \leq M_2\) and \(\|W^{-1}\| \leq M_3\).

(iii) For each \(t, s \in J\), the function \(g(t, s, \cdot) : X \to X\) is continuous and for each \(x \in X\), the function \(g(\cdot, \cdot, x) : J \times J \to X\) is strongly measurable.

(iv) For each \(t \in J\), the function \(f(t, \cdot, \cdot) : X \times X \to X\) is continuous and for each \(x, y \in X\), the function \(f(\cdot, x, y) : J \to X\) is strongly measurable.

(v) For every positive integer \(k\), there exists \(h_k \in L^1(J)\) such that for almost all \(t \in J\)

\[\sup_{\|x\| \leq k} \left\| f(t, x(t), \int_0^t g(t, s, x(s))ds) \right\| \leq h_k(t).\]

(vi) There exists a continuous function \(q : J \to R\) such that

\[\left\| \int_0^t g(t, s, x(s))ds \right\| \leq q(t)\Omega(\|x\|), \ t \in J, \ x \in X,\]

where \(\Omega : [0, \infty) \to (0, \infty)\) is a continuous nondecreasing function.

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There exists a continuous function \( p : J \to R \) such that
\[
\|f(t, x, y)\| \leq p(t) \Omega(\|x\|) + \|y\|, \quad t \in J, \quad x, y \in X.
\]

\( T(t), \quad t > 0 \) is compact.

\[
\int_0^b \tilde{m}(s)ds < \int_c^\infty \frac{ds}{1 + s + \Omega(s)},
\]

where \( c = M_1(\|x_0\|), \quad \tilde{m}(t) = \max\{\omega, M_1 N, M_1[p(t) + q(t)]\} \) and
\[
N = M_2 M_3[\|x_1\| + M_1 e^{\omega b}\|x_0\| + M_1 \int_0^b e^{\omega(b-s)}[p(s) + q(s)]\Omega(\|x(s)\|)ds].
\]

Then the system (2.2) has a mild solution of the following form
\[
x(t) = T(t)x_0 + \int_0^t T(t-s)\left[(Bu)(s) + f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds\right]ds. \quad (2.3)
\]

**Definition 2.1.** The system (2.2) is said to be controllable on the interval \( J \) if for every \( x_0, x_1 \in X \), there exists a control \( u \in L^2(J, U) \) such that the solution \( x(.) \) of (2.2) satisfies \( x(b) = x_1 \).

### 2.3. CONTROLLABILITY RESULT

**Theorem 2.1.** If the hypotheses (i) - (ix) are satisfied, then the system (2.2) is controllable on \( J \).

**Proof.** Consider the space \( C = C(J, X) \), the Banach space of all continuous functions from \( J \) into \( X \) with sup norm.

Using the hypothesis (ii) for an arbitrary function \( x(.) \), define the control
\[
u(t) = W^{-1}[x_1 - T(b)x_0 - \int_0^b T(b-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds]dt.
\]

Now we shall show that using this control the operator \( F : C \to C \) defined by
\[
(Fx)(t) = T(t)x_0 + \int_0^t T(t-\eta)BW^{-1}[x_1 - T(b)x_0
\]

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- \int_0^b T(b - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds \eta d\eta \\
+ \int_0^t T(t - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds \\

has a fixed point. This fixed point is then a solution of equation (2.3).

Clearly \((Fx)(b) = x_1\) which means that the control \(u\) steers the Volterra integro-differential system from the initial state \(x_0\) to \(x_1\) in time \(b\), provided we can obtain a fixed point of the nonlinear operator \(F\).

