CHAPTER II

STEADY STATE ANALYSIS OF A NON-MARKOVIAN BULK QUEUEING SYSTEM WITH REPAIR OF SERVICE STATION ON REQUEST AND MULTIPLE VACATIONS WITH EXCEPTIONAL LAST VACATION

Application of vacation models with additional service can be found in production line systems. There are queueing models in which a leaving customer may be dissatisfied with the low quality of service rendered and he may complain and request that the station be adjusted to its proper level of service. In some queueing models, a leaving customer may request for some additional service. A batch arrival queue with N-policy and single vacation is analysed by Lee et al.[61]. A $M^X/G/1$ queue with N-policy and multiple vacations is analysed by Lee et al.[60], in which, the arrival occur in bulk but service is done, one at a time. Krishna Reddy et al.[49] analysed a bulk queue with N-policy multiple vacations and setup times. Thangaraj [97] studied a queue with a pair of instantaneous independent Bernoulli feedback processes. Arumuganathan R. [4] analysed $M^X/G(a,b)/1$ queueing model with multiple vacations and repair of service station.

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This chapter deals with the analysis of a $M^X/G(a,b)/1$ queueing system with repair of service station on request and multiple vacations with exceptional last vacation. The leaving batch of customers may request for repair of service station with probability $\pi$ and it is assumed that more than one request will never be made by the same batch. After the repair time or service completion without request for repair, if the queue length is less than ‘$a$’, then the server avails multiple vacation till the queue length reaches ‘$a$’. After a vacation, if the server finds at least ‘$a$’ customers waiting for service (say) $\xi$, then the server requires an exceptional last vacation to start the service. After this exceptional last vacation, the server serves a batch of min $(\xi,b)$ customers, where $b \geq a$.

![Schematic Representation of the Model](image)

**Figure 2.1 Schematic Representation of the Model**

$Q$ – Queue length
A practical situation of this model occurs in machining of castings in CNC turning machines. A technician operates a CNC machine to machine a minimum of 200 castings to a maximum of 250 castings in a batch. If there are few castings in this batch which are found with run out, he is allowed to process the same with some additional service to remove the run out and complete the machining. After completion of service, if he finds that insufficient batch quantity is available, he can carry out preventive maintenance on the machine. After completing this (multiple) vacation, if the required number of quantity in a batch is available, the service starts always with an exceptional last vacation, namely, operating the machine manually to move the X axis and Z axis to their home position before starting the machining cycle.

For the model proposed using supplementary variable technique, the probability generating function of the steady state queue size at an arbitrary time is obtained. Expressions for the expected length of the queue, expected length of busy and idle periods are derived. Expected waiting time in the queue is also obtained. A cost model of the queueing system is developed. Numerical illustration is presented.
2.1 Mathematical Model

Let \( \lambda \) be the Poisson arrival rate. \( X \) be the group size random variable of the arrival, \( g_k \) be the probability that \( k \) customers arrive in a batch and \( X(\mathbf{z}) \) be the probability generation function of \( X \). Let \( G(\cdot) \), \( V(\cdot) \), \( R(\cdot) \) and \( S(\cdot) \) be the cumulative distribution functions of service time, vacation time, repair time and exceptional last vacation time, respectively. Let \( g(x) \), \( v(x) \), \( r(x) \) and \( s(x) \) be the probability density functions of \( G, V, R \) and \( S \), respectively. Let \( \tilde{G}(\theta), \tilde{V}(\theta), \tilde{R}(\theta) \) and \( \tilde{S}(\theta) \) denote Laplace-Stieltje's transform of \( G, V, R \) and \( S \) respectively. The remaining service time of a batch in service at an arbitrary time \( t \) is denoted by \( G^0(t) \). \( V^0(t) \) is the remaining vacation time at time \( t \). \( R^0(t) \) is the remaining repair time at time \( t \) and \( S^0(t) \) is the remaining last vacation time at time \( t \). \( N_q(t) \) and \( N_s(t) \) denote the number of customers in the queue and under service, respectively, at time \( t \). We define \( Y(t) \) such that \( Y(t) = 0 \), when the server is on vacation or in the exceptional last vacation. \( Y(t) = 1 \), when the server is attending repair, \( Y(t) = 2 \), when the server is busy with service. We define \( Z(t) \) such that \( Z(t) = j \) implies that the server is on \( j^{th} \) vacation.
Let
\[ P_{ij}(x,t)dt = \Pr(N_s(t) = i, N_q(t) = j, x < G^0(t) \leq x + dt, Y(t) = 2), \quad a \leq i \leq b, j \geq 0 \]
\[ Q_{jn}(x,t)dt = \Pr(N_q(t) = n, x \leq V^0(t) \leq x + dt, Y(t) = 0, Z(t) = j), \quad n \geq 0, j \geq 1 \]
\[ R_n(x,t)dt = \Pr(N_q(t) = n, x \leq R^0(t) \leq x + dt, Y(t) = 1), \quad n \geq 0 \]
\[ S_n(x,t)dt = \Pr(N_q(t) = n, x \leq S^0(t) \leq x + dt, Y(t) = 0), \quad n \geq a \]

