VI. CONTROLLABILITY OF NONLINEAR
SOBOLEV-TYPE DELAY INTEGRODIFFERENTIAL
SYSTEMS

6.1. INTRODUCTION

Controllability of Sobolev-type nonlinear integrodifferential systems in Banach spaces has been discussed by Balachandran and Dauer [5] with the help of the Schauder fixed point theorem. Recently Balachandran et al. [10] studied the existence of solutions of nonlinear integrodifferential equations of Sobolev-type with nonlocal conditions in Banach spaces by using the compact semigroups and the Schauder fixed point theorem. In this chapter, we establish sufficient conditions for the controllability of nonlinear Sobolev-type delay integrodifferential systems in Banach spaces. Further we derive a set of sufficient conditions for the controllability of Sobolev-type integrodifferential systems with time varying delay in Banach spaces. The results are obtained by using the Schaefer fixed point theorem.

6.2. SOBOLEV-TYPE DELAY
INTEGRODIFFERENTIAL SYSTEMS

6.2.1. Preliminaries

Consider the Sobolev-type delay integrodifferential system of the form

\[(Ex(t))' + Ax(t) = (Bu)(t) + \int_0^t f(s, x_s)ds, \quad t \in J = [0, b], \quad (6.1)\]

\[x_0 = \phi, \quad \text{on } [-r, 0],\]

where the state \(x(\cdot)\) takes values in a Banach space \(X\) and the control function \(u(\cdot)\) is given in \(L^2(J, U)\), a Banach space of admissible control functions, with \(U\) as a Banach space. Here \(A\) is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators \(T(t)\) in \(X\), \(B\) is a bounded linear operator from \(U\) into \(Y\), a Banach space and \(f : J \times C \to Y\), where \(C = C([-r, 0], X)\) is the Banach space of all continuous functions \(\phi : [-r, 0] \to X\) endowed with the norm \(\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}\). Also for \(x \in C([-r, b], X)\), we have \(x_t \in C\) for \(t \in [0, b]\), \(x_t(\theta) = x(t + \theta)\) for \(\theta \in [-r, 0]\). The norm of \(X\) is denoted by \(\|\cdot\|\) and \(Y\) by \(|\cdot|\).
The operators \( A : D(A) \subset X \to Y \) and \( E : D(E) \subset X \to Y \) satisfy the hypotheses \([C_i]\) for \( i = 1, \ldots, 4:\)

\[\begin{align*}
&C_1 \ A \text{ and } E \text{ are closed linear operators} \\
&C_2 \ D(E) \subset D(A) \text{ and } E \text{ is bijective} \\
&C_3 \ E^{-1} : Y \to D(E) \text{ is continuous} \\
&C_4 \text{ For each } t \in [0, b] \text{ and for some } \lambda \in \rho(-AE^{-1}), \text{ the resolvent set of } -AE^{-1}, \\
&\quad \text{the resolvent } R(\lambda, -AE^{-1}) \text{ is a compact operator.}
\end{align*}\]

The hypotheses \([C_1], [C_2]\) and the closed graph theorem imply the boundedness of the linear operator \( AE^{-1} : Y \to Y.\)

**Lemma [59].** Let \( S(t) \) be a uniformly continuous semigroup and let \( A \) be its infinitesimal generator. If the resolvent set \( R(\lambda : A) \) of \( A \) is compact for every \( \lambda \in \rho(A), \) then \( S(t) \) is a compact semigroup.

From the above fact, \(-AE^{-1}\) generates a compact semigroup \( T(t), \ t \geq 0, \) on \( Y.\)

We further assume the following hypotheses:

\[\begin{align*}
&C_5 \ -AE^{-1} \text{ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators } T(t) \text{ in } Y \text{ satisfying} \\
&\quad |T(t)| \leq M_1 e^{\omega t}, \ t \geq 0 \text{ for some } M_1 \geq 1 \text{ and } \omega \geq 0. \\
&C_6 \text{ The linear operator } W : L^2(J, U) \to X \text{ defined by} \\
&\quad Wu = \int_0^b E^{-1}(b-s)Bu(s)ds \\
&\quad \text{has an invertible operator } W^{-1} : X \to L^2(J, U) \setminus \ker W \text{ and there exist positive constants } M_2, M_3 \text{ such that } |B| \leq M_2 \text{ and } |W^{-1}| \leq M_3. \\
&C_7 \text{ For each } t \in J, \text{ the function } f(t, \cdot) : C \to X \text{ is continuous and for each } x \in C, \text{ the function } f(\cdot, x) : J \to X \text{ is strongly measurable.} \\
&C_8 \text{ For every positive integer } k, \text{ there exists } g_k \in L^1(0, b) \text{ such that} \\
&\quad \sup_{\|x\| \leq k} |f(t, x)| \leq g_k(t), \text{ for } t \in J \ a.e.
\end{align*}\]
[C_9] There exists an integrable function \( m : [0, b] \to [0, \infty) \) such that
\[
|f(t, \phi)| \leq m(t)\Omega(\|\phi\|), \quad 0 \leq t \leq b, \quad \phi \in \mathcal{C},
\]
where \( \Omega : [0, \infty) \to (0, \infty) \) is a continuous nondecreasing function.

\[[C_{10}]\]
\[
\int_0^b \tilde{m}(s)ds < \int_c^\infty \frac{ds}{1 + s + \Omega(s)},
\]
where \( c = \|E^{-1}\| M_1\|E\phi(0)\| \), \( \tilde{m}(t) = \max\{\omega, \|E^{-1}\| M_1 N, \|E^{-1}\| M_1 \int_0^t m(s)ds\} \)
and
\[
N = M_2 M_3[\|x_1\| + \|E^{-1}\| M_1 e^{\omega b}\|\phi\| + \|E^{-1}\| M_1 \int_0^b e^{\omega (b-s)} \int_0^s m(\tau)\Omega(\|x_\tau\|)d\tau ds].
\]

Then the system (6.1) has a mild solution of the following form
\[
x(t) = E^{-1}T(t)E\phi(0) + \int_0^t E^{-1}T(t-s)(Bu)(s) + \int_0^s f(\tau, x_\tau) d\tau ds, \quad t \in J,
\]
\[x_0 = \phi\]
and \( E x(t) \in C([0, b]; Y) \cap C'((0, b); Y) \).

