5.1. INTRODUCTION

Controllability of Sobolev-type differential systems in Banach spaces was first studied by Han [38] by means of the Schauder fixed point theorem. Brill [12] and Showalter [69] established the existence of solutions of semilinear evolution equations of Sobolev-type in Banach spaces and Han [38] studied the existence of solutions of Sobolev-type integrodifferential equations in Banach spaces. In this chapter we establish sufficient conditions for the controllability of nonlinear Sobolev-type integrodifferential systems in Banach spaces by using the Schaefer fixed point theorem. Motivation for this type of systems is found in [11,16,17,71]. The results generalize those of [38].

5.2. PRELIMINARIES

Consider the Sobolev-type nonlinear integrodifferential system of the form

\[
(Ex(t))' + Ax(t) = (Bu)(t) + f(t, x(t), \int_0^t g(t, s, x(s))ds), \quad t \in J = [0, b],
\]

\[
x(0) = x_0,
\]

where the state \(x(\cdot)\) takes values in a Banach space \(X\) and the control function \(u(\cdot)\) is given in \(L^2(J, U)\), a Banach space of admissible control functions with \(U\) as a Banach space, \(B\) is a bounded linear operator from \(U\) into a Banach space \(Y\), \(g: J \times J \times X \rightarrow X\) and \(f: J \times X \times X \rightarrow Y\) are given functions. The norm of \(X\) is denoted by \(\| \cdot \|\) and \(Y\) by \(| \cdot |\).

The operators \(A : D(A) \subseteq X \rightarrow Y\) and \(E : D(E) \subseteq X \rightarrow Y\) satisfy the hypotheses \((C_i)\) for \(i = 1, \ldots, 4:\)

\((C_1)\) \(A\) and \(E\) are closed linear operators

\((C_2)\) \(D(E) \subseteq D(A)\) and \(E\) is bijective

\((C_3)\) \(E^{-1} : Y \rightarrow D(E)\) is continuous

\((C_4)\) For each \(t \in [0, b]\) and for some \(\lambda \in \rho(-AE^{-1})\), the resolvent set of \(-AE^{-1}\), the resolvent \(R(\lambda, -AE^{-1})\) is a compact operator.
The hypotheses \((C_1), (C_2)\) and the closed graph theorem imply the boundedness of the linear operator \(AE^{-1} : Y \to Y\).

**Lemma [59].** Let \(S(t)\) be a uniformly continuous semigroup and let \(A\) be its infinitesimal generator. If the resolvent set \(R(\lambda : A)\) of \(A\) is compact for every \(\lambda \in \rho(A)\), then \(S(t)\) is a compact semigroup.

From the above fact, \(-AE^{-1}\) generates a compact semigroup \(T(t), t \geq 0,\) on \(Y\). Thus \(\max_{t \in J}|T(t)|\) is finite and so denote \(M_1 = \max_{t \in J}|T(t)|\).

\((C_5)\) The linear operator \(W : L^2(J, U) \to X\) defined by
\[
Wu = \int_0^b E^{-1}(b-s)Bu(s)ds
\]
has an invertible operator \(W^{-1} : X \to L^2(J, U) \setminus \ker W\) and there exist positive constants \(M_2, M_3\) such that \(|B| \leq M_2\) and \(|W^{-1}| \leq M_3\).

\((C_6)\) For each \((t, s) \in J \times J\), the function \(g(t, s, \cdot) : X \to X\) is continuous and for each \(x \in X\), the function \(g(\cdot, \cdot, x) : J \times J \to X\) is strongly measurable.

\((C_7)\) For each \(t \in J\), the function \(f(t, \cdot, \cdot) : X \times X \to Y\) is continuous and for each \(x, y \in X\), the function \(f(\cdot, x, y) : J \to Y\) is strongly measurable.

\((C_8)\) For every positive integer \(k\), there exists \(h_k \in L^1(J)\) such that, for almost all \(t \in J\),
\[
\sup_{\|x\| \leq k} \left| f(t, x(t), \int_0^t g(t, s, x(s))ds) \right| \leq h_k(t).
\]

\((C_9)\) There exists a continuous function \(m : J \times J \to [0, \infty)\) such that
\[
\|g(t, s, x)\| \leq m(t, s)\Omega(\|x\|), \quad t, s \in J, \quad x \in X,
\]
where \(\Omega : [0, \infty) \to (0, \infty)\) is a continuous nondecreasing function.