The following equations are obtained for the queueing system using supplementary variable technique:

\[ P_{i0}(x-\Delta t,t+\Delta t) = P_{i0}(x,t)(1-\lambda \Delta t) + (1-\pi) \sum_{m=a}^{b} P_{mi}(0,t)g(x)\Delta t \]
\[ + R_{i}(0,t)g(x)\Delta t + S_{i}(0,t)g(x)\Delta t, \quad a \leq i \leq b \]
\[ P_{ij}(x-\Delta t,t+\Delta t) = P_{ij}(x,t)(1-\lambda \Delta t) + \sum_{k=1}^{j} P_{i-j}(x,t)\lambda g_k \Delta t, \quad a \leq i \leq b-1, j \geq 1 \]
\[ P_{bj}(x-\Delta t,t+\Delta t) = P_{bj}(x,t)(1-\lambda \Delta t) + (1-\pi) \sum_{m=a}^{b} P_{b-j}(0,t)g(x)\Delta t \]
\[ + \sum_{k=1}^{j} P_{b-j}(x,t)\lambda g_k \Delta t + R_{b+j}(0,t)g(x)\Delta t + S_{b+j}(0,t)g(x)\Delta t, \quad j \geq 1 \]
\[
Q_{i0}(x - \Delta t, t + \Delta t) = Q_{i0}(x, t)(1 - \lambda \Delta t) + (1 - \pi) \sum_{m=a}^b P_{m0}(0, t) v(x) \Delta t + R_0(0, t)v(x) \Delta t
\]

\[
Q_{jn}(x - \Delta t, t + \Delta t) = Q_{jn}(x, t)(1 - \lambda \Delta t) + (1 - \pi) \sum_{m=a}^b P_{mn}(0, t) v(x) \Delta t + \sum_{k=1}^n Q_{j-k}(x, t) \lambda g_k \Delta t + R_n(0, t)v(x) \Delta t, \quad 1 \leq n \leq a - 1
\]

\[
Q_{jn}(x - \Delta t, t + \Delta t) = Q_{jn}(x, t)(1 - \lambda \Delta t) + \sum_{k=1}^n Q_{j-k}(x, t) \lambda g_k \Delta t, \quad n \geq a
\]

\[
Q_{j0}(x - \Delta t, t + \Delta t) = Q_{j0}(x, t)(1 - \lambda \Delta t) + Q_{j-1}0(0, t)v(x) \Delta t, \quad j \geq 2
\]

\[
Q_{j0}(x - \Delta t, t + \Delta t) = Q_{j0}(x, t)(1 - \lambda \Delta t) + \sum_{k=1}^n Q_{j-k}(x, t) \lambda g_k \Delta t + Q_{j-1}n(0, t)v(x) \Delta t, \quad 1 \leq n \leq a - 1
\]

\[
Q_{jn}(x - \Delta t, t + \Delta t) = Q_{jn}(x, t)(1 - \lambda \Delta t) + \sum_{k=1}^n Q_{j-k}(x, t) \lambda g_k \Delta t, \quad j \geq 2, n \geq a
\]

\[
R_0(x - \Delta t, t + \Delta t) = R_0(x, t)(1 - \lambda \Delta t) + \pi \sum_{m=a}^b P_{m0}(0, t) r(x) \Delta t
\]

\[
R_n(x - \Delta t, t + \Delta t) = R_n(x, t)(1 - \lambda \Delta t) + \pi \sum_{m=a}^b P_{mn}(0, t) r(x) \Delta t + \sum_{k=1}^n R_{n-k}(x, t) \lambda g_k \Delta t, \quad n \geq 1
\]
\[
S_n(x - At, l + At) = S_n(x, t) (1 - \lambda \Delta t) + \sum_{k=1}^{n} S_{n-k}(x, t) \lambda g_k \Delta t + \sum_{i=1}^{\infty} Q_{i0}(0, t) s(x) \Delta t,
\]
\[
n \geq a
\]

From the above equations, the steady state queue size equations are obtained as follows:

\[
- \frac{d}{dx} P_{i0}(x) = -\lambda P_{i0}(x) + (1 - \pi) \sum_{m=a}^{b} P_{mi}(0) g(x) + R_i(0) g(x) + S_i(0) g(x) \quad (2.1)
\]

\[
- \frac{d}{dx} P_{ij}(x) = -\lambda P_{ij}(x) + \lambda \sum_{k=1}^{j} P_{i-j-k}(x) g_k, \quad a \leq i \leq b - 1, \quad j \geq 1
\]

\[
- \frac{d}{dx} P_{bj}(x) = -\lambda P_{bj}(x) + (1 - \pi) \sum_{m=a}^{b} P_{bm}(0) g(x) + R_j(0) g(x) + S_j(0) g(x)
\]

\[
+ R_{b+j}(0) g(x) + S_{b+j}(0) g(x), \quad j \geq 1
\]

\[
- \frac{d}{dx} Q_{10}(x) = -\lambda Q_{10}(x) + (1 - \pi) \sum_{m=a}^{b} P_{m0}(0) v(x) + R_0(0) v(x) \quad (2.4)
\]

\[
- \frac{d}{dx} Q_{1n}(x) = -\lambda Q_{1n}(x) + (1 - \pi) \sum_{m=a}^{b} P_{mn}(0) v(x)
\]

