7. Lie Group Analysis of Natural Convection Heat Transfer in an Inclined Surface with Variable Thermal Conductivity

7.1 Introduction

Most of the existing analytical studies for the problem considered are based on the constant physical properties of the fluid. The process of heat transfer is also affected by physical properties of the fluid. These properties have definite magnitudes for each substance and some are functions of pressure and temperature. It has become a practice in the field of heat transfer to modify relations which have been developed for a constant property fluid in such a way that they also account for the effects caused by property changes. Here the fluid thermal conductivity is assumed to vary as a linear function of the temperature. Prasad et al. (2000) investigated the visco-elastic fluid flow and heat transfer in a porous medium over a non-isothermal stretching sheet. Thermal conductivity of the fluid is varying linearly with temperature. They considered two different cases, namely, prescribed surface temperature and prescribed heat flux. They find that the effect of the suction is to decrease the velocity and that of blowing is to increase the velocity in the flow field. The effect of porosity is to decrease the velocity in the boundary layer in both the cases of blowing and suction. The effect of the porosity is to increase the wall temperature gradient when there is suction and impermeability of the wall. The effect of increasing the visco-elastic parameter is to decrease the temperature profile.

The steady laminar boundary layer flow of water with an external force along a vertical isothermal plate is studied by Pantokratoras (2004). The external force may be produced either by the motion of the plate or by a free stream. He also consider non-linear density-temperature relationship and viscosity and thermal conductivity as functions of temperature. It is found that the wall heat transfer and the wall shear stress increase as the buoyancy parameter increases. The wall heat transfer of the moving plate is greater than that corresponding to the free stream case. Most of the existing studies on convective boundary layer flow problem are based on the constant
physical properties of the fluid. However it is known that these physical properties change with temperature. To accurately predict the fluid flow and heat transfer, it is necessary to take into account this variation of physical properties. This chapter describes the effect of thermal conductivity on natural convection heat transfer and fluid flow past an inclined surface using Lie group analysis.

7.2 Mathematical Analysis

Consider the natural convection heat transfer in laminar boundary layer flow of an incompressible viscous fluid along an inclined semi-infinite surface with an acute angle $\alpha$ from the vertical. The surface is maintained at a constant temperature $T_w$ which is higher than the constant temperature $T_\infty$ of the surrounding fluid. The fluid properties are assumed to be constant except the thermal conductivity. The governing equations of the mass, momentum and energy for the steady flow can be written as

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{7.1}
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty) \cos \alpha, \tag{7.2}
\]

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\partial}{\partial y} \left( k(T) \frac{\partial T}{\partial y} \right), \tag{7.3}
\]

with the boundary conditions

\[
u = v = 0, \quad T = T_w, \quad \text{at } y = 0,
\]

\[
u = 0, \quad T = T_\infty, \quad \text{as } y \to \infty. \tag{7.4}
\]

The thermal conductivity of the fluid $k$ is assumed to vary linearly with the temperature as

\[k(T) = k_o[a + b(T_w - T)],\]

where $k_o$ is the constant value of the coefficient of thermal conductivity away from the plate and $a$, $b$ are constants and $b > 0$.  

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The nondimensional variables are

\[ \bar{x} = \frac{x U_\infty}{\nu}, \quad \bar{y} = \frac{y U_\infty}{\nu}, \quad \bar{u} = \frac{u}{U_\infty}, \quad \bar{v} = \frac{v}{U_\infty}, \quad \theta = \frac{T - T_\infty}{T_w - T_\infty}. \]  

(7.5)

Substituting (7.5) into equations (7.1)-(7.4) and dropping the over bars, we obtain,

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  

(7.6)

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + Gr \theta \cos \alpha \]  

(7.7)

\[ u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{1}{Pr} \left[ (\alpha + A(1 - \theta)) \frac{\partial^2 \theta}{\partial y^2} - A \left( \frac{\partial \theta}{\partial y} \right)^2 \right]. \]  

