I. INTRODUCTION

1.1. INTEGRODIFFERENTIAL SYSTEMS

The study of integrodifferential equations has emerged, in recent years, as an independent branch of modern research because of its applications to many fields such as continuum mechanics, population dynamics, ecology, nuclear reactor dynamics and so on. A large class of scientific and engineering problems is modelled by partial differential equations, integral equations or coupled ordinary and partial differential equations which can be described as differential equations in infinite dimensional spaces using semigroups. In general, integrodifferential equations or nonlinear evolution equations, with or without delays, serve as an abstract formulation of many partial integrodifferential equations which arise in problems connected with heat-flow in materials with memory, viscoelasticity and many other physical phenomena. So it becomes important to study the controllability problem of such systems in infinite dimensional spaces. Since the object of the thesis is to study the controllability of abstract nonlinear integrodifferential systems which can be represented by nonlinear integrodifferential equations in various forms, we shall motivate our study briefly by giving the occurrence of these systems in different fields of study.

1.2. MOTIVATION

(A) Biology

In the study of enzyme membrane [41], the following partial differential equation arises

\[
\frac{\partial x(\tau, t)}{\partial t} = \frac{\partial^2 x(\tau, t)}{\partial \tau^2} + f(x(\tau, t)), \quad t > 0, \quad 0 < \tau < 1,
\]

\[
x(0, t) = \beta, \quad x(1, t) = \gamma, \quad t > 0,
\]

\[
x(\tau, 0) = 0, \quad 0 < \tau < 1,
\]

\[
f(x) = \frac{\sigma x}{1 + |x|},
\]

where \(\sigma, \beta, \gamma\) are positive parameters and \(x(\tau, t)\) denotes the concentration of substrate at \((\tau, t)\) in a one-dimensional membrane.
The abstract formulation of the above partial differential equation is

\[ x'(t) + Ax(t) = f(x(t)), \]
\[ x(0) = x_0, \]

where \(-A\) is the infinitesimal generator of an analytic semigroup in a Banach space \(X\) with \(x_0 \in X\) and \(f\) is a uniformly bounded nonlinear term.

(B) Viscoelasticity

Consider the partial integrodifferential equation of the form

\[ u(t, x) = u_{xx}(t, x) + \int_0^t \sigma(t, s, u(s, x), \int_0^s \sigma_1(s, \tau, u(\tau, x))d\tau)ds \]
\[ + \sigma_2(t, u(t, x)), \quad 0 < x < 1, \quad t > 0 \]

with the initial and boundary conditions

\[ u(0, t) = u(1, 0), \]
\[ u(x, 0) = u_0(x), \quad 0 < x < 1, \]

where \(\sigma, \sigma_1\) and \(\sigma_2\) are continuous, continuously differentiable with respect to the first argument, uniformly Lipschitz continuous in \(s, \tau\) and \(t\) respectively.

The above partial integrodifferential equation arises in the study of viscoelasticity and many other physical phenomena \([28,29,32,36,76]\) and it can be written in the abstract form as

\[ x'(t) + Ax(t) = f(t, x(t)) + \int_0^t g(t, s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau)ds, \quad t > 0, \]
\[ x(0) = x_0, \]

where \(-A\) is the infinitesimal generator of a \(C_0\) semigroup on a Banach space \(X\) and the nonlinear functions \(k, g\) and \(f\) satisfy appropriate conditions.
(C) Heat Equation

(i) Consider the classical heat equation [46] for materials with memory of the form

\[ q(t, x) = -Eu_t(t, x) - \int_0^t b(t - s)u_x(s, x)ds, \]
\[ u_t(t, x) = -\frac{\partial q(t, x)}{\partial x} + f(t, u), \]
\[ u(0, x) = u_0(x). \]

The first equation gives the heat flux and the second is the balance equation. The above equation can be written as (assuming \( E = 1 \))

\[ u_t(t, x) = \frac{\partial^2}{\partial x^2}[u(t, x) + \int_0^t b(t - s)u(s, x)ds] + f(t, u), \]
\[ u(0, x) = u_0(x), \]

where \( f \) and \( b \) are continuous functions.

The abstract formulation of the above equation is a semilinear integrodifferential equation

\[ x'(t) = A[x(t) + \int_0^t F(t - s)x(s)ds] + f(t, x(t)), \quad 0 \leq t \leq T, \]
\[ x(0) = x_0, \]

where \( A \) is the generator of a strongly continuous semigroup in a Banach space and \( F(t) \) is a bounded linear operator. For example \( A = \frac{\partial^2}{\partial x^2} \) on \( H^2(0,1) \cap H^1_0(0,1) \) generates a strongly continuous semigroup on \( L^2(0,1) \).