\[
+ \lambda \sum_{k=1}^{n} Q_{1-n-k}(x) g_k + R_n(0) v(x), \quad 1 \leq n \leq a - 1
\]

\[
- \frac{d}{dx} Q_{jn}(x) = -\lambda Q_{jn}(x) + \lambda \sum_{k=1}^{n} Q_{1-n-k}(x) g_k, \quad n \geq a
\]

\[
- \frac{d}{dx} Q_{j0}(x) = -\lambda Q_{j0}(x) + Q_{j-1} v(x), \quad j \geq 2
\]
\[-\frac{d}{dx}Q_{jn}(x) = -\lambda Q_{jn}(x) + Q_{j-1} n(\theta)v(x) + \lambda \sum_{k=1}^{n} Q_{j-k}(x)g_k, \quad j \geq 2, 1 \leq n \leq a - 1 \quad (2.8)\]

\[-\frac{d}{dx}Q_{jn}(x) = -\lambda Q_{jn}(x) + \lambda \sum_{k=1}^{n} Q_{j-k}(x)g_k, \quad j \geq 2, n \geq a \quad (2.9)\]

\[-\frac{d}{dx}R_0(x) = -\lambda R_0(x) + \pi \sum_{m=a}^{b} P_m(\theta)r(x) \quad (2.10)\]

\[-\frac{d}{dx}R_0(x) = -\lambda R_n(x) + \pi \sum_{m=a}^{b} P_{mn}(\theta)r(x) + \lambda \sum_{k=1}^{n} R_{n-k}(x)g_k, \quad n \geq 1 \quad (2.11)\]

\[-\frac{d}{dx}S_0(x) = -\lambda S_{n}(x) + \lambda \sum_{k=1}^{n} S_{n-k}(x)g_k + \sum_{l=1}^{\infty} Q_{ln}(\theta)s(x), \quad n \geq a \quad (2.12)\]

Taking Laplace Stieltje’s transform on the both sides of equations from (2.1) to (2.12), we get

\[\theta \ddot{P}_{i0}(\theta) - P_{i0}(\theta) = \lambda \ddot{P}_{i0}(\theta) - (1 - \pi) \sum_{m=a}^{b} P_{mi}(\theta)\ddot{G}(\theta) - R_{i}(\theta)\ddot{G}(\theta) - S_{i}(\theta)\ddot{G}(\theta) \quad (2.13)\]

\[\theta \ddot{P}_{ij}(\theta) - P_{ij}(\theta) = \lambda \ddot{P}_{ij}(\theta) - \lambda \sum_{k=1}^{j} \dot{P}_{i,j-k}(\theta)g_k, \quad a \leq i \leq b - 1, j \geq 1 \quad (2.14)\]

\[\theta \ddot{P}_{bj}(\theta) - P_{bj}(\theta) = \lambda \ddot{P}_{bj}(\theta) - (1 - \pi) \sum_{m=a}^{b} P_{mb+j}(\theta)\ddot{G}(\theta) \quad (2.15)\]
\[ \begin{align*}
\theta Q_{10}(\theta) - Q_{10}(0) &= \lambda \tilde{Q}_{10}(\theta) - (1 - \pi) \sum_{m=a}^{b} P_{m0}(0) \tilde{V}(\theta) - R_0(0) \tilde{V}(\theta) \\
\theta Q_{1n}(\theta) - Q_{1n}(0) &= \lambda \tilde{Q}_{1n}(\theta) - (1 - \pi) \sum_{m=a}^{b} P_{mn}(0) \tilde{V}(\theta) - \sum_{k=1}^{n} \tilde{Q}_1 n-k(\theta) \xi_k \\
- R_n(0) \tilde{V}(\theta), & \quad 1 \leq n \leq a - 1 \quad (2.17) \\
\theta Q_{jn}(\theta) - Q_{jn}(0) &= \lambda \tilde{Q}_{jn}(\theta) - \lambda \sum_{k=1}^{n} \tilde{Q}_j n-k(\theta) \xi_k, & \quad n \geq a \quad (2.18) \\
\theta S_n(\theta) - S_n(0) &= \lambda \tilde{S}_n(\theta) - \lambda \sum_{k=1}^{n} \tilde{S}_n n-k(\theta) \xi_k - \sum_{l=1}^{\infty} Q_{ln}(0) \tilde{S}(\theta), & \quad n \geq a \quad (2.24)
\end{align*} \]
2.2 Queue Size Distribution

We define the following probability generating functions:

\[
\tilde{p}_i(z, \theta) = \sum_{j=0}^{\infty} \tilde{p}_{ij}(\theta) z^j \quad \text{and} \quad \tilde{p}_i(z, 0) = \sum_{n=0}^{\infty} p_{ij}(0) z^j \quad a \leq i \leq b
\]

\[
\tilde{Q}_j(z, \theta) = \sum_{n=0}^{\infty} \tilde{Q}_{jn}(\theta) z^n \quad \text{and} \quad Q_j(z, 0) = \sum_{n=0}^{\infty} Q_{jn}(0) z^n \quad j \geq 1
\]

\[
\tilde{R}(z, \theta) = \sum_{n=0}^{\infty} \tilde{R}_n(\theta) z^n \quad \text{and} \quad R(z, 0) = \sum_{n=0}^{\infty} R_n(0) z^n
\]

\[
\tilde{S}(z, \theta) = \sum_{n=a}^{\infty} \tilde{S}_n(\theta) z^n \quad \text{and} \quad S(z, 0) = \sum_{n=a}^{\infty} S_n(0) z^n
\]