\((C_{10})\) There exists a continuous function \(p : J \to [0, \infty)\) such that
\[
|f(t, x, y)| \leq p(t)\Omega_0(\|x\| + \|y\|), \quad t \in J, \quad x, y \in X,
\]
where \(\Omega_0 : [0, \infty) \to (0, \infty)\) is a continuous nondecreasing function.

\((C_{11})\)
\[
\int_0^b \tilde{m}(s)ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)},
\]
where \( c = \|E^{-1}\|M_1|Ex_0| + Nb \), \( \hat{m}(t) = \max\{M_1\|E^{-1}\|p(t), m(t, t)\} \) and

\[
N = M_2M_0\|x_1\| + \|E^{-1}\|M_1|Ex_0|
+ M_1\|E^{-1}\| \int_0^b p(s)\Omega_0(\|x(s)\|) + \int_0^s m(s, \tau)\Omega(\|x(\tau)\|)d\tau)ds.
\]

Then the system (5.1) has a mild solution of the following form

\[
x(t) = E^{-1}T(t)Ex_0 + \int_0^t E^{-1}T(t-s)(Bu)(s)
+ f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds, \quad t \in J \quad (5.2)
\]

and \( E\tilde{x}(t) \in C([0, b]; Y) \cap C'((0, b); Y). \)

**Definition 5.1.** The system (5.1) is said to be **controllable** on the interval \( J \) if for every \( x_0 \in D(E), x_1 \in X \), there exists a control \( u \in L^2(J, U) \) such that the solution \( x(t) \) of (5.1) satisfies \( x(b) = x_1. \)

### 5.3. CONTROLLABILITY RESULT

**Theorem 5.1.** If the hypotheses \((C_1) - (C_{11})\) are satisfied, then the system (5.1) is controllable on \( J \).

**Proof.** Let \( C = C(J, X) \). Using the hypothesis \((C_3)\) for an arbitrary function \( x(\cdot) \), define the control

\[
u(t) = W^{-1}[x_1 - E^{-1}T(b)Ex_0 - \int_0^b E^{-1}T(b-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds](t).
\]

We shall now show that when using this control, the operator \( F : C \to C \) defined by

\[
(Fx)(t) = E^{-1}T(t)Ex_0 + \int_0^t E^{-1}T(t-\eta)BW^{-1}[x_1 - E^{-1}T(b)Ex_0
- \int_0^b E^{-1}T(b-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds](\eta)d\eta
+ \int_0^t E^{-1}T(t-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds
\]

has a fixed point. This fixed point is then a solution of equation (5.2).
Clearly \((Fx)(b) = x_1\), which means that the control \(u\) steers the Sobolev-type integrodifferential system from the initial state \(x_0\) to \(x_1\) in time \(b\), provided we can obtain a fixed point of the nonlinear operator \(F\).

In order to study the controllability problem of (5.1), we introduce a parameter \(\lambda \in (0, 1)\) and consider the following system

\[
(Ex(t))' + \lambda Ax(t) = \lambda (Bu)(t) + \lambda f(t, x(t), \int_0^t g(t, s, x(s))ds), \quad t \in J, (5.3)
\]

\[
x(0) = x_0.
\]

First we obtain \textit{a priori} bounds for the mild solution of the equation (5.3).

From

\[
x(t) = \lambda E^{-1}T(t)Ex_0 + \lambda \int_0^t E^{-1}T(t - \eta)BW^{-1}[x_1 - E^{-1}T(b)Ex_0 - \int_0^b E^{-1}T(b - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds](\eta)d\eta + \lambda \int_0^t E^{-1}T(t - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds,
\]

we have,

\[
\|
x(t)\| \leq \|E^{-1}\|M_1\|Ex_0\| + \int_0^t \|E^{-1}\|M_1M_2M_3\|x_1\| \|E^{-1}\|M_1\|Ex_0\|
\]

\[
+ \|E^{-1}\|M_1\int_0^b p(s)\Omega_0(\|x(s)\|) + \int_0^s m(s, \tau)\Omega(\|x(\tau)\|)d\tau)ds)d\eta
\]

\[
+ \|E^{-1}\|M_1\int_0^t p(s)\Omega_0(\|x(s)\|) + \int_0^s m(s, \tau)\Omega(\|x(\tau)\|)d\tau)ds
\]

\[
\leq \|E^{-1}\|M_1\|Ex_0\| + \int_0^t \|E^{-1}\|M_1Nds
\]

\[
+ \|E^{-1}\|M_1\int_0^t p(s)\Omega_0(\|x(s)\|) + \int_0^s m(s, \tau)\Omega(\|x(\tau)\|)d\tau)ds
\]

\[
\leq \|E^{-1}\|M_1\|Ex_0\| + \|E^{-1}\|M_1N + \|E^{-1}\|M_1\int_0^t p(s)\Omega_0(\|x(s)\|
\]

\[
+ \int_0^s m(s, \tau)\Omega(\|x(\tau)\|)d\tau)ds.
\]