**Definition 6.1.** The system (6.1) is said to be controllable on the interval \( J \) if for every continuous initial function \( \phi \in \mathcal{C} \), there exists a control \( u \in L^2(J, U) \) such that the solution \( x(t) \) of (6.1) satisfies \( x(b) = x_1 \).

### 6.2.2. Controllability Result

**Theorem 6.1.** If the hypotheses \([C_1] - [C_{10}]\) are satisfied, then the system (6.1) is controllable on \( J \).

**Proof.** Using the hypothesis \([C_9]\) for an arbitrary function \( x(.,.) \), define the control
\[
u(t) = W^{-1}[x_1 - E^{-1}T(b)E\phi(0) - \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, x_\tau) d\tau ds](t).
\]
For \( \phi \in \mathcal{C} \), define \( \hat{\phi} \in \mathcal{C}_b \), \( \mathcal{C}_b = C([-r, b], X) \) by
\[
\hat{\phi}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0, \\ E^{-1}T(t)E\phi(0), & 0 \leq t \leq b \end{cases}
\]
\[57\]
If \( x(t) = y(t) + \phi(t) \), \( t \in [-r, b] \), it is easy to see that \( y \) satisfies

\[
y_0 = 0, \quad y(t) = \int_0^t E^{-1}(t-\eta)BW^{-1}[x_1 - E^{-1}(b)E\phi(0)]
- \int_0^b E^{-1}(b-\eta) \int_0^s f(\tau, y_\tau + \phi_\tau)d\tau ds)(\eta)d\eta \\
+ \int_0^t E^{-1}(t-\eta) \int_0^s f(\tau, y_\tau + \phi_\tau)d\tau ds, \quad 0 \leq t \leq b \quad (6.2)
\]
if and only if \( x \) satisfies

\[
x(t) = E^{-1}(t)E\phi(0) + \int_0^t E^{-1}(t-\eta)BW^{-1}[x_1 - E^{-1}(b)E\phi(0)]
- \int_0^b E^{-1}(b-\eta) \int_0^s f(\tau, x_\tau + \phi_\tau)d\tau ds)(\eta)d\eta \\
+ \int_0^t E^{-1}(t-\eta) \int_0^s f(\tau, x_\tau + \phi_\tau)d\tau ds
\]
and \( x_0 = \phi \).

Define \( C^0_b = \{ y \in C_b : y_0 = 0 \} \) and we now show that when using the control, the operator \( F : C^0_b \to C^0_b \), defined by

\[
(Fy)(t) = \begin{cases} 
0, & -r \leq t \leq 0, \\
\int_0^t E^{-1}(t-\eta)BW^{-1}[x_1 - E^{-1}(b)E\phi(0)] \\
- \int_0^b E^{-1}(b-\eta) \int_0^s f(\tau, y_\tau + \phi_\tau)d\tau ds)(\eta)d\eta \\
+ \int_0^t E^{-1}(t-\eta) \int_0^s f(\tau, y_\tau + \phi_\tau)d\tau ds, & 0 \leq t \leq b
\end{cases}
\]
has a fixed point. This fixed point is then a solution of equation (6.2).

Clearly \( x(b) = x_1 \) which means that the control \( u \) steers the system (6.1) from the initial function \( \phi \) to \( x_1 \) in time \( b \), provided we can obtain a fixed point of the nonlinear operator \( F \).

In order to study the controllability problem of (6.1), we introduce a parameter \( \lambda \in (0,1) \) and consider the following system

\[
(Ex(t))' + \lambda Ax(t) = \lambda (Bu)(t) + \lambda \int_0^t f(s, x_s)ds, \quad t \in J = [0, b], \quad (6.3)
\]
\[
x_0 = \phi.
\]
First we obtain \( a \text{ \textit{priori} bounds} \) for the mild solution of the equation (6.3).

From

\[
x(t) = \lambda E^{-1}T(t)E\phi(0) + \lambda \int_0^t E^{-1}T(t-\eta)BW^{-1}[x_1 - E^{-1}T(b)E\phi(0) \]

\[
- \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, x_\tau) d\tau ds (\eta) d\eta
\]

\[
+ \lambda \int_0^t E^{-1}T(t-s) \int_0^s f(\tau, x_\tau) d\tau ds,
\]

we have,

\[
\|x(t)\| \leq \|E^{-1}\| M_1 e^{\omega t} |E\phi(0)| + \int_0^t \|E^{-1}\| |T(t-\eta)| M_2 M_3 \|x_1\|
\]

\[
+ \|E^{-1}\| M_1 e^{\omega b} |E\phi(0)| + \int_0^b \|E^{-1}\| M_1 e^{\omega (b-s)} \int_0^s m(\tau) \Omega(\|x_\tau\|) d\tau ds d\eta
\]

\[
+ \|E^{-1}\| M_1 e^{\omega t} \int_0^t e^{-\omega s} \int_0^s m(\tau) \Omega(\|x_\tau\|) d\tau ds,
\]

\( t \in [0, b] \).