In order to study the controllability problem of (2.2), we introduce a parameter \(\lambda \in (0, 1)\) and consider the following system

\[
x'(t) = \lambda Ax(t) + \lambda (Bu)(t) + \lambda f(t, x(t), \int_0^t g(t, s, x(s))ds), \quad t \in J, \quad (2.4)
\]

\[
x(0) = x_0.
\]

First we obtain a priori bounds for the mild solution of the equation (2.4). From

\[
x(t) = \lambda T(t)x_0 + \lambda \int_0^t T(t - \eta)BW^{-1}[x_1 - T(b)x_0
\]

\[
- \int_0^b T(b - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds \eta d\eta
\]

\[
+ \lambda \int_0^t T(t - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds,
\]

we have

\[
\|x(t)\| \leq M_1e^{\omega t}\|x_0\| + \int_0^t \|T(t - \eta)\|M_2M_3\|x_1\| + M_1e^{\omega b}\|x_0\|
\]

\[
+ M_1\int_0^b e^{(b - s)}(p(s)\Omega(\|x(s)\|) + q(s)\Omega(\|x(s)\|))ds\eta
\]

\[
+ M_1e^{\omega t}\int_0^t e^{-\omega s}[p(s)\Omega(\|x(s)\|) + q(s)\Omega(\|x(s)\|))]ds
\]

\[
\leq M_1e^{\omega t}\|x_0\| + \int_0^t \|T(t - s)\|N ds
\]

\[
+ M_1e^{\omega t}\int_0^t e^{-\omega s}[p(s) + q(s)]\Omega(\|x(s)\|)ds
\]

\[
\leq M_1e^{\omega t}\|x_0\| + M_1Ne^{\omega t}\int_0^t e^{-\omega s}ds
\]

\[
+ M_1e^{\omega t}\int_0^t e^{-\omega s}[p(s) + q(s)]\Omega(\|x(s)\|)ds
\]

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\[ e^{-\omega t}\|x(t)\| \leq M_1\|x_0\| + M_1N \int_0^t e^{-\omega s} ds \]
\[ + M_1 \int_0^t e^{-\omega s}(p(s) + q(s))\Omega(\|x(s)\|) ds. \]

Denoting by \( v(t) \) the right-hand side of the above inequality, we have \( v(0) = M_1(\|x_0\|) = c, \|x(t)\| \leq e^{\omega t}v(t) \) and

\[ v'(t) = M_1 N e^{-\omega t} + M_1 e^{-\omega t}[p(t) + q(t)]\Omega(\|x(t)\|) \]
\[ \leq M_1 N e^{-\omega t} + M_1 e^{-\omega t}[p(t) + q(t)]\Omega(e^{\omega t}v(t)) \]

or

\[ e^{\omega t}v'(t) \leq M_1 N + M_1[p(t) + q(t)]\Omega(e^{\omega t}v(t)). \]

We remark that

\[ (e^{\omega t}v(t))' = \omega e^{\omega t}v(t) + e^{\omega t}v'(t) \]
\[ \leq \omega e^{\omega t}v(t) + M_1 N + M_1[p(t) + q(t)]\Omega(e^{\omega t}v(t)) \]
\[ \leq \hat{m}(t)[e^{\omega t}v(t) + 1 + \Omega(e^{\omega t}v(t))]. \]

This implies

\[ \int_0^{e^{\omega t}v(t)} \frac{ds}{1 + s + \Omega(s)} \leq \int_0^b \hat{m}(s) ds < \int_0^\infty \frac{ds}{1 + s + \Omega(s)}, \quad t \in J. \]

This inequality implies that there is a constant \( K \) such that \( v(t) \leq K, \ t \in J, \) and hence \( \|x(t)\| \leq K, t \in J, \) where \( K \) depends only on \( b \) and on the functions \( \hat{m} \) and \( \Omega. \)

Next we must prove that the operator \( F \) is a completely continuous operator.