(2.25)

Multiplying the equations (2.16) by \(z^0\), (2.17) by \(z^n\) \((1 \leq n \leq a - 1)\), (2.18) by \(z^n\) \((n > a)\) and using (2.25), we get

\[
(\theta - \lambda + z.X(z)) \tilde{Q}_1(z, \theta) = Q_1(z, 0) - \tilde{V}(\theta) \sum_{n=0}^{a-1} \left( (1 - \pi) \sum_{m=a}^{b} p_{mn}(0) z^n + R_n(0) z^n \right)
\]

(2.26)

Multiplying the equations (2.19) by \(z^0\), (2.20) by \(z^n\) \((1 \leq n \leq a - 1)\), (2.21) by \(z^n\) \((n \geq a)\) and using (2.25), we get

\[
(\theta - \lambda + z.X(z)) \tilde{Q}_j(z, \theta) = Q_j(z, 0) - \tilde{V}(\theta) \sum_{n=0}^{a-1} \left( Q_{j-1, n}(0) z^n \right), \quad j \geq 2
\]

(2.27)
Multiplying the equations (2.13) by $z^0$ and (2.14) by $z^j$ ($j \geq 1$) and using (2.25), we get

$$
(\theta - \lambda + \dot{\lambda}X(z)) \tilde{P}_1(z,\theta) = P_1(z,0) - \tilde{G}(\theta) \left[ (1 - \pi) \sum_{m=a}^b P_{mj}(0) + R(0) + S(0) \right],
$$

$$
a \leq i \leq b - 1 \quad \text{(2.28)}
$$

Multiplying the equations (2.13) by $z^0$, (2.15) by $z^j$ ($j \geq 1$) and using (2.25), we get

$$
z^b (\theta - \lambda + \dot{\lambda}X(z)) \tilde{P}_b(z,0) = z^b P_b(z,0) - \tilde{G}(\theta) \left[ (1 - \pi) \sum_{m=a}^b P_m(z,0) - \sum_{j=0}^{b-1} P_{mj}(0)z^j \right]
$$

$$
+ \left[ R(z,0) - \sum_{n=0}^{b-1} R_n(0)z^n \right] + \left[ S(z,0) - \sum_{n=0}^{b-1} S_n(0)z^n \right] \quad \text{(2.29)}
$$

Multiplying the equations (2.22) by $z^0$, (2.23) by $z^n$ ($n \geq 1$) and using (2.25), we get

$$
(\theta - \lambda + \dot{\lambda}X(z)) \tilde{R}(z,\theta) = R(z,0) - \pi \sum_{m=a}^b P_m(z,0) \tilde{R}(\theta)
$$

$$
\quad \text{(2.30)}
$$

Multiplying the equations (2.24) by $z^n$ ($n \geq a$) and using (2.25), we get

$$
(\theta - \lambda + \dot{\lambda}X(z)) \tilde{S}(z,\theta) = S(z,0) - \tilde{S}(\theta) \sum_{l=1}^\infty \left[ Q_l(z,0) - \sum_{n=0}^{a-1} Q_{ln}(0)z^n \right]
$$

$$
\quad \text{(2.31)}
$$
Substituting $\theta = \lambda - \lambda X(z)$ in equations (2.26), (2.27), (2.28), (2.30), (2.31), we get

\[ Q_j(z,0) = \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} \left[ (1 - \pi) \sum_{m=a}^{b} \tilde{P}_{mn}(0)z^n + R_n(0)z^n \right] \tag{2.32} \]

\[ Q_j(z,0) = \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} \left[ Q_{j-1}(0)z^n \right], \quad j \geq 2 \tag{2.33} \]

\[ P_j(z,0) = G(\lambda - \lambda X(z)) \left[ (1 - \pi) \sum_{m=a}^{b} \tilde{P}_{mi}(0) + R_i(0) + S_i(0) \right], \quad a \leq i \leq b - 1 \tag{2.34} \]

\[ R(z,0) = \pi \sum_{m=a}^{b} \tilde{P}_{m}(z,0) \tilde{R}(\lambda - \lambda X(z)) \tag{2.35} \]

\[ S(z,0) = \sum_{i=1}^{\infty} \left( Q_i(z,0) - \sum_{n=0}^{a-1} Q_{ln}(0)z^n \right) \tilde{S}(\lambda - \lambda X(z)) \tag{2.36} \]