(ii) A very special model for one dimensional heat flow in materials with memory [54] is the partial functional delay integrodifferential equation of the form

\[ w_t(x, t) = w_{xx}(x, t) + \int_0^t f(s, w(x, s - r))ds, \quad 0 < x < 1, \quad t > 0, \]
\[ w(0, t) = w(1, t) = 0, \quad t > 0, \]
\[ w(x, t) = \phi(x, t), \quad -r \leq t \leq 0, \]

3
and it can be written in abstract form as

\[
x'(t) = Ax(t) + \int_0^t f(s, x_s)ds, \quad 0 \leq t \leq b,
\]

\[
x(t) = \phi(t), \quad -r \leq t \leq 0,
\]

where \( A \) is the infinitesimal generator of a strongly continuous semigroup in a Banach space \( X \), \( f \) is a continuous function and \( \phi \) is an initial function.

(iii) Consider the partial integrodifferential equation of the form

\[
w_t(x, t) = w_{xx}(x, t) + \int_0^t k(t - s, w(x, s))ds + h(x, t), \quad 0 < x < 1, \quad t > 0,
\]

\[
w(0, t) = w(1, t) = 0, \quad t > 0,
\]

\[
w(x, 0) = w_0(x), \quad 0 < x < 1,
\]

where \( k \) and \( h \) are continuous functions.

The above equation has many physical applications and arises in such problems as heat flow in materials with memory [16,17,47] and it can be written as

\[
u'(t) = Au(t) + \int_0^t g(t - s, u(s))ds + f(t), \quad t \geq 0,
\]

\[
u(0) = u_0 \in X,
\]

where \( A \) is the infinitesimal generator of a strongly continuous semigroup in a Banach space \( X \), \( g \) is a nonlinear unbounded operator.

(D) Physics

The following partial functional integrodifferential equation arises in almost all phases of physics and other areas of applied mathematics [23,27]

\[
\frac{\partial z(x, t)}{\partial t} - (a(x, t)z_x(x, t))_x = G(t, z(x, t - r), \int_0^t H(t, s, z(x, s - r))ds),
\]

\[
0 \leq x \leq 1, \quad t \in [0, b],
\]

with initial and boundary conditions

\[
z(0, t) = z(1, t) = 0, \quad t \in [0, b],
\]

\[
z(x, t) = \phi(x, t), \quad 0 \leq x \leq 1, \quad -r \leq t \leq 0,
\]
where \( a(x, t) \) is a positive continuous function, \( H \) and \( G \) are continuous functions and \( \phi \) is a given initial function and it can be written as

\[
x'(t) + A(t)x(t) = f(t, x_t, \int_0^t g(t, s, x_s)ds), \quad t \in [0, b],
\]

\[
x(t) = \phi(t), \quad -r \leq t \leq 0,
\]

where \(-A(t)\) is the infinitesimal generator of an analytic semigroup of linear operators, \( f \) and \( g \) are continuous functions.

(E) Thermodynamics

(i) Consider the initial boundary value problem for the pseudoparabolic partial differential equation of the form

\[
\frac{\partial P u(x, t)}{\partial t} + Qu(x, t) = f(x, t, (D^\alpha u(x, t))_{|\alpha| \leq 2m-1}), \quad \text{in } \Omega \times (0, b),
\]

\[
B_i u(x, t) = 0 \quad \text{in } \partial \Omega \times (0, b) \quad \text{for } 1 \leq i \leq m,
\]

\[
u(x, 0) = u_0(x) \quad \text{in } \Omega,
\]

where \( P \) and \( Q \) are linear elliptic differential operators of orders \( 2m \) and \( 2l \) respectively with \( m \geq l \). Here \( \Omega \) be a nonempty bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \) and \( f \) is a continuous function. To know more about this equation, one can refer the paper [12].

The above equation occurs in thermodynamics [18], in the flow of fluid through fissured rocks [11], in the shear in second order fluids [71] and in soil mechanics. The abstract representation is

\[
(Bu(t))' + Au(t) = g(u(t), t), \quad t \in [0, b],
\]

\[
u(0) = u_0,
\]

where \( A, B \) are closed linear operators with domains contained in a Banach space \( X \) and ranges contained in a Banach space \( Y \) and \( g \) is a continuous function.

(ii) Consider the partial functional differential equation of the form [45]

\[
\frac{\partial}{\partial t} (w(x, t) - w_{xx}(x, t)) - w_{xx}(x, t) = f(t, w(x, t - r)), \quad t \in [0, b],
\]

\[
w(0, t) = w(1, t) = 0,
\]

\[
w(x, t) = \phi(x, t), \quad -r \leq t \leq 0, \quad x \in [0, 1],
\]
where \( f : [0, b] \times R \to R \) is continuous and \( \phi \) is a given initial function.

