VII. CONTROLLABILITY OF NONLINEAR NEUTRAL INTEGRODIFFERENTIAL SYSTEMS

7.1. INTRODUCTION

Controllability of nonlinear neutral systems in finite dimensional spaces has been studied by Gahl [30] and Underwood and Chukwu [75]. Balachandran [2] discussed the controllability of neutral Volterra integrodifferential systems and infinite delay neutral Volterra systems in finite dimensional spaces. Balachandran and Balasubramaniam [3] established controllability of a larger class of nonlinear neutral Volterra integrodifferential systems and Balachandran and Dauer [4] investigated the relative controllability of nonlinear neutral Volterra integrodifferential systems in finite dimensional spaces. O'Connor [58] extensively studied the controllability of linear neutral systems in abstract spaces. However no work has been reported regarding the controllability of nonlinear neutral integrodifferential systems in Banach spaces. In this chapter we derive a new set of sufficient conditions for the controllability of nonlinear neutral integrodifferential systems in Banach spaces by using the Schaefer fixed point theorem.

7.2. PRELIMINARIES

Consider the nonlinear neutral integrodifferential system of the form

$$\frac{d}{dt} [x(t) - g(t, x_t)] = Ax(t) + Bu(t) + \int_0^t f(s, x_s) ds, \quad t \in J = [0, b], \quad (7.1)$$

where the state $x(\cdot)$ takes values in a reflexive Banach space $X$ with the norm $| \cdot |$ and the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with $U$ as a Banach space. Here, $A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in $X$, $B$ is a bounded linear operator from $U$ into $X$, $f : J \times C \to X$ and $g : J \times C \to X$ are continuous functions, where $C = C([-r, 0], X)$ is the Banach space of all continuous functions $\phi : [-r, 0] \to X$ endowed with the norm $\| \phi \| = \sup \{|\phi(\theta)| : -r \leq \theta \leq 0\}$. Also for $x \in C([-r, b], X)$ we have $x_t \in C$ for $t \in [0, b]$, $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.

We assume the following hypotheses:
(i) $A$ is the infinitesimal generator of a uniformly continuous compact semigroup of bounded linear operators $T(t)$ in $X$ such that

$$|T(t)| \leq M_1,$$

for some $M_1 \geq 1$ and $|AT(t)| \leq M$, where $M > 0$.

(ii) The linear operator $W : L^2(J,U) \rightarrow X$ defined by

$$Wu = \int_0^b T(b - s)Bu(s)ds$$

has an invertible operator $W^{-1} : X \rightarrow L^2(J,U) \setminus \ker W$ and there exist positive constants $M_2, M_3$ such that $|B| \leq M_2$ and $|W^{-1}| \leq M_3$.

(iii) For each $t \in J$, the function $f(t,\cdot) : C \rightarrow X$ is continuous and for each $x \in C$, the function $f(\cdot,x) : J \rightarrow X$ is strongly measurable.

(iv) For every positive integer $k$, there exists $\alpha_k \in L^1(J)$ such that

$$\sup_{\|x\| \leq k} |f(t,x)| \leq \alpha_k(t) \text{ for } t \in J \text{ a.e.}$$

(v) The function $g$ is completely continuous and such that the operator

$$G : C([-r,0],X) \rightarrow C([0,b],X)$$

defined by $(G\phi)(t) = g(t,\phi)$ is compact.

(vi) There exist constants $c_1 < 1$ and $c_2 > 0$ such that

$$|g(t,\phi)| \leq c_1\|\phi\| + c_2, \quad t \in J, \quad \phi \in C.$$

(vii) There exists an integrable function $m : [0,b] \rightarrow [0,\infty)$ such that

$$|f(t,\phi)| \leq m(t)\Omega(\|\phi\|), \quad 0 \leq t \leq b, \quad \phi \in C,$

where $\Omega : [0,\infty) \rightarrow (0,\infty)$ is a continuous nondecreasing function.