From equation (2.29), we get

\[ z^b p_b(z,0) = G(\lambda - \lambda X(z)) \left[ (1 - \pi) \sum_{m=a}^{b} \left( P_m(z,0) - \sum_{j=0}^{b-1} P_{mj}(0)z^j \right) + \left( R(z,0) - \sum_{n=0}^{b-1} R_n(0)z^n \right) \right] \]

\[ + \left( S(z,0) - \sum_{n=0}^{b-1} S_n(0)z^n \right) \tag{2.37} \]
Let \( p_i(\theta, \lambda X(z)) = T_{mi}(0), R_i = R_i(0), S_i = S_i(0) \) \( m = a \) \( g_i \) \( (1 - \pi) \sum_{m=a}^{b-l} P_m(z, 0) + S(z, 0) + R(z, 0) - \sum_{n=0}^{b-1} ((1 - \pi)p_n + R_n + S_n)z^n \]

\[ p_b(z, 0) = \frac{\tilde{G}(\lambda - \lambda X(z))/(1 - \pi)}{(z^{b} - \tilde{G}(\lambda - \lambda X(z))(1 - \pi)} \]

\[ = \frac{\tilde{G}(\lambda - \lambda X(z)) f(z)}{(z^{b} - \tilde{G}(\lambda - \lambda X(z))(1 - \pi)} \]

where

\[ f(z) = (1 - \pi) \sum_{m=a}^{b-l} P_m(z, 0) + S(z, 0) + R(z, 0) - \sum_{n=0}^{b-1} ((1 - \pi)p_i + R_i + S_i) \]

From equations (2.26) and (2.32), we get

\[ \tilde{Q}_1(z, \theta) = (\tilde{V}(\lambda - \lambda X(z)) - \tilde{V}(0)) \left[ \sum_{n=0}^{a-1} (1 - \pi) \sum_{m=a}^{b} P_{mn}(0)z^n + R_n(0)z^n \right] \]

\[ \frac{(0 - \lambda + \lambda X(z))}{(0 - \lambda + \lambda X(z))} \]

From equations (2.27) and (2.33), we get

\[ \tilde{Q}_j(z, \theta) = \frac{(\tilde{V}(\lambda - \lambda X(z)) - \tilde{V}(0)) \left[ \sum_{n=0}^{a-1} Q_{j-1, n}(0)z^n \right]}{(0 - \lambda + \lambda X(z))}, \quad j \geq 2 \]
From equations (2.31) and (2.36), we get

\[ s(z,0) = \left( \sum_{j=1}^{a} \left( Q_j(z,0) - \sum_{n=0}^{a-j} Q_{j+n}(0)z^n \right) \right) \left( 0 - \lambda + \lambda X(z) \right) \]

(2.44)

From equations (2.28) and (2.34), we get

\[ \tilde{p}_j(z,0) = \frac{\left( \sum_{m=a}^{b} \left( P_{m1}(0) + R_1(0) + S_1(0) \right) \right)}{\left( 0 - \lambda + \lambda X(z) \right)} \]

(2.45)

From equations (2.29) and (2.40), we get

\[ \tilde{p}_h(z,0) = \frac{\tilde{G}(\lambda - \lambda X(z)) - \tilde{G}(0)}{\left( \lambda^b - \tilde{G}(\lambda - \lambda X(z))(1 - \pi) \right) \left[ 0 - \lambda + \lambda X(z) \right]} \]

(2.46)

From equations (2.30) and (2.35), we get

\[ \tilde{R}_h(z,0) = \frac{\left( \tilde{R}(\lambda - \lambda X(z)) - \tilde{R}(0) \right) \pi \sum_{m=a}^{b} \left( P_{m2}(z,0) \right)}{\left[ 0 - \lambda + \lambda X(z) \right]} \]

(2.47)
Let \( P(z) \) be the probability generating function of the queue size at an arbitrary time epoch. Then

\[
P(z) = \sum_{i=a}^{b-1} \tilde{P}_i(z,0) + \tilde{P}_b(z,0) + \sum_{j=1}^{\infty} \tilde{Q}_j(z,0) + \tilde{S}(z,0) + \tilde{R}(z,0) \quad (2.48)
\]

Using equations (2.42), (2.43), (2.44), (2.45), (2.46) and (2.47) in (2.48), we get

\[
\begin{align*}
P(z) &= \frac{(\tilde{G}(\lambda - \lambda X(z)) - 1) \sum_{i=a}^{b-1} [(1 - \pi)p_i + R_i + S_i]}{-\lambda + \lambda X(z)} \\
&\quad + \frac{\pi(\tilde{R}(\lambda - \lambda(z)) - 1) \tilde{G}(\lambda - \lambda X(z)) \sum_{i=a}^{b-1} [(1 - \pi)p_i + R_i + S_i]}{-\lambda + \lambda X(z)} \\
&\quad + \frac{(\tilde{V}(\lambda - \lambda X(z)) - 1) \sum_{i=0}^{a-1} [(1 - \pi)p_i + R_i + S_i]z^i}{-\lambda + \lambda X(z)} \\
&\quad + \frac{(\tilde{S}(\lambda - \lambda X(z)) - 1)(\tilde{V}(\lambda - \lambda X(z))) \sum_{i=0}^{a-1} [(1 - \pi)p_i + R_i + S_i]z^i}{-\lambda + \lambda X(z)} \\
&\quad + \frac{\tilde{G}(\lambda - \lambda X(z)) - 1)f(z) + \pi(\tilde{R}(\lambda - \lambda X(z)) - 1) \tilde{G}(\lambda - \lambda X(z)) - 1)f(z)}{-\lambda + \lambda X(z)[z^b - (\tilde{G}(\lambda - \lambda X(z))(1 - \pi)]} \quad (2.49)
\end{align*}
\]
Substituting for \( f(z) \) from equation (2.41) and using \((1-\pi)P_i + R_i + S_i = d_i\) in equation (2.49) and simplifying, we get