The abstract formulation of the above partial differential equation is a Sobolev-type functional differential equation

\[
(Bu(t))' + Au(t) = g(t, u_t), \quad t > 0, \\
u(t) = \phi(t), \quad -r \leq t \leq 0.
\]

(F) Population Dynamics

Consider the following boundary value problem of the form

\[
\begin{align*}
\frac{\partial}{\partial t} \left[ u(t, \xi) + \int_{-\infty}^{t} \int_{0}^{\pi} b(s - t, \eta, \xi) u(s, \eta) d\eta ds \right] \\
= \frac{\partial^2}{\partial \xi^2} u(t, \xi) + a_0(\xi) u(t, \xi) + \int_{-\infty}^{t} a(s - t) u(s, \xi) ds + a_1(t, \xi),
\end{align*}
\]

\( t \geq 0, \quad 0 \leq \xi \leq \pi, \)

\( u(t, 0) = u(t, \pi) = 0, \quad t \geq 0, \)

\( u(\theta, \xi) = \phi(\theta, \xi), \quad \theta \leq 0, \quad 0 \leq \xi \leq \pi, \)

where the functions \( a_0, a_1, b \) and \( \phi \) satisfy appropriate conditions in order to guarantee existence of solutions as well as additional properties.

The above problem arises from control systems described by an abstract retarded functional differential equations with a feedback control governed by a proportional integrodifferential law and it occurs in population dynamics for spatially distributed populations [37] and it can be written as

\[
\frac{d}{dt} [x(t) + F(t, x_t)] = A x(t) + G(t, x_t), \quad t \geq \sigma, \\
x_{\sigma} = \phi,
\]

where the initial function \( \phi, F \) and \( G \) satisfy appropriate conditions. For more examples of neutral differential systems, one can see [3,58].

Motivation for various types of models represented by different kinds of integrodifferential equations can be found in [13,21,33-35,57].
1.3. CONTROLLABILITY PROBLEM AND METHODS

Controllability is one of the most important properties of dynamical systems. The problem of controllability is to show the existence of a control function which steers the solution of the system from its initial state to a final state, where the initial and final states may vary over the entire space. There are various approaches to the study of controllability of nonlinear systems which may be classified as follows:

i) methods based on the stability theory of Lyapunov

ii) methods for systems defined on a manifold

iii) methods which are geometrical in nature

iv) fixed point methods.

Of all the above methods, the fixed point method can be effectively used to study the controllability of nonlinear systems. In this method, the controllability problem is transformed to a fixed point problem for an appropriate nonlinear operator in a function space. An essential part of this approach is to guarantee the existence of an invariant subset for this operator. The fixed point methods for studying the controllability problem has been extended recently to general Banach spaces by using the theory of semigroups.

The theory of semigroups of bounded linear operators has applications in many branches of analysis and in particular to the solution of initial value problems for partial differential equations [31,59]. It is closely related to the solution of ordinary differential equations in Banach spaces. In recent years, the theory of semigroups of bounded linear operators has been extensively applied to study the existence problems in differential equations [14,15,39,40,55,65,66] and the controllability problems in control theory [1,5,7,8,19,48,72,80].

1.4. CONTROLLABILITY OF NONLINEAR INTEGRODIFFERENTIAL SYSTEMS

Controllability of nonlinear systems represented by ordinary differential equations in finite dimensional spaces has been extensively studied. Several authors [7,19,38,61-63,73,74] have extended the concept to infinite dimensional systems represented by abstract equations in Banach spaces. Controllability of linear

Brill [12] and Showalter [69,70] established the existence of solutions of semilinear evolution equations of Sobolev-type in Banach spaces. This type of equations arises in various applications such as in the flow of fluids through fissured rocks, thermodynamics and shear in second order fluids. So it is interesting to study the controllability problem for such systems in infinite dimensional spaces. In this direction, Han [38] and Balachandran and Dauer [5] studied the controllability of Sobolev-type systems in Banach spaces. In the present work, we mainly apply fixed point methods to study the controllability of various classes of nonlinear integrodifferential systems including systems of Sobolev-type in Banach spaces. The basic tools used here are the semigroup theory, Schauder's fixed point theorem and the Schaefer fixed point theorem.

1.5. CONTRIBUTIONS OF THE AUTHOR

In the light of the above, the author has obtained some significant results on the following topics.

1. Controllability of nonlinear Volterra integrodifferential systems.

2. Controllability of nonlinear integrodifferential systems.

3. Controllability of nonlinear delay integrodifferential systems.
4. Controllability of nonlinear Sobolev-type integrodifferential systems.

5. Controllability of nonlinear Sobolev-type delay integrodifferential systems.

6. Controllability of nonlinear neutral integrodifferential systems.

The rest of the thesis contains a detailed account of the above topics.

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