We consider the function \( \mu \) given by

\[ \mu(t) = \sup\{\|x(s)\| : -r < s < t\}, \quad 0 \leq t \leq b. \]

Let \( t^* \in [-r, t] \) be such that \( \mu(t) = \|x(t^*)\| \). If \( t^* \in [0, b] \), by the previous inequality, we have

\[ e^{-\omega t} \mu(t) \leq \|E^{-1}\| M_1 |E\phi(0)| + \|E^{-1}\| M_1 N \int_0^t e^{-\omega s} ds
\]

\[ + \|E^{-1}\| M_1 \int_0^t e^{-\omega s} \int_0^s m(\tau) \Omega(\|x_\tau\|) d\tau ds
\]

\[ \leq \|E^{-1}\| M_1 |E\phi(0)|
\]

\[ + \|E^{-1}\| M_1 N \int_0^t e^{-\omega s} ds + \|E^{-1}\| M_1 \int_0^t e^{-\omega s} \int_0^s m(\tau) \Omega(\mu(\tau)) d\tau ds.
\]

If \( t^* \in [-r, 0] \), then \( \mu(t) = \|E\phi\| \) and the previous inequality holds since \( M_1 \geq 1 \).

Denoting by \( v(t) \), the right-hand side of the above inequality, we have

\[ c = v(0) = \|E^{-1}\| M_1 |E\phi(0)|, \quad \mu(t) \leq e^{\omega t} v(t), \quad 0 \leq t \leq b \]

and

\[ v'(t) = \|E^{-1}\| M_1 N e^{-\omega t} + \|E^{-1}\| M_1 e^{-\omega t} \int_0^t m(s) \Omega(\mu(s)) ds
\]

\[ \leq \|E^{-1}\| M_1 N e^{-\omega t} + \|E^{-1}\| M_1 e^{-\omega t} \int_0^t m(s) \Omega(e^{\omega s} v(s)) ds.
\]
We remark that
\[
(e^{t\sigma}v(t))' = \omega e^{t\sigma}v(t) + e^{t\sigma}v'(t) \\
\leq \omega e^{t\sigma}v(t) + \| E^{-1} \| M_1 N + \| E^{-1} \| M_1 \int_0^t m(s)\Omega(e^{s\sigma}v(s))\,ds \\
\leq \omega e^{t\sigma}v(t) + \| E^{-1} \| M_1 N + \| E^{-1} \| M_1 \Omega(e^{t\sigma}v(t)) \int_0^t m(s)\,ds \\
\leq \tilde{\tau}(t)(e^{t\sigma}v(t) + 1 + \Omega(e^{t\sigma}v(t))).
\]
This implies
\[
\int_{\nu(0)}^{e^{t\sigma}v(t)} \frac{ds}{1 + s + \Omega(s)} \leq \int_0^b \tilde{\tau}(s)\,ds < \int_c^\infty \frac{ds}{1 + s + \Omega(s)}, \quad 0 \leq t \leq b.
\]
This inequality implies that there is a constant $K$ such that $v(t) \leq K$ and hence $\mu(t) \leq K$, $t \in [0,b]$. Since $\| x(t) \| \leq \mu(t)$, $t \in [0,b]$, we have
\[
\| x \|_1 = \sup\{\| x(t) \| : -r \leq t \leq b \} \leq K,
\]
where $K$ depends only on $b$ and on the functions $m$ and $\Omega$.

Next we must prove that the operator $F$ is a completely continuous operator. Let $B_k = \{ y \in C_0^b : \| y \|_1 \leq k \}$ for some $k \geq 1$.

We first show that the set $\{ Fy : y \in B_k \}$ is equicontinuous. Let $y \in B_k$ and $t_1, t_2 \in [0,b]$. Then if $0 < t_1 < t_2 < b$,
\[
\| (Fy)(t_1) - (Fy)(t_2) \| \\
\leq \| \int_0^{t_1} E^{-1}[T(t_1 - \eta) - T(t_2 - \eta)]BW^{-1}[x_1 - E^{-1}T(b)E\dot{\phi}(0)] \\
- \int_0^b E^{-1}T(b - s) \int_0^s f(\tau, y_\tau + \dot{\phi}_\tau)\,d\tau\,ds(\eta)\,d\eta \| \\
+ \| \int_{t_1}^{t_2} E^{-1}T(t_2 - \eta)BW^{-1}[x_1 - E^{-1}T(b)E\dot{\phi}(0)] \\
- \int_0^b E^{-1}T(b - s) \int_0^s f(\tau, y_\tau + \dot{\phi}_\tau)\,d\tau\,ds(\eta)\,d\eta \| \\
+ \| \int_0^{t_1} E^{-1}[T(t_1 - s) - T(t_2 - s)] \int_0^s f(\tau, y_\tau + \dot{\phi}_\tau)\,d\tau\,ds \| \\
+ \| \int_{t_1}^{t_2} E^{-1}T(t_2 - s) \int_0^s f(\tau, y_\tau + \dot{\phi}_\tau)\,d\tau\,ds \| \\
\leq \int_0^{t_1} \| E^{-1}[T(t_1 - \eta) - T(t_2 - \eta)]M_kM_3\| x_1 \| + \| E^{-1} | M_1 e^{\omega h} | E\dot{\phi}(0) \|
\[\begin{align*}
&= \| E^{-1} \| M_1 b \int_0^b e^{\kappa(b-s)} \int_0^s g_{\kappa}(\tau)d\tau ds d\eta \\
&+ \int_{t_1}^{t_2} \| E^{-1} \| |T(t_2 - \eta)| M_2 M_3 || x_1 || + \| E^{-1} \| M_1 e^{\kappa} |E\phi(0)| \\
&\quad + \| E^{-1} \| M_1 b \int_0^b e^{\kappa(b-s)} \int_0^s g_{\kappa}(\tau)d\tau ds d\eta \\
&\quad + \int_{t_1}^{t_2} \| E^{-1} \| |T(t_2 - s) - T(t_1 - s)| \int_0^s g_{\kappa}(\tau)d\tau ds \\
&\quad + \int_{t_1}^{t_2} \| E^{-1} \| |T(t_2 - s)| \int_0^s g_{\kappa}(\tau)d\tau ds,
\end{align*}\]

where \( k' = k + \| \phi \| \). The right hand side is independent of \( y \in B_k \) and tends to zero as \( t_2 - t_1 \to 0 \), since the compactness of \( T(t) \), for \( t > 0 \), implies the continuity in the uniform operator topology.