(viii)

$$\int_0^b \dot{m}(s)ds \leq \int_c^\infty \frac{ds}{s + \Omega(s)},$$

where $c = \frac{1}{1 - c_1}[M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 + M_2c_2b + M_1Nb]$,

$$\dot{m}(t) = \max\{\frac{1}{1 - c_1}Mc_1, \frac{M_1}{Mc_1}m(t)\} \quad \text{and}$$

$$N = M_2M_3|x_1| + M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_1\|x_2\| + c_2$$

$$+ M \int_0^b (c_1\|x_2\| + c_2)ds + M_1 \int_0^t \int_0^s m(\tau)\Omega(\|x_\tau\|)d\tau ds].$$

73
Then the system (7.1) has a mild solution of the following form [37]

\[ x(t) = T(t)[\phi(0) - g(0, \phi)] + g(t, x_t) + \int_0^t AT(t - s)g(s, x_s)ds + \int_0^t T(t - s)(Bu)(s) + \int_0^s f(\tau, x_\tau)d\tau]ds, \quad t \in J, \]

\[ x_0 = \phi. \]

**Definition 7.1.** The system (7.1) is said to be controllable on the interval \( J \) if for every continuous initial function \( \phi \in C \), there exists a control \( u \in L^2(J, U) \) such that the solution \( x(t) \) of (7.1) satisfies \( x(b) = x_1 \).

### 7.3. CONTROLLABILITY RESULT

**Theorem 7.1.** If the hypotheses (i) - (viii) are satisfied, then the system (7.1) is controllable on \( J \).

**Proof.** Using the hypothesis (ii) for an arbitrary function \( x(\cdot) \), define the control

\[ u(t) = W^{-1}[x_1 - T(b)[\phi(0) - g(0, \phi)] - g(b, x_b) - \int_0^b AT(b - s)g(s, x_s)ds - \int_0^b T(b - s)\int_0^s f(\tau, x_\tau)d\tau ds](t). \]

For \( \phi \in C \), define \( \hat{\phi} \in C_b, \ C_b = C([-r, b], X) \) by

\[ \hat{\phi}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0, \\ T(t)\phi(0), & 0 \leq t \leq b \end{cases} \]

If \( x(t) = y(t) + \hat{\phi}(t), \ t \in [-r, b] \), it is easy to see that \( y \) satisfies

\[ y_0 = 0 \]

\[ y(t) = -T(t)g(0, \phi) + g(t, y_t + \hat{\phi}_t) + \int_0^t AT(t - s)g(s, y_s + \hat{\phi}_s)ds + \int_0^t T(t - \eta)(Bu)[x_1 - T(b)[\phi(0) - g(0, \phi)] - g(b, y_b + \hat{\phi}_b) - \int_0^b AT(b - s)g(s, y_s + \hat{\phi}_s)ds - \int_0^b T(b - s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)d\tau ds](\eta)d\eta 
+ \int_0^t T(t - s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)d\tau ds \quad (7.2) \]

74
if and only if $x$ satisfies

$$x(t) = T(t)[\phi(0) - g(0, \phi)] + g(t, x_t) + \int_0^t A T(t-s) g(s, x_s) ds$$

$$+ \int_0^t T(t-\eta) BW^{-1}[x_1 - T(b)[\phi(0) - g(0, \phi)] - g(b, x_b)] ds$$

$$- \int_0^b AT(b-s) g(s, x_s) ds$$

$$- \int_0^b T(b-s) \int_0^s f(\tau, x_\tau) d\tau ds] d\eta$$

$$+ \int_0^t T(t-s) \int_0^s f(\tau, x_\tau) d\tau ds$$

and $x_0 = \phi$.

Define $C^0 = \{ y \in C_b : y_0 = 0 \}$ and $F : C^0 \rightarrow C^0$, by

$$(Fy)(t) = 0, \quad -r \leq t \leq 0,$$

$$(Fy)(t) = -T(t) g(0, \phi) + g(t, y_t + \hat{\phi}_t) + \int_0^t A T(t-s) g(s, y_s + \hat{\phi}_s) ds$$

$$+ \int_0^t T(t-\eta) BW^{-1}[x_1 - T(b)[\phi(0) - g(0, \phi)] - g(b, y_b + \hat{\phi}_b)] ds$$

$$- \int_0^b AT(b-s) g(s, y_s + \hat{\phi}_s) ds$$

$$- \int_0^b T(b-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds] d\eta$$

$$+ \int_0^t T(t-s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau) d\tau ds, \quad 0 \leq t \leq b$$

has a fixed point. This fixed point is then a solution of equation (7.2).

Clearly $x(b) = x_1$ which means that the control $u$ steers the neutral integrodifferential system from the initial function $\phi$ to $x_1$ in time $b$, provided we can obtain a fixed point of the nonlinear operator $F$.

In order to study the controllability problem of (7.1) we introduce a parameter $\lambda \in (0,1)$ and consider the following system

$$\frac{d}{dt} [x(t) - \lambda g(t, x_t)] = \lambda Ax(t) + \lambda Bu(t) + \lambda \int_0^t f(s, x_s) ds, \quad t \in J = [0, b], (7.3)$$

$$x_0 = \phi.$$

First we obtain a priori bounds for the mild solution of the equation (7.3).