(7.8)

with the boundary conditions

\[ u = v = 0, \quad \theta = 1, \quad \text{at} \ y = 0, \]

\[ u = 0, \quad \theta = 0, \quad \text{as} \ y \to \infty. \]  

(7.9)

7.3 Symmetry Groups of Equations

The symmetry groups of equations (7.6)-(7.8) are calculated using classical Lie group approach, see Bluman and Kumei (1989). The one-parameter infinitesimal Lie group of transformations leaving (7.6)-(7.8) invariant is assumed as

\[ x^* = x + \epsilon \xi_1(x, y, u, v, \theta) \]

\[ y^* = y + \epsilon \xi_2(x, y, u, v, \theta) \]

\[ u^* = u + \epsilon \eta_1(x, y, u, v, \theta) \]

\[ v^* = v + \epsilon \eta_2(x, y, u, v, \theta) \]

\[ \theta^* = \theta + \epsilon \eta_3(x, y, u, v, \theta). \]

(7.10)

By carrying out a straightforward but tedious algebra, we finally obtain the form of the infinitesimals as

\[ \xi_1 = 2c_1 x - c_2 \]

\[ \xi_2 = \frac{1}{2} c_1 y - f(x) \]  

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\eta_1 = c_1 u \\
\eta_2 = uf'(x) - \frac{1}{2} c_1 v \\
\eta_3 = 0. \tag{7.11}

Imposing the restrictions from boundaries and from the boundary conditions on the infinitesimals, we obtain the following form for equations (7.11):

\begin{align*}
\xi_1 &= 2c_1 x - c_2 \\
\xi_2 &= \frac{1}{2} c_1 y \\
\eta_1 &= c_1 u \\
\eta_2 &= -\frac{1}{2} c_1 v \\
\eta_3 &= 0, \tag{7.12}
\end{align*}

where the parameters \( c_1 \) represents the scaling transformation and parameter \( c_2 \) represents translation in the \( x \) coordinate.

### 7.4 Reduction to Ordinary Differential Equations

In this section, parameter \( c_1 \) is taken to be arbitrary and all other parameters are zero in (7.12). The characteristic equations are

\[
\frac{dx}{2x} = \frac{dy}{(1/2)y} = \frac{du}{u} = \frac{dv}{-(1/2)v} = \frac{d\theta}{0}, \tag{7.13}
\]

from which the similarity variables, the velocities and temperature turn out to be of the form

\[
\eta = x^{-1/4} y, \quad u = x^{1/2} F_1(\eta), \quad v = x^{-1/4} F_2(\eta), \quad \theta = F_3(\eta). \tag{7.14}
\]

Substituting (7.18) into equations (7.6)-(7.8), we finally obtain the system of nonlinear ordinary differential equations

\[
\begin{align*}
F_1'' &= \frac{1}{2} F_1^2 - \frac{1}{4} \eta F_1 F_1' + F_2 F_1' - Gr F_3 \cos \alpha \\
F_2' &= \frac{1}{4} \eta F_1' - \frac{1}{2} F_1 \\
F_3'' &= \frac{Pr}{\left[a + A(1 - F_3)\right]} \left(F_2 F_3' - \frac{1}{4} \eta F_1 F_3' + \Delta F_2^2\right). \tag{7.15}
\end{align*}
\]
The appropriate boundary conditions are expressed as

\[ F_1 = F_2 = 0, \quad F_3 = 1, \quad \text{at } \eta = 0, \]

\[ F_1 = 0, \quad F_3 = 0, \quad \text{as } \eta \to \infty. \]  

(7.16)

7.5 Numerical methods for solutions

Since the equations are highly nonlinear, a numerical treatment would be more appropriate. The system of transformed equations (7.15) together with the boundary conditions (7.16) is numerically solved by employing a fourth order Runge-Kutta method and shooting techniques with a systematic guessing of \( F'(0) \) and \( F''(0) \). The procedure is repeated until we get the results up to the desired degree of accuracy, namely \( 10^{-5} \). A code is written in MATHEMATICA [Wolfram (1999)] package and solutions are presented graphically.