Let \( B_k = \{ x \in C : \|x\| \leq k \} \) for some \( k \geq 1. \)

We first show that \( F \) maps \( B_k \) into an equicontinuous family. Let \( x \in B_k \) and \( t_1, t_2 \in J \) and \( \epsilon > 0. \) Then if \( 0 < \epsilon < t_1 < t_2 \leq b \)

\[ \|(Fx)(t_1) - (Fx)(t_2)\| \]
\[ \leq ||T(t_1) - T(t_2)||\|x_0\| \]
\[ + \int_0^{t_1-\epsilon} [T(t_1 - \eta) - T(t_2 - \eta)]BW^{-1}[x_1 - T(b)x_0 \]
\[ - \int_0^b T(b - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds][\eta]d\eta \| \]

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The right-hand side tends to zero as $t_2 - t_1 \to 0$ and $\epsilon$ is sufficiently small, since the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology.

Thus $F$ maps $B_k$ into an equicontinuous family of functions. It is easy to see that the family $FB_k$ is uniformly bounded.
Next we show $FB_k$ is compact. Since we have shown $FB_k$ is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that $F$ maps $B_k$ into a precompact set in $X$.

Let $0 < t \leq b$ be fixed and $\epsilon$ a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$
(F\epsilon x)(t) = T(t)x_0 + \int_0^{t-\epsilon} [T(t-\eta)BW^{-1}|x_1 - T(b)x_0
$$

$$
- \int_0^b T(b-s)f(s,x(s), \int_0^s g(s,\tau, x(\tau))d\tau)ds](\eta)\eta
$$

$$
+ \int_0^{t-\epsilon} T(t-s)f(s,x(s), \int_0^s g(s,\tau, x(\tau))d\tau)ds
$$

$$
= T(t)x_0 + T(\epsilon) \int_0^{t-\epsilon} [T(t-\eta-\epsilon)BW^{-1}|x_1 - T(b)x_0
$$

$$
- \int_0^b T(b-s)f(s,x(s), \int_0^s g(s,\tau, x(\tau))d\tau)ds](\eta)\eta
$$

$$
+ T(\epsilon) \int_0^{t-\epsilon} T(t-s-\epsilon)f(s,x(s), \int_0^s g(s,\tau, x(\tau))d\tau)ds.
$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(F\epsilon x)(t) : x \in B_k\}$ is precompact in $X$ for every $\epsilon$, $0 < \epsilon < t$. Moreover for every $x \in B_k$, we have

$$
\| (Fx)(t) - (F\epsilon x)(t) \|
$$

$$
\leq \int_{t-\epsilon}^t \| T(t-\eta)BW^{-1}|x_1 - T(b)x_0
$$

$$
- \int_0^b T(b-s)f(s,x(s), \int_0^s g(s,\tau, x(\tau))d\tau)ds](\eta)\eta
$$

$$
+ \int_{t-\epsilon}^t \| T(t-s)f(s,x(s), \int_0^s g(s,\tau, x(\tau))d\tau)\|ds
$$

$$
\leq \int_{t-\epsilon}^t \| T(t-\eta)\| M_2M_3 [\|x_1\| + M_1 e^{\omega b}\|x_0\|
$$

$$
+ M_1 \int_0^b e^{\omega(b-s)}h_k(s)ds]\eta
$$

$$
+ \int_{t-\epsilon}^t \| T(t-s)\| h_k(s)ds.
$$

Therefore there are precompact sets arbitrarily close to the set $\{(F\epsilon x)(t) : x \in B_k\}$. Hence the set $\{(Fx)(t) : x \in B_k\}$ is precompact in $X$.

It remains to be shown that $F : C \to C$ is continuous. Let $\{x_n\}_0^\infty \subseteq C$ with $x_n \to x$ in $C$. Then there is an integer $r$ such that $\|x_n(t)\| \leq r$ for all $n$ and
t ∈ J, so \( x_n \in B_r \) and \( x \in B_r \).