From

$$x(t) = \lambda T(t)[\phi(0) - g(0, \phi)] + \lambda g(t, x_t) + \lambda \int_0^t A T(t-s) g(s, x_s) ds$$

75
we have,

\[ |x(t)| \leq M_1[||\phi|| + c_1||\phi|| + c_2] + c_1||x_i|| + c_2 + M \int_0^t (c_1||x_s|| + c_2)ds +
M_1 \int_0^t M_2 M_3[|x_1| + M_1(||\phi|| + c_1||\phi|| + c_2) + c_1||x_0|| + c_2
+ M \int_0^t (c_1||x_s|| + c_2)ds + M_1 \int_0^t \int_0^s m(\tau)\Omega(||x_\tau||)d\tau ds]d\eta
+ M_1 \int_0^t \int_0^s m(\tau)\Omega(||x_\tau||)d\tau ds.

We consider the function \( \mu \) given by

\[ \mu(t) = \sup\{|x(s)| : -r < s < t\}, \quad 0 < t < b. \]

Let \( t^* \in [-r, t] \) be such that \( \mu(t) = |x(t^*)| \). If \( t^* \in [0, b] \) by the previous inequality we have

\[ \mu(t) \leq M_1[||\phi|| + c_1||\phi|| + c_2] + c_1\mu(t) + c_2
+ M c_1 \int_0^t \mu(s)ds + M c_2 b + M_1 Nb
+ M_1 \int_0^t \int_0^s m(\tau)\Omega(\mu(\tau))d\tau ds
\leq M_1[||\phi|| + c_1||\phi|| + c_2] + c_1\mu(t) + c_2
+ M c_1 \int_0^t \mu(s)ds + M c_2 b + M_1 Nb
+ M_1 \int_0^t \int_0^s m(\tau)\Omega(\mu(\tau))d\tau ds
\]

or

\[ \mu(t) \leq \frac{1}{1 - c_1} \left\{ M_1[||\phi|| + c_1||\phi|| + c_2] + c_2 + M c_2 b + M_1 Nb
+ M c_1 \int_0^t \mu(s)ds + M_1 \int_0^t \int_0^s m(\tau)\Omega(\mu(\tau))d\tau ds \right\}. \]
If \( t^* \in [-r, 0] \) then \( \mu(t) = \|\phi\| \) and the previous inequality holds since \( M_1 \geq 1 \).

Denoting by \( v(t) \) the right-hand side of the above inequality, we have
\[
c = v(0) = \frac{1}{1 - c_1} \{ M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_2 + + M_2 b + M_1 N b \}, \quad \mu(t) \leq v(t),
\]
\( 0 \leq t \leq b \) and
\[
v'(t) = \frac{1}{1 - c_1} M_1 \mu(t) + \frac{1}{1 - c_1} M_1 \int_0^t m(\tau) \Omega(\mu(\tau)) d\tau
\]
\[
\leq \frac{1}{1 - c_1} M_1 v(t) + \frac{1}{1 - c_1} M_1 \int_0^t m(\tau) \Omega(v(\tau)) d\tau
\]
\[
\leq \frac{1}{1 - c_1} M_1 \left\{ v(t) + \frac{M_1}{M_c} \int_0^t m(\tau) \Omega(v(\tau)) d\tau \right\}.
\]

Let \( w(t) = v(t) + \frac{M_1}{M_c} \int_0^t m(\tau) \Omega(v(\tau)) d\tau \).
Then \( w(0) = v(0) \), \( v(t) \leq w(t) \), and
\[
w'(t) = v'(t) + \frac{M_1}{M_c} m(t) \Omega(v(t))
\]
\[
\leq \frac{1}{1 - c_1} M_1 w(t) + \frac{M_1}{M_c} m(t) \Omega(w(t))
\]
\[
\leq \dot{m}(t) \{ w(t) + \Omega(w(t)) \}.
\]

This implies
\[
\int_{w(0)}^{w(t)} \frac{ds}{s + \Omega(s)} \leq \int_0^b \dot{m}(s) ds < \int_c^\infty \frac{ds}{s + \Omega(s)} \quad 0 \leq t \leq b.
\]

This inequality implies that there is a constant \( K \) such that \( v(t) \leq K, t \in [0, b] \)
and hence \( \mu(t) \leq K, \quad t \in [0, b], \quad \|x_t\| \leq \mu(t) \), we have
\[
\|x\|_1 = \sup\{|x(t)| : -r \leq t \leq b\} \leq K,
\]
where \( K \) depends only on \( b \) and on the functions \( m \) and \( \Omega \).