Thus the set \( \{ Fy; y \in B_k \} \) is equicontinuous.

Notice that we considered here only the case \( 0 < t_1 < t_2 \), since the other cases \( t_1 < t_2 < 0 \) or \( t_1 < 0 < t_2 \) are very simple.

It is easy to see that the family \( FB_k \) is uniformly bounded. Next we show that \( FB_k \) is compact. Since we have shown that \( FB_k \) is an equicontinuous collection, it suffices, by the Arzela-Ascoli theorem, to show that \( F \) maps \( B_k \) into a precompact set in \( X \).

Let \( 0 < t \leq b \) be fixed and \( \epsilon \) a real number satisfying \( 0 < \epsilon < t \). For \( y \in B_k \), we define

\[
(F \epsilon y)(t) = \int_{t-\epsilon}^t E^{-1}T(t-\eta)BW^{-1}[x_1 - E^{-1}T(b)E\phi(0)] \\
- \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, y_{\tau + \hat{\phi}_{\tau}})d\tau ds(\eta)d\eta \\
+ \int_{t-\epsilon}^{t-\epsilon} E^{-1}T(t-s) \int_0^s f(\tau, y_{\tau + \hat{\phi}_{\tau}})d\tau ds(\eta)d\eta.
\]

Since \( T(t) \) is a compact operator, the set \( Y_\epsilon(t) = \{(F \epsilon y)(t) : y \in B_k \} \) is precompact in \( X \) for every \( \epsilon \), \( 0 < \epsilon < t \). Moreover for every \( y \in B_k \) we have

\[
\|(F \epsilon y)(t) - (F \epsilon y)(t)\| \\
\leq \int_{t-\epsilon}^t \| E^{-1}T(t-\eta)BW^{-1}[x_1 - E^{-1}T(b)E\phi(0)] \\
- \int_0^b E^{-1}T(b-s) \int_0^s f(\tau, y_{\tau + \hat{\phi}_{\tau}})d\tau ds(\eta)d\eta \\
+ \int_{t-\epsilon}^{t-\epsilon} E^{-1}T(t-s) \int_0^s f(\tau, y_{\tau + \hat{\phi}_{\tau}})d\tau ds(\eta)d\eta.
\]

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\[
\leq \int_{t-\epsilon}^{t} \| E^{-1} \| \| T(t-\eta) \| M_2 M_3 \| x_1 \| + | E^{-1} | M_1 e^{a | \phi(0) |}
+ \int_{t-\epsilon}^{t} \| E^{-1} \| M_1 \int_0^\delta e^{\omega(b-s)} \int_0^\delta g_{\nu}(\tau) d\tau ds d\eta
+ \int_{t-\epsilon}^{t} \| E^{-1} \| \| T(t-s) \| \int_0^\delta g_{\nu}(\tau) d\tau ds.
\]

Therefore there are precompact sets arbitrarily close to the set \( \{(Fy)(t) : y \in B_k \} \). Hence the set \( \{(Fy)(t) : y \in B_k \} \) is precompact in \( X \).

It remains to be shown that \( F : C^0_b \rightarrow C^0_b \) is continuous. Let \( \{y_n\} \subseteq C^0_b \) with \( y_n \rightarrow y \) in \( C^0_b \). Then there is an integer \( r \) such that \( \| y_n(t) \| \leq r \) for all \( n \) and \( t \in J \), so \( y_n \in B_r \) and \( y \in B_r \). By \( [C\gamma] \), \( f(t, y_n(t) + \hat{\phi}_t) \rightarrow f(t, y(t) + \hat{\phi}_t) \) for each \( t \in J \) and since \( |f(t, y_n(t) + \hat{\phi}_t) - f(t, y(t) + \hat{\phi}_t)| \leq 2g_{\nu}(t) \), \( r' = r + \| \hat{\phi} \| \), we have, by dominated convergence theorem,

\[
\| Fy_n - Fy \| = \sup_{t \in J} \int_0^t E^{-1} T(t-\eta) BW^{-1} [\int_0^b T(b-s) d\eta
+ \int_0^t E^{-1} T(t-s) \int_0^\delta [f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)] d\tau ds d\eta
+ \int_0^b \int_0^\delta [f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)] d\tau ds]
\leq \int_0^b E^{-1} \| \| T(t-\eta) \| M_2 M_3 M_1 \int_0^b e^{\omega(b-s)} d\eta
+ \int_0^b \int_0^\delta [f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)] d\tau ds d\eta
+ \int_0^b \int_0^\delta [f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)] d\tau ds
\rightarrow 0
\]
as \( n \rightarrow \infty \).

Thus \( F \) is continuous. This completes the proof that \( F \) is completely continuous.

