4. Lie Group Analysis of Natural Convection Heat and Mass Transfer in an Inclined Surface

4.1 Introduction

The heat, mass and momentum transport on a surface has several applications in design of chemical processing equipment, formation and dispersion of fog, distribution of temperature and moisture over agricultural fields and groves of fruit trees, damage of crops due to freezing, food processing and cooling towers. Murthy (2000) presented the similarity solution for the problem of hydrodynamic dispersion in mixed convection heat and mass transfer from vertical surface embedded in porous medium. He analyzed the heat and mass transfer in the boundary layer region for aiding and opposing buoyancies. It is found that the heat transfer coefficient always increases with the flow driving parameter. The Lewis number has a complex impact on the heat and mass transfer mechanism. As Lewis number increases, the effect of solutal dispersion on the non dimensional mass transfer coefficient becomes less predictable in both aiding and opposing buoyancies.

Chen (2004) studied the momentum, heat and mass transfer characteristics of magnetohydrodynamic natural convection flow over a permeable, inclined surface with variable wall temperature and concentration with the effects of ohmic heating and viscous dissipation. Power-law temperature and concentration are considered at the inclined surface. It is found that the velocity, the local Nusselt number and local Sherwood number are decreased in the presence of a magnetic field. Increasing the angle of inclination has the effect to decrease the local friction coefficient, the Nusselt number and the Sherwood number. The viscous dissipation effect shows a considerable reduction in the heat transfer rate. The local Nusselt number is increased by increasing the values of the Prandtl number. The local Nusselt number and Sherwood number are increased when suction is present at the permeable wall whereas the opposite is true for the case of injection. This Chapter describes the natural convection heat and mass transfer flow past an inclined plate for various parameters using Lie group analysis.
4.2 Mathematical Analysis

Consider the steady natural convection in laminar boundary layer flow and mass transfer of a viscous incompressible fluid past a heated semi-infinite inclined plate with an acute angle \( \alpha \) from the vertical. The temperature and species concentration at the surface are \( T_w \) and \( C_w \), where \( T_\infty \) and \( C_\infty \) are the temperature and the species concentration of the free stream. All the fluid properties are assumed to be constant. The governing equations of the mass, momentum, energy and concentration for the steady flow can be written as,

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} &= \nu \frac{\partial^2 u}{\partial y^2} + g \beta (T - T_\infty) \cos \alpha - g \beta^* (C - C_\infty) \cos \alpha, \\
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2}, \\
\frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} &= \frac{D}{\partial^2 C}{\partial y^2}.
\end{align*}
\]

The boundary conditions are

\[
\begin{align*}
u = v &= 0, & T &= T_w, & C &= C_w & \text{at } y = 0, \\
u &= u = 0, & T &= T_\infty, & C &= C_\infty & \text{as } y \to \infty.
\end{align*}
\]

The dimensionless quantities are

\[
\bar{x} = \frac{x U_\infty}{\nu}, \quad \bar{y} = \frac{y U_\infty}{\nu}, \quad \bar{u} = \frac{u}{U_\infty}, \quad \bar{v} = \frac{v}{U_\infty}, \quad \bar{T} = \frac{T - T_\infty}{T_w - T_\infty}, \quad \bar{C} = \frac{C - C_\infty}{C_w - C_\infty}. \tag{4.6}
\]

Substituting (4.6) into equations (4.1)-(4.4) and dropping the bars, we obtain,

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \tag{4.7} \\
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{\partial^2 u}{\partial y^2} + Gr \theta \cos \alpha - Gc \phi \cos \alpha, \tag{4.8} \\
\frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} &= \frac{1}{Pr} \frac{\partial^2 \theta}{\partial y^2}, \tag{4.9} \\
\frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} &= \frac{1}{Sc} \frac{\partial^2 \phi}{\partial y^2}. \tag{4.10}
\end{align*}
\]
The boundary conditions are expressed as

\[
\begin{align*}
    u &= v = 0, \quad \theta = 1, \quad \phi = 1 \quad \text{at } y = 0, \\
    u &= 0, \quad \theta = 0, \quad \phi = 0 \quad \text{as } y \to \infty.
\end{align*}
\]  

(4.11)

4.3 Symmetry Groups of Equations

The symmetry groups of equations (4.7)-(4.10) are calculated using classical Lie group approach, see Bluman and Kumei (1989). The one-parameter infinitesimal Lie group of transformations leaving (4.7)-(4.10) invariant is assumed as

\[
\begin{align*}
    x^* &= x + \epsilon_1(x, y, u, v, \theta, \phi) \\
    y^* &= y + \epsilon_2(x, y, u, v, \theta, \phi) \\
    u^* &= u + \epsilon_3(x, y, u, v, \theta, \phi) \\
    v^* &= v + \epsilon_4(x, y, u, v, \theta, \phi) \\
    \theta^* &= \theta + \epsilon_5(x, y, u, v, \theta, \phi) \\
    \phi^* &= \phi + \epsilon_6(x, y, u, v, \theta, \phi).
\end{align*}
\]  

(4.12)

Carrying out the straightforward algebra, we obtain the following results

\[
\begin{align*}
    \xi_1 &= 2c_1x - c_2x - c_3 \\
    \xi_2 &= \frac{1}{2}c_1y - \frac{1}{2}c_2y - f(x) \\
    \eta_1 &= c_1u \\
    \eta_2 &= -uf'(x) - \frac{1}{2}c_1v + \frac{1}{2}c_2v \\
    \eta_3 &= c_3\theta - \frac{gc}{Gr}c_4 \\
    \eta_4 &= c_2\phi + c_4.
\end{align*}
\]  

(4.13)

Imposing the restrictions from boundaries and from the boundary conditions on the infinitesimals, we obtain the following form for equations (4.13):

\[
\begin{align*}
    \xi_1 &= 2c_1x - c_2x - c_3 \\
    \xi_2 &= \frac{1}{2}c_1y - \frac{1}{2}c_2y
\end{align*}
\]
where the parameters $c_1$ and $c_2$ represent the scaling transformations and parameter $c_3$ represents translation in the $x$ coordinate.