Let \( B_k = \{ y \in C_b^0 : \|y\|_1 \leq k \} \) for some \( k \geq 1 \). It will now be shown that \( F \)
is a completely continuous operator.

We first show that the set \( \{ F y : y \in B_k \} \) is equicontinuous. Let \( y \in B_k \)
and \( t_1, t_2 \in [0, b] \). Then if \( 0 < t_1 < t_2 \leq b \),
\[
\|(F y)(t_1) - (F y)(t_2)\|.
\]

77
\[ \leq |T(t_1) - T(t_2)||g(0, \phi)| + |g(t_1, y_1 + \hat{\phi}_1) - g(t_2, y_2 + \hat{\phi}_2)| \\
+ \int_{t_1}^{t_2} |A[T(t_1 - s) - T(t_2 - s)]| |(c_1 ||y_s + \hat{\phi}_s|| + c_2)ds \\
+ \int_{t_1}^{t_2} |AT(t_2 - s)|(c_1 ||y_s + \hat{\phi}_s|| + c_2)ds \\
+ \int_{t_1}^{t_1} |T(t_1 - \eta) - T(t_2 - \eta)| M_2 M_3 |x_1| + M_1 \{|\phi(0) - g(0, \phi)|\} \\
+ c_1 ||y_0 + \hat{\phi}_0|| + c_2 + M \int_0^{b} (c_1 ||y_s + \hat{\phi}_s|| + c_2)ds \\
+ M_1 \int_0^{b} \int_{0}^{\tau} \alpha_{\kappa}(\tau) d\tau ds |\eta \\
+ \int_{t_1}^{t_2} |T(t_2 - \eta)| M_2 M_3 |x_1| + M_1 \{|\phi(0) - g(0, \phi)|\} \\
+ c_1 ||y_0 + \hat{\phi}_0|| + c_2 + M \int_0^{b} (c_1 ||y_s + \hat{\phi}_s|| + c_2)ds \\
+ M_1 \int_0^{b} \int_{0}^{\tau} \alpha_{\kappa}(\tau) d\tau ds |\eta \]
\[ + \int_{t_1}^{t_2} |T(t_1 - s) - T(t_2 - s)| \int_0^s \alpha_k(\tau) d\tau ds \]
\[ + \int_{t_1}^{t_2} |T(t_2 - s)| \int_0^s \alpha_k(\tau) d\tau ds, \]

where \( k' = k + \| \hat{\phi} \| \). The right hand side is independent of \( y \in B_k \) and tends to zero as \( t_2 - t_1 \to 0 \), since \( g \) is completely continuous and the compactness of \( T(t) \) for \( t > 0 \) implies the continuity in the uniform operator topology.

Thus the set \( \{ Fy : y \in B_k \} \) is equicontinuous.

Notice that we considered here only the case \( 0 < t_1 < t_2 \), since the other cases \( t_1 < t_2 < 0 \) or \( t_1 < 0 < t_2 \) are very simple.

It is easy to see that the family \( FB_k \) is uniformly bounded. Next, we show \( FB_k \) is compact. Since we have shown \( FB_k \) is equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that \( F \) maps \( B_k \) into a precompact set in \( X \).

Let \( 0 < t \leq b \) be fixed and \( \epsilon \) a real number satisfying \( 0 < \epsilon < t \). For \( y \in B_k \) we define

\[
(F_\epsilon y)(t) = -T(t)g(0, \phi) + g(t - \epsilon, y_{t-\epsilon} + \hat{\phi}_{t-\epsilon}) + \int_{t-\epsilon}^{t-\epsilon} AT(t - s)g(s, y_s + \hat{\phi}_s)ds \\
+ \int_{0}^{t-\epsilon} T(t - \eta)BW^{-1}[x_1 - T(b)[\phi(0) - g(0, \phi)] - g(b, y_b + \hat{\phi}_b) \\
- \int_{0}^{b} AT(b - s)g(s, y_s + \hat{\phi}_s)ds - \int_{0}^{b} T(b - s) \int_{0}^{s} f(\tau, y_{\tau} + \hat{\phi}_{\tau})d\tau d\eta \\
+ \int_{0}^{t-\epsilon} T(t - s) \int_{0}^{s} f(\tau, y_{\tau} + \hat{\phi}_{\tau})d\tau ds. \\
\]