Finally the set \( \zeta(F) = \{ y \in C^0_b : y = \lambda Fy, \lambda \in (0,1) \} \) is bounded, since for every solution \( y \) in \( \zeta(F) \), the function \( x = y + \hat{\phi} \) is a mild solution of (6.3) for which we have proved that \( \| x \|_1 \leq K \) and hence

\[
\| y \|_1 \leq K + \| \hat{\phi} \|.
\]

Consequently, by Schaefer's theorem, the operator \( F \) has a fixed point in \( C^0_b \). This means that any fixed point of \( F \) is a mild solution of (6.1) on \( J \) satisfying \( (Fx)(t) = x(t) \). Thus the system (6.1) is controllable on \( J \).
6.2.3. Example

Consider the following partial integrodifferential equation of the form
\[
\frac{\partial}{\partial t}(z(t,x) - z_{xx}(t,x)) - z_{xx}(t,x) = Bu(t) + \int_0^t p(s, z(x, s - r))ds, \quad (6.4)
\]
\[
0 < x < 1, \quad t > 0,
\]
with
\[
z(0, t) = z(1, t) = 0, \quad t > 0,
\]
\[
z(x, t) = \phi(x, t), \quad -r < t < 0,
\]
where \( \phi \) is continuous and \( u \in L^2(J, U) \) with \( U \subset J \).

Assume that the following conditions hold with \( X = Y = L^2[0, 1] \).

[A1] The operator \( B : U \to Y \), with \( U \subset J \), is a bounded linear operator.

[A2] The linear operator \( W : L^2(J, U) \to X \) defined by
\[
Wu = \int_0^b E^{-1}(b-s)Bu(s)ds
\]
has an bounded invertible operator \( W^{-1} \) which takes values in \( L^2(J, U) \setminus \ker W \).

[A3] Further the function \( p : J \times X \to Y \) is continuous, bounded and strongly measurable.

[A4] Let \( f(t, w_t)(x) = p(t, w(t-x)) \), \( 0 < x < 1 \).

Define the operators \( A : D(A) \subset X \to Y \), \( E : D(E) \subset X \to Y \) by
\[
Aw = -w_{xx}, \quad Ew = w - w_{xx}
\]
respectively, where each domain \( D(A), D(E) \) is given by
\[
\{w \in X, w, w_x \text{ are absolutely continuous}, \ w_{xx} \in X, w(0) = w(1) = 0 \}.
\]
Then \( A \) and \( E \) can be written respectively as
\[
Aw = \sum_{n=1}^{\infty} n^2(w, w_n)w_n, \quad w \in D(A),
\]
\[
Ew = \sum_{n=1}^{\infty} (1 + n^2)(w, w_n)w_n, \quad w \in D(E),
\]

where \(w_n(x) = \sqrt{2}\sin nx\), \(n = 1, 2, 3, \ldots\), is the orthogonal set of eigenvectors of \(A\). Furthermore, for \(w \in X\), we have

\[
E^{-1}w = \sum_{n=1}^{\infty} \frac{1}{1 + n^2} (w, w_n)w_n,
\]

\[
-\lambda E^{-1}w = \sum_{n=1}^{\infty} \frac{-n^2}{1 + n^2} (w, w_n)w_n,
\]

\[
T(t)w = \sum_{n=1}^{\infty} e^{i\lambda n^2 t} (w, w_n)w_n.
\]

It is easy to see that \(-\lambda E^{-1}\) generates a strongly continuous semigroup \(T(t)\) on \(Y\) and \(T(t)\) is compact such that \(|T(t)| < e^{-t}\) for each \(t > 0\).

\[A5\] The function \(p\) satisfies the following conditions:

There exists an integrable function \(q : J \rightarrow [0, \infty)\) such that

\[
|p(t, w(t - y))| \leq q(t)\Omega_1(\|w\|),
\]

where \(\Omega_1 : [0, \infty) \rightarrow (0, \infty)\) is continuous and nondecreasing.

Also we have

\[
\int_0^b \hat{n}(s)ds < \int_0^\infty \frac{ds}{1 + s + \Omega_1(s)},
\]

where \(c = |E^{-1}|e^{-t}|E\phi(0)|\), and \(\hat{n}(t) = \max\{-1, |E^{-1}|e^{-t}N, |E^{-1}|e^{-t} \int_0^t q(s)ds\}\).

Here \(N\) depends on \(E\), \(A\), \(B\), and \(p\). Further all the conditions stated in Theorem 6.1. are satisfied. Hence the system (6.4) is controllable on \(J\).

6.3. SOBOLEV TYPE INTEGRODIFFERENTIAL SYSTEMS WITH TIME VARYING DELAY

6.3.1. Preliminaries

In this section we consider the following Sobolev-type integrodifferential system with time varying delay of the form

\[
(Ex(t))' + Ax(t)
= (Bu)(t) + f(t, x(\sigma_1(t)), \int_0^t l(t, s)g(s, x(\sigma_2(s)))ds), t \in J = [0, b], (6.5)
\]

\(x(t) = \phi(t), \quad t \in [-h, 0]\)
where \( \sigma_i(t) = t - h_i(t) \) with \( h_i(t) \geq 0 \), \( h = \max\{h_i(t) : t \in J\} \), \( h_i : J \to J \), \( i = 1,2 \) are continuous functions, the state \( x(\cdot) \) takes values in a Banach space \( X \) and the control function \( u(\cdot) \) is given in \( L^2(J,U) \), a Banach space of admissible control functions, with \( U \) as a Banach space. Here \( A \) is the infinitesimal generator of a strongly continuous semigroup \( T(t), t \geq 0 \), in the Banach space \( X \), \( B \) is a bounded linear operator from \( U \) into a Banach space \( Y \), \( g : J \times X \to X, f : J \times X \times X \to Y, l : J \times J \to R \) are given functions and \( \phi \in Z = C([-h,0], X) \). The norm of \( X \) is denoted by \( \| \cdot \| \) and \( Y \) by \( | \cdot | \).

The following conditions are similar to those in the previous section, however for the sake of completeness, we are stating them here.