Denoting by \(v(t)\), the right-hand side of the above inequality, we have \(c = v(0) = \|E^{-1}\|M_1\|Ex_0\| + Nb\), \(\|x(t)\| \leq v(t)\) and

\[
v'(t) = M_1\|E^{-1}\|p(t)\Omega_0(\|x(t)\|) + \int_0^t m(t, \tau)\Omega(\|x(\tau)\|)d\tau)
\]

\[
\leq M_1\|E^{-1}\|p(t)\Omega_0(v(t)) + \int_0^t m(t, \tau)\Omega(v(\tau))d\tau).
\]

49
Let
\[ w(t) = v(t) + \int_0^t m(t, \tau)\Omega(v(\tau))d\tau. \]
Then \( w(0) = v(0) = c, \quad v(t) \leq w(t), \)
and
\[ w'(t) = v'(t) + m(t, t)\Omega(v(t)) \leq M_1 \| E^{-1} \| p(t)\Omega_0(w(t)) + m(t, t)\Omega(w(t)) \]
\[ \leq \hat{m}(t)[\Omega_0(w(t)) + \Omega(w(t))]. \]

This implies
\[ \int_{w(0)}^{w(t)} \frac{ds}{\Omega_0(s) + \Omega(s)} \leq \int_0^k \hat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)}, \quad t \in J. \]

This inequality implies that there is a constant \( K \) such that \( w(t) \leq K \) and hence \( \| x(t) \| \leq K, t \in J \), where \( K \) depends only on \( b \) and on the functions \( \hat{m}, \Omega_0 \) and \( \Omega \).

Let \( B_k = \{ x \in C : \| x \| \leq k \} \) for some \( k \geq 1 \).

Next we must prove that the operator \( F : B_k \to B_k \) is a completely continuous operator.

We now show that the set \( \{ Fx : x \in B_k \} \) is equicontinuous. Let \( x \in B_k \) and \( t_1, t_2 \in J \). Then if \( 0 < t_1 < t_2 < b \),
\[ \| (Fx)(t_1) - (Fx)(t_2) \| \]
\[ \leq \| E^{-1} \| |T(t_1) - T(t_2)| |Ex_0| \]
\[ + \| \int_0^{t_1} E^{-1}[T(t_1 - \eta) - T(t_2 - \eta)]BW^{-1}[x_1 - E^{-1}T(b)Ex_0 \]
\[ - \int_0^b E^{-1}T(b - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)d\eta)d\eta| \]
\[ + \| \int_{t_1}^{t_2} E^{-1}T(t_2 - \eta)BW^{-1}[x_1 - E^{-1}T(b)Ex_0 \]
\[ - \int_0^b E^{-1}T(b - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)d\eta)d\eta| \]
\[ + \| \int_0^{t_1} E^{-1}[T(t_1 - s) - T(t_2 - s)]f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)d\sigma| \]
\[ + \| \int_0^{t_2} E^{-1}T(t_2 - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)d\sigma| \]
\[
\leq \|E^{-1}\| |T(t_1) - T(t_2)| \|Ex_0|
+ \int_{t_1}^{t_2} \|E^{-1}\| |T(t_1 - \eta) - T(t_2 - \eta)| \|x_1\| + \|E^{-1}\| \|M_1|Ex_0|
+ \|E^{-1}\| \|M_1\| \int_0^b h_k(s) ds d\eta
+ \int_{t_1}^{t_2} \|E^{-1}\| |T(t_2 - \eta)| \|M_2M_3\| \|x_1\| + \|E^{-1}\| \|M_1|Ex_0|
+ \|E^{-1}\| \|M_1\| \int_0^b h_k(s) ds d\eta
+ \int_{t_1}^{t_2} \|E^{-1}\| |T(t_1 - s) - T(t_2 - s)| h_k(s) ds
+ \int_{t_1}^{t_2} \|E^{-1}\| |T(t_2 - s)| h_k(s) ds.
\]

The right-hand side tends to zero as \(t_2 - t_1 \to 0\), since the compactness of \(T(t)\), for \(t > 0\), implies the continuity in the uniform operator topology.