\[
P(z) = \left[ G(\lambda - \lambda X(z)) - 1 + \pi \hat{G}(\lambda - \lambda X(z)) \hat{R}(\lambda - \lambda X(z)) - \pi \hat{G}(\lambda - \lambda X(z)) \right] \sum_{i=a}^{b-l} \left( z^{b-i} d_i \right)
\]

\[
= \left[ G(\lambda - \lambda X(z)) - 1 + \pi \hat{G}(\lambda - \lambda X(z)) \hat{R}(\lambda - \lambda X(z)) - \pi \hat{G}(\lambda - \lambda X(z)) \right] \sum_{i=a}^{b-l} \left( z^{b-i} b_i \right)
\]

\[
- \left[ S(\lambda - \lambda X(z)) \right] \left[ \hat{V}(\lambda - \lambda X(z)) - 1 \right] \sum_{i=0}^{a-1} \left( z^i d_i \right)
\]

\[
- \left[ S(\lambda - \lambda X(z)) \right] (\hat{V}(\lambda - \lambda X(z)) - 1) \sum_{i=0}^{a-1} \left( q_i z^i \right)
\]

\[
(\lambda - \lambda X(z)) \left[ z^{b} - (1-\pi)\hat{G}(\lambda - \lambda X(z)) - \pi \hat{R}(\lambda - \lambda X(z)) \hat{G}(\lambda - \lambda X(z)) \right]
\]

(2.50)

Probability generating function of the number of customers involving only ‘b’ unknowns \( d_0, d_1, d_2, \ldots, d_{b-1} \). The probability generating function \( P(z) \) has to satisfy \( P(1)=1 \). In order to satisfy this condition, applying L’Hospital’s rule and evaluating \( \lim_{z \to 1} P(z) \) and equating the expression to 1, we have to satisfy

\[
E(G) + \pi E(R) \sum_{i=a}^{b-l} (b-i)d_i + bE(V) \sum_{i=0}^{a-l} d_i = b - \lambda E(X)[E(G) + \pi E(R)]
\]

39
Since \( p, q, \) and \( G \), are the probabilities of \( 'i' \) customers being in the queue at a customer departure epoch, vacation completion epoch and at a repair completion epoch, respectively, it follows that \( c_n = (1 - \pi)p_n + R_n + S_n, n = 0 \) to \( b-1 \) are to be positive quantities and hence left hand side of the above expression should be positive. Thus \( P(1) = 1 \) is satisfied iff \( b - \lambda E(X)[E(G) + \pi E(R)] > 0 \) and defining \( \rho \) by

\[
\rho = \frac{\lambda E(X)[E(G) + \pi E(R)]}{b}
\]

we get that \( \rho < 1 \) is the condition to be satisfied for the existence of steady state for the model under consideration.

By Rouche's theorem of complex variables, it follows that

\[
[z^b - (1 - \pi)G(\lambda - X(z)) - \pi G(\lambda - \lambda X(z))\tilde{R}(\lambda - \lambda X(z))] \]

has \( (b-1) \) zeros inside and one on the unit circle \(|z| = 1\). Since \( P(z) \) is analytic within and on the unit circle, the numerator must vanish at these points, which gives \( b \) equations in \( b \) unknowns. These equations can be solved by any suitable numerical technique.
Lemma 2.1

Let \( \alpha_i \) be the probability that ‘i’ customers arrive during a vacation. Then, the probability generating function of \( \alpha_i \) is given by

\[
\sum_{i=0}^{\infty} \alpha_i z^i = V(\lambda - \lambda X(z))
\]

Proof:

Conditioning on the actual vacation length, number of arrivals and the group size, we get

\[
\alpha_i = \int_0^\infty \sum_{m=0}^i \frac{(e^{-\lambda t})(\lambda t)^m g_i^{(m)}}{m!} \, dt, \quad \text{where } g_i^{(m)} \text{ is the m-fold convolution of } g_i \text{ with itself (i.e., total of } m \text{ arrivals make ‘i’ customers).}
\]

Multiplying the above equation by \( z^i \) and taking the summation from \( i = 0 \) to \( \infty \), we get

\[
\sum_{i=0}^{\infty} \alpha_i z^i = e^{-\lambda t} \int_0^\infty \sum_{m=0}^\infty \frac{(\lambda t)^m}{m!} \sum_{i=m}^\infty g_i^{(m)} z^i \, dt = V(\lambda - \lambda X(z))
\]