By (iv), \( f(t, x_n(t), \int_0^t g(t, s, x_n(s))ds) \to f(t, x(t), \int_0^t g(t, s, x(s))ds) \)
for each \( t \in J \) and since

\[
\left\| f(t, x_n(t), \int_0^t g(t, s, x_n(s))ds) - f(t, x(t), \int_0^t g(t, s, x(s))ds) \right\| \leq 2h_r(t),
\]

we have by dominated convergence theorem,

\[
\| Fx_n - Fx \| = \sup_{t \in J} \left\| \int_0^t \left( T(t - \eta)BW^{-1}\left[ f(s, x_n(s), \int_0^s g(s, \tau, x_n(\tau))d\tau) - f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)\right]\right) d\eta \right\|
\]

\[
\leq \int_0^b \| T(t - \eta)\| M_1 \int_0^b e^{\omega(b-s)} \| f(s, x_n(s), \int_0^s g(s, \tau, x_n(\tau))d\tau) - f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau) \| d\eta
\]

\[
+ \int_0^b \| T(t - s)\| \| f(s, x_n(s), \int_0^s g(s, \tau, x_n(\tau))d\tau) - f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau) \| ds \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus \( F \) is continuous. This completes the proof that \( F \) is completely continuous.

Finally the set \( \zeta(F) = \{ x \in C : x = \lambda Fx, \lambda \in (0,1) \} \) is bounded, as we
proved in the first step. Consequently by Schaefer’s theorem, the operator \( F \)
has a fixed point in \( C \). This means that any fixed point of \( F \) is a mild solution
of (2.2) on \( J \) satisfying \((Fx)(t) = x(t)\). Thus the system (2.2) is controllable on \( J \).

### 2.4. EXAMPLE

Consider the partial integrodifferential system of the form

\[
\begin{align*}
    z_t(t, y) &= z_{yy}(t, y) + Bu(t) + \mu_2(t, z(t, y), \int_0^t \mu_1(t, s, z(s, y))ds), \\
    z(t, 0) &= z(t, 1) = 0, \quad t \in J, \\
    z(0, y) &= z_0(y), \quad 0 < y < 1,
\end{align*}
\]
where $B : U \to X$, with $U \subset J$ and $X = L^2([0,1], R)$, is a linear operator such that there exists an invertible operator $W^{-1}$ takes values in $L^2(J, U) \setminus \ker W$, where $W$ is defined by

$$Wu = \int_0^b T(b-s)Bu(s)ds,$$

and $T(t)$ is a strongly continuous semigroup. Further the functions

$$\mu_1 : J \times J \times X \to X,$$

$$\mu_2 : J \times X \times X \to X$$

are all continuous, bounded and strongly measurable such that

$$\|\mu_2(t, z(t, y), \int_0^t \mu_1(t, s, z(s, y))ds\| \leq I(t)\Omega(\|z\|),$$

where $l : J \to R$ is continuous and $\Omega : [0, \infty) \to (0, \infty)$ is continuous and non-decreasing.

Let $g(t, s, z)(y) = \mu_1(t, s, z(t, y)), f(t, z, \sigma)(y) = \mu_2(t, z(t, y), \sigma(t, y))$ and $A : X \to X$ be defined by $Az = \frac{\partial^2 z}{\partial y^2}$, with domain $D(A) = \{ z \in X, z, z_y$ are absolutely continuous, $z_{yy} \in X, z(0) = z(1) = 0 \}$.

Then

$$Az = \sum_{n=1}^{\infty} n^2(z, z_n)z_n, \ z \in D(A),$$

where $z_n(s) = \sqrt{2}\sin ns, \ n = 1, 2, 3, \ldots$, is the orthogonal set of eigenvectors of $A$. It is well known that $A$ is the infinitesimal generator of an analytic semigroup $T(t), \ t \geq 0, \ in \ X$ and is given by $[76]$

$$T(t)z = \sum_{n=1}^{\infty} \exp(-n^2 t)(z, z_n)z_n, \ z \in X,$$

where $T(t)$ satisfies the hypothesis (i) and since the analytic semigroup $T(t)$ is compact, it also satisfies (viii). Further all the conditions of Theorem 2.1. are satisfied. Hence the system (2.5) is controllable on $J$.

**Remark.** Examples in which the operator $W$ in hypothesis (ii) has an invertible operator are discussed by Quinn and Carmichael [64].