Thus \(\{Fx : x \in B_k\}\) is equicontinuous. It is easy to see that the family \(FB_k\) is uniformly bounded.

Next we show \(FB_k\) is compact. Since we have shown \(FB_k\) is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem, to show \(\{(Fx)(t) : x \in B_k\}\) is precompact in \(X\) for any \(t \in [0, b]\).

Let \(0 < t \leq b\) be fixed and \(\epsilon\) a real number satisfying \(0 < \epsilon < t\). For \(x \in B_k\), we define

\[
(F_\epsilon x)(t) = E^{-1}T(t)Ex_0 + \int_0^{t-\epsilon} E^{-1}T(t - \eta)BW^{-1}|x_1 - E^{-1}T(b)Ex_0
- \int_0^b E^{-1}T(b - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds)(\eta) d\eta
+ \int_0^{t-\epsilon} E^{-1}T(t - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds.
\]

Since \(T(t)\) is a compact operator, the set \(Y_\epsilon(t) = \{(Fx)(t) : x \in B_k\}\) is precompact in \(X\) for every \(\epsilon\), \(0 < \epsilon < t\). Moreover for every \(x \in B_k\), we have

\[
\|(Fx)(t) - (F_\epsilon x)(t)\|
\leq \int_{t-\epsilon}^t \|E^{-1}T(t - \eta)BW^{-1}|x_1 - E^{-1}T(b)Ex_0
- \int_0^b E^{-1}T(b - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds)(\eta) d\eta
+ \int_{t-\epsilon}^t \|E^{-1}T(t - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds\| ds
\]
\[
\leq \int_{t-\varepsilon}^{t} \|E^{-1}\| M_1 M_2 M_3 \|x_1\| + \|E^{-1}\| M_1 |E x_0|
\]
\[
+ \|E^{-1}\| M_1 \int_{0}^{b} h_k(s) ds d\eta + \int_{t-\varepsilon}^{t} \|E^{-1}\| M_1 h_k(s) ds.
\]

Therefore there are precompact sets arbitrarily close to the set \{(Fx)(t) : x \in B_k\}. Hence the set \{(Fx)(t) : x \in B_k\} is precompact in \(X\).

It remains to be shown that \(F : C \to C\) is continuous. Let \(\{x_n\}_{n=0}^{\infty} \subseteq C\) with \(x_n \to x\) in \(C\). Then there is an integer \(r\) such that \(\|x_n(t)\| \leq r\) for all \(n\) and \(t \in J\), so \(x_n \in B_r\) and \(x \in B_r\). By \((C_I)\),

\[
f(t, x_n(t), \int_{0}^{t} g(t, s, x_n(s)) ds) \to f(t, x(t), \int_{0}^{t} g(t, s, x(s)) ds) \quad \text{for each} \quad t \in J,
\]
and since

\[
|f(t, x_n(t), \int_{0}^{t} g(t, s, x_n(s)) ds) - f(t, x(t), \int_{0}^{t} g(t, s, x(s)) ds)| \leq 2h_r(t),
\]
we have, by dominated convergence theorem,

\[
\|F x_n - F x\| = \sup_{t \in J} \| \int_{t}^{t} E^{-1} T(t - \eta) BW^{-1} \left( \int_{0}^{b} T(b - s) \right) [f(s, x_n(s), \int_{0}^{s} g(s, r, x_n(r)) dr) - f(s, x(s), \int_{0}^{s} g(s, r, x(r)) dr)](\eta) d\eta
\]
\[
+ \int_{0}^{t} E^{-1} T(t - s)[f(s, x_n(s), \int_{0}^{s} g(s, r, x_n(r)) dr) - f(s, x(s), \int_{0}^{s} g(s, r, x(r)) dr)] ds
\]
\[
\leq \int_{0}^{b} \|E^{-1}\| M_1 M_2 M_3 \left[ \int_{0}^{b} [f(s, x_n(s), \int_{0}^{s} g(s, r, x_n(r)) dr) - f(s, x(s), \int_{0}^{s} g(s, r, x(r)) dr)] d\eta
\]
\[
+ \int_{0}^{b} \|E^{-1}\| M_1 [f(s, x_n(s), \int_{0}^{s} g(s, r, x_n(r)) dr) - f(s, x(s), \int_{0}^{s} g(s, r, x(r)) dr)] ds \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus \(F\) is continuous. This completes the proof that \(F\) is completely continuous.