The operators \( A : D(A) \subset X \to Y \) and \( E : D(E) \subset X \to Y \) satisfy the hypotheses \([C_i]\) for \( i = 1,\ldots,4 \):

\[ [C_1] \quad A \text{ and } E \text{ are closed linear operators}, \]

\[ [C_2] \quad D(E) \subset D(A) \text{ and } E \text{ is bijective}, \]

\[ [C_3] \quad E^{-1} : Y \to D(E) \text{ is continuous}, \]

\[ [C_4] \quad \text{For each } t \in [0,b] \text{ and for some } \lambda \in \rho(-AE^{-1}), \text{ the resolvent set of } -AE^{-1}, \text{ the resolvent } R(\lambda, -AE^{-1}) \text{ is a compact operator}. \]

The hypotheses \([C_1],[C_2]\) and the closed graph theorem imply the boundedness of the linear operator \( AE^{-1} : Y \to Y \).

**Lemma [59]**. Let \( S(t) \) be a uniformly continuous semigroup and let \( A \) be its infinitesimal generator. If the resolvent set \( R(\lambda : A) \) of \( A \) is compact for every \( \lambda \in \rho(A) \), then \( S(t) \) is a compact semigroup.

From the above fact, \( -AE^{-1} \) generates a compact semigroup \( T(t), t \geq 0 \), on \( Y \). Thus \( \max_{t \in J}|T(t)| \) is finite and so denote \( M_1 = \max_{t \in J}|T(t)| \).

\[ [C_5] \quad \text{The linear operator } W : L^2(J,U) \to X \text{ defined by} \]

\[ Wu = \int_0^b E^{-1}T(b-s)Bu(s)ds \]

has an invertible operator \( W^{-1} : L^2(J,U) \setminus \ker W \to X \) and there exist positive constants \( M_2,M_3 \) such that \( \|B\| \leq M_2 \) and \( \|W^{-1}\| \leq M_3 \).
For each $t \in J$, the function $g(t, \cdot) : X \to X$ is continuous and for each $x \in X$, the function $g(\cdot, x) : J \to X$ is strongly measurable.

For each $t \in J$, the function $f(t, \cdot, \cdot) : X \times X \to X$ is continuous and for each $x, y \in X$, the function $f(\cdot, x, y) : J \to X$ is strongly measurable.

For every positive integer $k$, there exists $\alpha_k \in L^1(0, b)$ such that for almost all $t \in J$,

$$\sup_{\|x\| \leq k} |f(t, x, y)| \leq \alpha_k(t).$$

There exists a continuous function $m : J \to [0, \infty)$ such that

$$\|g(t, x)\| \leq m(t)\Omega(\|x\|), \quad t \in J, x \in X,$$

where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

There exists a continuous function $p : J \to [0, \infty)$ such that

$$|f(t, x, y)| \leq p(t)\Omega_0(\|x\| + \|y\|), \quad t \in J, \ x, y \in X,$$

where $\Omega_0 : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

There exists a constant $L$ such that

$$|l(t, s)| \leq L \quad \text{for} \ t \geq s \geq 0.$$

$$\int_0^b \hat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)},$$

where $c = \|E^{-1}\|M_1[p(t) + N\bar{b}], \ \hat{m}(t) = \max\{\|E^{-1}\|M_1p(t), Lm(t)\}$ and

$$N \cdot = M_2M_3[\|x_1\| + \|E^{-1}\|M_1|E\phi(0)|]$$

$$+ M_1\|E^{-1}\| \int_0^b p(s)\Omega_0(\|x(s)\| + L \int_s^\infty m(\tau)\Omega(\|x(\tau)\|)d\tau)d\tau].$$

Then the system (6.5) has a mild solution of the following form

$$x(t) = E^{-1}T(t)E\phi(0) + \int_0^t E^{-1}T(t - s)[(Bu)(s)$$

$$+ f(s, x(\sigma_1(s)), \int_s^\tau l(\tau, \sigma_2(\tau))g(\tau, x(\sigma_2(\tau)))d\tau)]ds, \quad t \in J, \quad (6.6)$$
and \( Ex(t) \in C([0,b]; Y) \cap C'((0,b); Y) \).

### 6.3.2. Controllability Result

**Theorem 6.2.** If the hypotheses \( [C_1] - [C_{12}] \) are satisfied, then the system (6.5) is controllable on \( J \).

**Proof.** Let \( C = C(J, X) \). Using the hypothesis \( [C_5] \) for an arbitrary function \( x(\cdot) \), define the control

\[
u(t) = W^{-1}\left[ x_1 - E^{-1}T(b)E\phi(0) - \int_0^b E^{-1}T(b-s) \
    f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x(\sigma_2(\tau)))d\tau ds\right](t).
\]

We now show that using this control, the operator \( F : C \rightarrow C \) defined by

\[
(Fx)(t) = E^{-1}T(t)E\phi(0) + \int_0^t E^{-1}T(t-\eta)BW^{-1}[x_1 - E^{-1}T(b)E\phi(0) \
- \int_0^b E^{-1}T(b-s)f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x(\sigma_2(\tau)))d\tau ds](\eta)d\eta \\
+ \int_0^t E^{-1}T(t-s)f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x(\sigma_2(\tau)))d\tau ds)ds
\]

has a fixed point. This fixed point is then a solution of equation (6.6).

Clearly \( (Fx)(b) = x_1 \) which means that the control \( u \) steers the system (6.5) from the initial function \( \phi \) to \( x_1 \) in time \( b \), provided we can obtain a fixed point of the nonlinear operator \( F \).