7.6 Results and discussions

Numerical solutions are obtained for various values of the Prandtl number, Grashof number and the thermal conductivity parameter. The Prandtl number \( Pr \) is varied from 0.71 (air) to 13.67 (water), the Grashof number \( Gr \) from 0.1 to 3 and the thermal conductivity parameter \( A \) from 0 to 1 with the angle of inclination \( \alpha \) taking the values 0°, 30° and 45°. The numerical results are depicted graphically in the form of velocity and temperature profiles. Most of the investigations are carried out for \( \alpha = 45^\circ \). Some results are taken for \( \alpha = 0^\circ \) (vertical plate case) and 30°.

Figures 7.1(a and b) show the velocity and the temperature profiles for different values of the thermal conductivity parameters with \( Pr = 0.71 \) corresponding to the fluid, air and \( Gr = 1 \). Here it is found from Figure 7.1(a) that the velocity increases slightly as the thermal conductivity parameter \( A(= 0, 0.5, 1) \) increases in the region \( y \in [0, 15] \), but near the surface it increases to reach a maximum and then decreases to finally approach zero. Clearly it is seen that there is no considerable effect on the thermal boundary layer when changes are made in the thermal conductivity parameter. Figures 7.2(a and b) show the velocity and temperature distributions for the
same governing parameters as in Figure 7.1 except the Grashof number \((Gr = 3)\). Comparing these two figures, there is no considerable change in the velocity and temperature distributions.

The temperature and velocity distributions for different values of the Prandtl number with \(Gr = 1, A = 0.1\) are presented in Figures 7.3 (a and b). From Figure 7.3 (a), it is clear that the velocity decreases by increasing the Prandtl number. The effect of the Prandtl number is a very important property in the temperature profile. The temperature distribution decreases with increase in the Prandtl number. This is in agreement with the physical fact that the thermal boundary layer thickness decreases with increase in the Prandtl number. Figures 7.4(a and b) depict the velocity and temperature distributions for various Grashof numbers with \(Pr = 0.71\) and \(A = 1\). It is seen from the Figure that the velocity increases along the boundary layer with the Grashof number. Since the favourable buoyancy force accelerates the fluid in the boundary layer, the velocity increases and suddenly falls, near the boundary layer. The temperature profiles decrease with the Grashof number; consequently the thermal boundary layer thickness is reduced.

The effect of the inclination of the surface for different parameters is depicted in Figure 7.5. For a fixed value of the thermal conductivity parameter, the velocity decreases with the angle of inclination \(\alpha\). But the fluid has higher velocity when the surface is vertical than when it is inclined because of the fact that the buoyancy force decreases due to gravity components \((g \cos \alpha)\) as the plate is inclined. The behaviour of thermal boundary layer is displayed in Figure 7.5(b). Here the temperature distribution along the boundary layer is compared with that for the vertical plate, that is, \(\alpha = 0\). Temperature distribution along the boundary layer is better when the surface is inclined than when it is vertical. It is observed that the increase in the thermal conductivity parameter increases both velocity and temperature distributions for all angles. Consequently both momentum and thermal boundary layer thicknesses are increased.
Fig. 7.1. The velocity and temperature profiles for Pr = 0.71 and Gr = 1.

Fig. 7.2. The velocity and temperature profiles for Pr = 0.71 and Gr = 3.
Fig. 7.3. The velocity and temperature profiles for $Gr = 1$ and $A = 0.1$.

Fig. 7.4. The velocity and temperature profiles for $Pr = 0.71$ and $A = 1$. 

Fig. 7.3. The velocity and temperature profiles for $Gr = 1$ and $A = 0.1$.

Fig. 7.4. The velocity and temperature profiles for $Pr = 0.71$ and $A = 1$. 

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Fig. 7.5. The velocity and temperature profiles for $Pr = 0.71$ and $Gr = 1$. 