Theorem 2.1

\[
q_n = \sum_{i=0}^{n} K_{n-i} d_i \quad \text{where } K_0 = \frac{\alpha_0}{1 - \alpha_0} \quad \text{and} \quad K_n = \frac{\alpha_n + \sum_{j=1}^{n} \alpha_j K_{n-j}}{(1 - \alpha_0)}
\]

where \( \alpha_i \) is the probability that ‘i’ customers arrive during a vacation.
Using equations (2.32) and (2.33) \( \sum_{j=1}^{\infty} Q_j(z,0) \) simplifies to

\[
\sum_{n=0}^{\infty} q_n z^n = \sum_{n=0}^{\infty} \left( \sum_{i=0}^{a-1} ((1 - \pi) p_i + R_i + q_i) z^n \right)
\]

\[
= \sum_{n=0}^{\infty} \alpha_n z^n \left[ \sum_{i=0}^{a-1} \left( (1 - \pi) p_i + R_i + q_i \right) \right] z^n \] [using Lemma 2.1]

\[
= \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \left( (1 - \pi) p_i + R_i + q_i \right) \alpha_n z^n
\]

\[
+ \sum_{n=a}^{\infty} \sum_{i=0}^{a-1} \left( (1 - \pi) p_i + R_i + q_i \right) \alpha_{n-i} z^n
\]

(2.51)

Equating the coefficient of \( z^n \), \( n = 0,1,2, \ldots, (a-1) \) on both sides of equation (2.51) we get

\[
q_n = \sum_{i=0}^{n} K_{n-i} \left[ (1 - \pi) p_i + R_i \right]
\]

\[
= \sum_{i=0}^{n} K_{n-i} d_i, \quad n = 0,1,2, \ldots, a-1
\]

where \( K_0 = \frac{\alpha_0}{1 - \alpha_0} \) and \( K_n = \frac{\alpha_n + \sum_{j=1}^{n} \alpha_j K_{n-j}}{(1 - \alpha_0)} \), \( n = 0,1,2, \ldots, a-1 \)

and \( \alpha_i \) is the probability that 'i' customers arrive during a vacation. \( \square \)
Theorem 2.2

\[ S_i = \sum_{n=a}^{n+b-1} \beta_{i-n} \left( \sum_{u=0}^{a-1} \alpha_{n-u} + \sum_{j=0}^{a-1-u} K_j \alpha_{n-j-u} \right) \] 

where \( \beta_i \) is the probability that \( 'i' \) customers arrive during the exceptional last vacation time.

Using equations (2.25) and (2.36), we get

\[ s(z) = s(a - a \cdot x(z)) \quad (2.52) \]

\[ \sum_{n=a}^{\infty} S_n z^n = \left[ \sum_{n=0}^{\infty} \beta_n z^n \right] \left[ \sum_{n=a}^{\infty} q_n z^n \right] \quad (2.53) \]

Equating coefficient of \( z^i \), \( i = a, a+1, \ldots, b-1 \) on both sides of equation (2.53), we get

\[ S_i = \sum_{n=a}^{n+b-1} q_n \beta_{i-n}, \quad n = a, a+1, \ldots, b-1 \quad (2.54) \]

Substituting for \( q_n \) from equation (2.51) for \( n = a, a+1, \ldots, b-1 \) in equation (2.54) and simplifying, we get

\[ S_i = \sum_{u=0}^{a-1} \sum_{n=a}^{n+b-1} \left( \begin{array}{c} i \\ n \end{array} \right) \beta_{i-n} \left( \sum_{u=0}^{a-1} \alpha_{n-u} + \sum_{j=0}^{a-1-u} K_j \alpha_{n-j-u} \right) \] 

\[ d_u = (1-\pi) p_u + R_u \quad \square \quad (2.55) \]
2.3 Expected Length of the Idle Period

Let $I$ be the idle period random variable. Then, the expected length of the idle period is given by,

$$E(I) = E(I_1) + E(S)$$

where $I_1$ is the idle period random variable due to multiple vacation process and $E(S)$ is the expected length of exceptional last vacation time. Define a random variable $U$ as

- $U = 0$, if the server finds at least 'a' customers after the first vacation.
- $U = 1$, if the server finds less than 'a' customers after the first vacation.

$$E(I_1) = E(I_1/U = 0) P(U = 0) + E(I_1/U = 1) P(U = 1)$$

$$= E(V) P(U = 0) + [E(V) + E(I_1)] P(U = 1)$$

(2.56)

Solving for $E(I_1)$, we get

$$E(I_1) = \frac{E(V)}{P(U = 0)}$$

(2.57)

Using equation (2.32), we get

$$P(U = 0) = 1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \alpha_i ((1 - \pi)p_{n-i} + R_{n-i})$$

(2.58)

where $\alpha_i$ = probability that 'i' customers arrive during a vacation. Using (2.58), the mean idle period $E(I)$ is given by

$$E(I) = \frac{E(V)}{\sum_{n=0}^{a-1} \sum_{i=0}^{n} \alpha_i ((1 - \pi)p_{n-i} + R_{n-i})} + E(S)$$

(2.59)
2.4 Expected Length of Busy Period

Let $B$ be the busy period random variable. Let $T$ be the residence time that the server is rendering service or under repair. Then

$$ E(T) = E(G) + \pi E(R). \quad (2.60) $$

We define a random variable $J$ as

$J = 0$, if the server finds less than 'a' customers after the residence time

$J = 1$, if the server finds at least 'a' customers after the residence time.