In order to study the controllability problem of (6.5), we introduce a parameter \( \lambda \in (0,1) \) and consider the following system

\[
(Ex(t))' + \lambda Ax(t) = \lambda(Bu)(t) + \lambda f(t, x(\sigma_1(t)), \int_0^t l(t, s)g(s, x(\sigma_2(s)))ds), \quad (6.7)
\]

\[x(t) = \phi(t), \quad t \in [-h,0].\]

First we obtain a priori bounds for the mild solution of the equation (6.7). From

\[
x(t) = \lambda E^{-1}T(t)E\phi(0) + \lambda \int_0^t E^{-1}T(t-\eta)BW^{-1}[x_1 - E^{-1}T(b)E\phi(0) \
- \int_0^b E^{-1}T(b-s)f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x(\sigma_2(\tau)))d\tau ds](\eta)d\eta \\
+ \int_0^t E^{-1}T(t-s)f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x(\sigma_2(\tau)))d\tau ds),
\]

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we have

\[ ||x(t)|| \leq \| E^{-1} \| M_1 |E\Phi(0)| \]
\[ + \int_0^t \| E^{-1} \| |T(t-\eta)| M_2 M_3 [||x_1|| + \| E^{-1} \| M_1 |E\Phi(0)| \]
\[ + \| E^{-1} \| M_1 \int_0^b p(s)\Omega_0(||x(s)|| + L \int_0^s m(\tau)\Omega(||x(\tau)||) d\tau) ds d\eta \]
\[ + \| E^{-1} \| M_1 \int_0^t p(s)\Omega_0(||x(s)|| + L \int_0^s m(\tau)\Omega(||x(\tau)||) d\tau) ds \]
\[ \leq \| E^{-1} \| M_1 |E\Phi(0)| + \int_0^t \| E^{-1} \| M_1 N ds \]
\[ + \| E^{-1} \| M_1 \int_0^t p(s)\Omega_0(||x(s)|| + L \int_0^s m(\tau)\Omega(||x(\tau)||) d\tau) ds \]
\[ \leq \| E^{-1} \| M_1 |E\Phi(0)| + \| E^{-1} \| M_1 N b \]
\[ + \| E^{-1} \| M_1 \int_0^t p(s)\Omega_0(||x(s)|| + L \int_0^s m(\tau)\Omega(||x(\tau)||) d\tau) ds. \]

Denoting by \( v(t) \), the right-hand side of the above inequality, we have \( c = v(0) = \| E^{-1} \| M_1 |E\Phi(0)| + N b \), \( ||x(t)|| \leq v(t) \) and

\[ v'(t) = \| E^{-1} \| M_1 p(t)\Omega_0(||x(t)|| + L \int_0^t m(\tau)\Omega(||x(\tau)||) d\tau) \]
\[ \leq \| E^{-1} \| M_1 p(t)\Omega_0(v(t)) + L \int_0^t m(\tau)\Omega(v(\tau)) d\tau. \]

Let

\[ w(t) = v(t) + L \int_0^t m(\tau)\Omega(v(\tau)) d\tau. \]

Then \( w(0) = v(0) = c, \ v(t) \leq w(t) \), and

\[ w'(t) = v'(t) + L m(t)\Omega(v(t)) \]
\[ \leq \| E^{-1} \| M_1 p(t)\Omega_0(w(t)) + L m(t)\Omega(w(t)) \]
\[ \leq \hat{m}(t) [\Omega_0(w(t)) + \Omega(w(t))]. \]

This implies

\[ \int_{w(0)}^{w(t)} \frac{ds}{\Omega_0(s) + \Omega(s)} \leq \int_0^b \hat{m}(s)ds < \int_0^\infty \frac{ds}{\Omega_0(s) + \Omega(s)}, \quad t \in J. \]

This inequality implies that there is a constant \( K \) such that \( w(t) \leq K \) and hence \( ||x(t)|| \leq K, \ t \in J \), where \( K \) depends only on \( b \) and on the functions \( \hat{m}, \Omega_0 \).
and \( \Omega \).

Next we prove that the operator \( F : C \rightarrow C \) is a completely continuous operator.

Let \( B_k = \{ x \in C : \| x \| \leq k \} \) for some \( k \geq 1 \).

We first show that the set \( \{ Fx : x \in B_k \} \) is equicontinuous. Let \( x \in B_k \) and \( t_1, t_2 \in J \). Then if \( 0 < t_1 < t_2 \leq b \),

\[
\| (Fx)(t_1) - (Fx)(t_2) \| \\
\leq \| E^{-1}\| | T(t_1) - T(t_2) | E \phi(0) | \\
+ \| \int_0^{t_1} E^{-1}[T(t_1 - \eta) - T(t_2 - \eta)]BW^{-1}[x_1 - E^{-1}T(b)E \phi(0)] \\
- \int_0^b E^{-1}T(b - s)f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(x(\sigma_2(\tau)))d\tau)ds] \eta d\eta \| \\
+ \| \int_{t_1}^{t_2} E^{-1}T(t_2 - \eta)BW^{-1}[x_1 - E^{-1}T(b)E \phi(0)] \\
- \int_0^b E^{-1}T(b - s)f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(x(\sigma_2(\tau)))d\tau)ds] \eta d\eta \| \\
+ \| \int_0^{t_1} E^{-1}[T(t_1 - s) - T(t_2 - s)]f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(x(\sigma_2(\tau)))d\tau)ds || \\
+ \| \int_{t_1}^{t_2} E^{-1}T(t_2 - s)f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(x(\sigma_2(\tau)))d\tau)ds || \\
\leq \| E^{-1}\| | T(t_1) - T(t_2) | E \phi(0) | \\
+ \int_0^{t_1} \| E^{-1}\| | T(t_1 - \eta) - T(t_2 - \eta) ||M_2M_3||x_1|| + \| E^{-1}\| M_1 | E \phi(0) | \\
+ \| E^{-1}\| M_1 \int_0^b \alpha_k(s)ds | \eta \|
+ \int_{t_1}^{t_2} \| E^{-1}\| | T(t_2 - \eta) ||M_2M_3||x_1|| + \| E^{-1}\| M_1 | E \phi(0) | \\
+ \| E^{-1}\| M_1 \int_0^b \alpha_k(s)ds | \eta \|
+ \int_0^{t_1} \| E^{-1}\| | T(t_1 - s) - T(t_2 - s) | \alpha_k(s)ds \\
+ \int_{t_1}^{t_2} \| E^{-1}\| | T(t_2 - s) | \alpha_k(s)ds.
\]

The right-hand side tends to zero as \( t_2 - t_1 \rightarrow 0 \) and \( \epsilon \) sufficiently small, since the compactness of \( T(t) \), for \( t > 0 \), implies the continuity in the uniform operator topology.
Thus the set \( \{ Fx : x \in B_k \} \) is equicontinuous. It is easy to see that the family \( FB_k \) is uniformly bounded.