Finally, the set \(\zeta(F) = \{x \in C : x = \lambda F x, \lambda \in (0, 1)\}\) is bounded, as we proved in the first step. Consequently by Schaefer’s theorem the operator \(F\)
has a fixed point in $C$. This means that any fixed point of $F$ is a mild solution of (5.1) on $J$ satisfying $(Fx)(t) = x(t)$. Thus the system (5.1) is controllable on $J$.

5.4. EXAMPLE

Consider the following partial integrodifferential equation of the form

$$\frac{\partial}{\partial t}(z(t, x) - z_{xx}(t, x)) - z_{xx}(t, x) = Bu(t) + \mu_1(t, z(t, x), \int_0^t \mu_2(t, s, z(s, x))ds)$$

(5.4)

with

$$z(t, 0) = z(t, 1) = 0, \quad z(0, x) = z_0(x), \quad 0 < x < 1, \quad t \in J.$$

Assume that the following conditions hold with $X = Y = L^2[0, 1]$.

[A1] The operator $B : U \to Y$, with $U \subset J$, is a bounded linear operator.

[A2] The linear operator $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^b E^{-1}T(b - s)Bu(s)ds$$

has a bounded invertible operator $W^{-1}$ which takes values in $L^2(J, U) \setminus \ker W$.

[A3] Further the functions

$$\mu_2 : J \times J \times X \to X, \quad \mu_1 : J \times X \times X \to Y$$

are all continuous, bounded and strongly measurable.

[A4] Let $g(t, s, w)(x) = \mu_2(t, s, w(x))$ and $f(t, w, \sigma)(x) = \mu_1(t, w(x), \sigma(x))$.

Define the operators $A : D(A) \subset X \to Y, \quad E : D(E) \subset X \to Y$ by

$$Aw = -w'', \quad Ew = w - w''$$

respectively, where each domain $D(A), D(E)$ is given by

$$\{w \in X, w, w' \text{ are absolutely continuous, } w'' \in X, w(0) = w(1) = 0\}.$$
Then $A$ and $E$ can be written respectively as
\[
Aw = \sum_{n=1}^{\infty} n^2(w, w_n)w_n, \quad w \in D(A),
\]
\[
Ew = \sum_{n=1}^{\infty} (1 + n^2)(w, w_n)w_n, \quad w \in D(E),
\]
where $w_n(x) = \sqrt{2}\sin nx$, $n = 1, 2, 3, \ldots$, is the orthogonal set of eigenvectors of $A$. Furthermore for $w \in X$, we have
\[
E^{-1}w = \sum_{n=1}^{\infty} \frac{1}{1 + n^2}(w, w_n)w_n,
\]
\[
-AE^{-1}w = \sum_{n=1}^{\infty} \frac{-n^2}{1 + n^2}(w, w_n)w_n,
\]
\[
T(t)w = \sum_{n=1}^{\infty} e^{-\frac{n^2}{1 + n^2}t}(w, w_n)w_n.
\]
It is easy to see that $-AE^{-1}$ generates a strongly continuous semigroup $T(t)$ on $Y$ and $T(t)$ is compact such that $\|T(t)\| \leq e^{-t}$ for each $t > 0$.

[A5] The functions $\mu_1$ and $\mu_2$ satisfy the following conditions:
(i) There exists a continuous function $q : J \times J \to [0, \infty)$ such that
\[
\|\mu_2(t, s, w)\| \leq q(t, s)\Omega_2(|w|),
\]
where $\Omega_2 : [0, \infty) \to (0, \infty)$ is continuous and nondecreasing.

(ii) There exists a continuous function $l : J \times \to [0, \infty)$ such that
\[
|\mu_1(t, w, \mu_2)| \leq l(t)\Omega_3(|w|),
\]
where $\Omega_3 : [0, \infty) \to (0, \infty)$ is continuous and nondecreasing.

Also we have
\[
\int_0^b \hat{n}(s)ds < \int_c^{\infty} \frac{ds}{\Omega_2(s) + \Omega_3(s)},
\]
where $c = \|E^{-1}\|e^{-t}[|Ex_0| + Nb]$, and $\hat{n}(t) = \max\{e^{-t}\|E^{-1}\|l(t), q(t, t)\}$. Here $N$ depends on $E$, $A$, $B$, $\mu_1$, and $\mu_2$. Further all the conditions stated in Theorem 5.1. are satisfied. Hence the system (5.4) is controllable on $J$. 

\*

54