4.4 Reductions to Ordinary Differential Equations

In this section, parameter $c_1$ is taken to be arbitrary and all other parameters are zero in (4.14).

The characteristic equations are

$$\frac{dx}{2x} = \frac{dy}{(1/2)y} = \frac{du}{u} = \frac{dv}{(-1/2)v} = \frac{d\theta}{0} = \frac{d\phi}{0},$$

from which the similarity variable, the velocities, the temperature and the concentration turn out to be of the form

$$\eta = x^{-1/4}, \quad u = x^{1/2}F_1(\eta), \quad v = x^{-1/4}F_2(\eta), \quad \theta = F_3(\eta), \quad \phi = F_4(\eta).$$

Substituting (4.16) into equations (4.7)-(4.10), we finally obtain the system of nonlinear ordinary differential equations

$$F_1'' = \frac{1}{2}F_1^2 - \frac{1}{4}\eta F_1 F_1' + F_2 F_1' - Gr F_3 \cos \alpha + Gc F_4 \cos \alpha,$$

$$F_2' = \frac{1}{4}\eta F_1' - \frac{1}{2}F_1,$$

$$F_3'' = Pr (F_2 F_3' - \frac{1}{4}\eta F_1 F_3'),$$

$$F_4'' = Sc (F_2 F_4' - \frac{1}{4}\eta F_1 F_4').$$

The appropriate boundary conditions are expressed as

$$F_1 = F_2 = 0, \quad F_3 = 1, \quad F_4 = 1 \quad \text{at } \eta = 0,$$

$$F_1 = 0, \quad F_3 = 0, \quad F_4 = 0 \quad \text{as } \eta \to \infty.$$
4.5 Numerical methods for solutions

Since the equations are highly nonlinear, a numerical treatment would be more appropriate. The system of transformed equations (4.17) together with the boundary conditions (4.18) is numerically solved by employing a fourth order Runge-Kutta method and shooting techniques with a systematic guessing of \( F'_L(0), F'_S(0) \) and \( F'_T(0) \). The procedure is repeated until we get the results up to the desired degree of accuracy, namely \( 10^{-5} \). A code is written in MATHEMATICA package [Wolfram (1999)] and solutions are presented graphically.

4.6 Results and Discussions

Numerical solutions are presented for different values of the Prandtl number, thermal Grashof number, solutal Grashof number and Schmidt number. The Prandtl number \( Pr \) is varied from 0.1 to 13.67, the thermal Grashof number \( Gr \) from 0.1 to 2.5, the solutal Grashof number \( Gc \) from 0.01 to 0.1 and the Schmidt number \( Sc \) from 1 to 10 with the angle of inclination \( \alpha \) taking the values 0°, 30° and 45°. The numerical results are presented graphically in the form of velocity, temperature and concentration profiles. Most of the investigations are carried out for \( \alpha = 45^\circ \). Some results are found for \( \alpha = 0^\circ \) (vertical plate case) and 30°.

Figures 4.1(a-c) show the effect of Schmidt number on the velocity, temperature and concentration of the boundary layer for \( Pr = 0.1, Gr = Gc = 0.1 \). It is clearly seen that the velocity is increased by increasing the Schmidt number. The thickness of the temperature and concentration boundary layer is decreased. Figures 4.2(a-c) show the velocity, temperature and concentration profiles for \( Pr = 0.71, Gr = 0.1 \) and \( Sc = 1 \). Increasing the solutal Grashof number decreases the velocity whereas it increases the temperature and concentration.

The effect of the thermal Grashof number on heat and mass fluid flow behaviour is depicted in Figures 4.3(a-c). It is found that the velocity increases rapidly and suddenly falls near the boundary due to favourable buoyancy force. The thermal and solutal boundary layer thicknesses are monotonically decreased on increasing the thermal Grashof number. In the presence of uniform Schmidt number it is seen that
the increase in the Prandtl number leads to a fall in the velocity and temperature of the fluid and a rise in the concentration of the fluid along the inclined surface as shown in Figures 4.4(a-c).

The effect of inclination of the surface for different parameters is depicted in Figures 4.5 (a-c). At a fixed value of the Schmidt number, the velocity is decreased for all angles. The fluid has higher velocity when the surface is vertical than when inclined because of the fact that the buoyancy effect decreases due to gravity components \((g \cos \alpha)\) as the plate is inclined. For a fixed value of the Schmidt number, the fluid has higher temperature when \(\alpha = 30^\circ\). Increasing the Schmidt number decreases the temperature and the concentration of the fluid along the surface. The inclination angle \(\alpha = 30^\circ\) gives the enhanced heat and mass distribution of the convective fluid.
Fig. 4.1. The velocity, temperature and concentration profiles for $Pr = 0.1$, $Gr = 0.1$ and $Gc = 0.1$
Fig. 4.2. The velocity, temperature and concentration profiles for $Pr = 0.71$, $Gr = 0.1$ and $Sc = 1$.
Fig. 4.3. The velocity, temperature and concentration profiles for $Pr = 0.71$, $Gr = 0.1$ and $Sc = 1$. 
Fig. 4.4. The velocity, temperature and concentration profiles for $Gr = 1, Gc = 0.1$ and $Sc = 1$.
Fig. 4.5. The velocity, temperature and concentration profiles for $Pr = 0.71$, $Gr = 0.1$ and $Gc = 0.1$