Next we show \( FB_k \) is compact. Since we have shown \( FB_k \) is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem, to show that \( F \) maps \( B_k \) into a precompact set in \( X \).

Let \( 0 < t \leq b \) be fixed and \( \epsilon \) a real number satisfying \( 0 < \epsilon < t \). For \( x \in B_k \), we define

\[
(F,x)(t) = E^{-1}T(t)E\phi(0) + \int_0^{t-\epsilon} E^{-1}T(t-\eta)BW^{-1}[x_1 - E^{-1}T(b)E\phi(0)] - \int_0^b E^{-1}T(b-s)f(s,x(\sigma_1(s))), \int_0^\epsilon l(s,\tau)g(\tau,x(\sigma_2(\tau)))d\tau ds][\eta]d\eta + \int_0^{t-\epsilon} E^{-1}T(t-s)f(s,x(\sigma_1(s))), \int_0^\epsilon l(s,\tau)g(\tau,x(\sigma_2(\tau)))d\tau ds.
\]

Since \( T(t) \) is a compact operator, the set \( Y_\epsilon(t) = \{(F,x)(t) : x \in B_k \} \) is precompact in \( X \) for every \( \epsilon, 0 < \epsilon < t \). Moreover for every \( x \in B_k \), we have

\[
\| (Fx)(t) - (Fx)(t) \| \leq \int_0^{t-\epsilon} \| E^{-1}T(t-\eta)BW^{-1}[x_1 - E^{-1}T(b)E\phi(0)] - \int_0^b E^{-1}T(b-s)f(s,x(\sigma_1(s))), \int_0^\epsilon l(s,\tau)g(\tau,x(\sigma_2(\tau)))d\tau ds][\eta]d\eta + \int_0^{t-\epsilon} \| E^{-1}||T(t-\eta)||M_2M_3[\|x_1\| + \|E^{-1}||M_1||E\phi(0)||
\]

\[
+ \|E^{-1}||M_1\int_0^b \alpha_k(s)ds||d\eta
\]

\[
+ \int_0^{t-\epsilon} \| E^{-1}||T(t-s)||\alpha_k(s)ds.d\eta.
\]

Therefore there are precompact sets arbitrarily close to the set \( \{(Fx)(t) : x \in B_k \} \). Hence the set \( \{(Fx)(t) : x \in B_k \} \) is precompact in \( X \).

It remains to be shown that \( F : C \to C \) is continuous. Let \( \{x_n\}_0^\infty \subseteq C \) with \( x_n \to x \) in \( C \). Then there is an integer \( \tau \) such that \( \|x_n(t)\| \leq \tau \) for all \( n \) and \( t \in J \), so \( x_n \in B_\tau \) and \( x \in B_\tau \). By \([C_7]\),

\[
f(t,x_n(\sigma_1(t)), \int_0^t l(t,s)g(s,x_n(\sigma_2(s)))ds) \to f(t,x(\sigma_1(t)), \int_0^t l(t,s)g(s,x(\sigma_2(s)))ds)
\]
for each $t \in J$ and since

$$|f(t, x_n(\sigma_1(t)), \int_0^t l(t, s)g(s, x_n(\sigma_2(s)))ds) - f(t, x(\sigma_1(t)), \int_0^t l(t, s)g(s, x(\sigma_2(s)))ds)| \leq 2\alpha_r(t),$$

we have, by dominated convergence theorem,

$$||Fx_n - Fx||$$

$$= \sup_{t \in J} \left| \int_0^t E^{-1}T(t - \eta)BW^{-1}[\int_0^b T(b - s)$$

$$[f(s, x_n(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x_n(\sigma_2(\tau)))d\tau)$$

$$- f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x(\sigma_2(\tau)))d\tau)](\eta)d\eta$$

$$+ \int_0^t E^{-1}T(t - s)[f(s, x_n(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x_n(\sigma_2(\tau)))d\tau)$$

$$- f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x(\sigma_2(\tau)))d\tau)]ds||$$

$$\leq \int_0^b ||E^{-1}||M_1M_2M_3$$

$$[M_1 \int_0^b ||f(s, x_n(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x_n(\sigma_2(\tau)))d\tau)$$

$$- f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x(\sigma_2(\tau)))d\tau)||d\eta]$$

$$+ \int_0^b ||E^{-1}||M_1[f(s, x_n(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x_n(\sigma_2(\tau)))d\tau)$$

$$- f(s, x(\sigma_1(s)), \int_0^s l(s, \tau)g(\tau, x(\sigma_2(\tau)))d\tau)]ds \to 0 \text{ as } n \to \infty.$$}

Thus $F$ is continuous. This completes the proof that $F$ is completely continuous.

Finally the set $\zeta(F) = \{x \in C : x = \lambda Fx, \ \lambda \in (0, 1)\}$ is bounded, as we proved in the first step. Consequently, by Schaefer’s theorem, the operator $F$ has a fixed point in $C$. This means that any fixed point of $F$ is a mild solution of (6.5) on $J$ satisfying $(Fx)(t) = x(t)$. Thus the system (6.5) is controllable on $J$.

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