Since \( T(t) \) is a compact operator, the set \( Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_k \} \) is precompact in \( X \) for every \( \epsilon, 0 < \epsilon < t \). Moreover for every \( y \in B_k \) we have

\[ \|(Fy)(t) - (F_\epsilon y)(t)\| \]
\[
\begin{align*}
&\leq |g(t, y_t + \hat{\phi}_t) - g(t - \epsilon, y_{t-\epsilon} + \hat{\phi}_{t-\epsilon})| + \int_{t-\epsilon}^{t} |AT(t-s)g(s, y_s + \hat{\phi}_s)|ds \\
&+ \int_{t-\epsilon}^{t} |T(t-\eta)BW^{-1}[x_1 - T(b)[\phi(0) - g(0, \phi)] - g(b, y_b + \hat{\phi}_b) \\
&- \int_{0}^{b} AT(b-s)g(s, y_s + \hat{\phi}_s)ds - \int_{0}^{b} T(b-s) \int_{0}^{\bar{\eta}} f(\tau, y_{\tau} + \hat{\phi}_\tau) d\tau ds(\eta)|d\eta \\
&+ \int_{t-\epsilon}^{t} |T(t-s)\int_{0}^{\bar{\eta}} f(\tau, y_{\tau} + \hat{\phi}_\tau) d\tau|ds
\end{align*}
\]

Therefore there are precompact sets arbitrarily close to the set \{\((Fy)(t) : y \in B_k\)\}. Hence the set \{\((Fy)(t) : y \in B_k\)\} is precompact in \(X\).

It remains to show that \(F : C^0_0 \to C^0_0\) is continuous. Let \(\{y_n\}_0^\infty \subseteq C^0_0\) with \(y_n \to y\) in \(C^0_0\). Then there is an integer \(r\) such that \(|y_n(t)| \leq r\) for all \(n\) and \(t \in J\), so \(y_n \in B_r\) and \(y \in B_r\). By (iii), \(f(t, y_n(t) + \hat{\phi}_t) \to f(t, y(t) + \hat{\phi}_t)\) for each \(t \in J\) and since \(|f(t, y_n(t) + \hat{\phi}_t) - f(t, y(t) + \hat{\phi}_t)| \leq 2\alpha_r(t), \quad r' = r + \|\hat{\phi}\|\) and also \(g\) is completely continuous, we have by dominated convergence theorem,

\[
\|Fy_n - Fy\| = \sup_{t \in J} \|g(t, y_n(t) + \hat{\phi}_t) - g(t, y(t) + \hat{\phi}_t)\| + \int_{0}^{1} AT(t-s)[\|g(s, y_n(s) + \hat{\phi}_s) - g(s, y(s) + \hat{\phi}_s)\|ds
\]

\[
+ \int_{0}^{1} T(t-\eta)BW^{-1}[\int_{0}^{1} AT(b-s)[\|g(s, y_n(s) + \hat{\phi}_s) - g(s, y(s) + \hat{\phi}_s)\|ds
\]

\[
+ \int_{0}^{1} T(b-s)\int_{0}^{\bar{\eta}} [f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)]d\tau ds(\eta)|d\eta \\
+ \int_{0}^{1} T(t-s)\int_{0}^{\bar{\eta}} [f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)]d\tau ds
\]

\[
\leq |g(t, y_n(t) + \hat{\phi}_t) - g(t, y(t) + \hat{\phi}_t)| + \int_{0}^{b} |AT(t-s)| [\|g(s, y_n(s) + \hat{\phi}_s) - g(s, y(s) + \hat{\phi}_s)\|ds
\]
Thus $F$ is continuous. This completes the proof that $F$ is completely continuous.

Finally the set $\zeta(F) = \{y \in C^0_b : y = \lambda F y, \lambda \in (0,1)\}$ is bounded, since for every solution $y$ in $\zeta(F)$ the function $x(t) = y(t) + \phi(t)$ is a mild solution of (7.3), for which we have proved that $\|x\| \leq K$ and hence

$$\|y\|_1 \leq K + \|\phi\|.$$  

Consequently by Schaefer's theorem, the operator $F$ has a fixed point in $C^0_b$. This means that any fixed point of $F$ satisfying $(Fx)(t) = x(t)$ is a mild solution of the problem (7.1) on $J$. Hence the system (7.1) is controllable on $J$.

\[\star \star \star \star\]