Then expected length of busy period $E(B)$ is given by

$$ E(B) = E(B/J=0) P(J=0) + E(B/J=1) P(J=1) = E(T) P(J=0) + (E(T) + E(B)) P(J=1) $$

Solving for $E(B)$, we get

$$ E(B) = \frac{E(T)}{P(J=0)} = \frac{E(T)}{\sum_{i=0}^{a-1} (1-\pi)p_i + R_j} \quad (2.61) $$
2.5 Expected Queue Length

The mean queue length $E(Q)$ at an arbitrary time epoch is obtained by differentiating $P(z)$ at $z=1$ and is given by

$$E(Q) =$$

$$f_1(X, G, R) \left[ \sum_{i=a}^{b-1} (b(b-1)-i(i-1)) (d_i + S_i) \right]$$

$$+ \left[ f_2(X, G, R) + f_6(X, G, R) \right] \left[ \sum_{i=a}^{b-1} (b-i)(d_i + S_i) \right]$$

$$+ \left[ f_3(X, S, G, R, V) + f_7(X, S, G, R, V) \right] \left[ \sum_{i=0}^{b-1} d_i \right]$$

$$+ \left[ f_5(X, G, R, V) + f_8(X, G, R, V) \right] \left[ \sum_{i=0}^{a-1} q_i \right]$$

$$+ \left[ f_4(X, G, R, V) \right] \left[ \sum_{i=0}^{a-1} q_{i+1} \right]$$

$$\frac{4T_1(T_2 + 2T_1)(b-T)^2}{(2.62)}$$

The functions $f_1$ to $f_8$ are given by

$$f_1 = 6 T_1 (b - T)$$

$$f_2 = 6 T_1 (b - T) (R_2 - G_1 + 2\pi G_1 R_1 + T_1 T_3)$$

$$f_3 = 6 T_1 b (b - T) (S_2 + V_2 + 2 S_1 V_1)$$

$$f_4 = 2 T_1 b (b - T) (3 V_1 - V_1 T_1)$$

$$f_5 = 2 T_1 b (b - T) (4 S_1 V_1 + 2(b - 1) V_1 + V_2)$$

$$f_6 = 6 T (T_1 T_4 + T_2 (b - T))$$

$$f_7 = 6 b (T_1 T_4 + T_2 (b - T)) (S_1 + V_1)$$

$$f_8 = 6 b (T_1 T_4 + T_2 (b - T)) V_1$$
where

\[ \Gamma = G_1 + \pi R_1; \quad T_1 = \lambda E(X); \quad T_2 = \lambda E(X^2) \]
\[ \Gamma_3 = \lambda E(G^2); \quad T_4 = b(b-1) - G_2 - \pi R_2 - 2\pi G_1 R_1 \]
\[ S_1 = \lambda E(X) E(S); \quad S_2 = \lambda E(X^2) E(S) + \lambda^2 [E(X)]^2 E(S^2) \]
\[ R_1 = \lambda E(X) E(R); \quad R_2 = \lambda E(X^2) E(R) + \lambda^2 [E(X)]^2 E(R^2) \]
\[ V_1 = \lambda E(X) E(V); \quad V_2 = \lambda E(X^2) E(V) + \lambda^2 [E(X)]^2 E(V^2) \]
\[ G_1 = \lambda E(X) E(G); \quad G_2 = \lambda E(X^2) E(G) + \lambda^2 [E(X)]^2 E(G^2) \]

2.6 Cost Model

The total average cost is obtained with the following assumptions:

- \( C_s \) : Start up cost.
- \( C_h \) : Holding cost per customer.
- \( C_o \) : Operating cost per unit time.
- \( C_r \) : Reward cost per unit time due to vacation.
- \( C_u \) : Setup cost per unit time.
- \( C_{rp} \) : Repair cost per unit time.

Since the length of the cycle is the sum of the idle period and busy period, from equations (2.58), (2.59) and (2.61), the expected length of cycle \( E(T_C) \) is given by

\[
E(T_C) = \frac{E(V)}{P(U = 0)} + E(S) + \frac{E(T)}{\sum_{i=0}^{a-1} [(1 - \pi)p_i + R_i]} \quad (2.63)
\]
The total average cost per unit time is obtained as

\[
\text{Total cost} = \text{Start up cost per cycle} + \text{repair cost per cycle} + \text{Setup cost per cycle} + \text{Holding cost of number of customers in the queue} + \text{Operating cost} \cdot \rho - \text{Reward due to vacation per cycle}
\]

\[
= \left[ C_s + \frac{C_r \, E(R) \, E(B)}{E(T)} + C_u \, E(S) - \frac{C_r \, E(V)}{P(U = 0)} \right] \frac{1}{E(T_c)} + C_h E(Q) + C_o \rho
\]

(2.64)

where \( \rho = \frac{\lambda \, E(X) \left[ E(G) + \pi E(R) \right]}{b} \)

2.7 Expected Waiting Time

Expected waiting time in the queue is obtained as

\[
E(W) = \frac{E(Q)}{\lambda \, E(X)}
\]

2.8 Numerical Example

A numerical model is analysed with the following assumptions:

1. Service time distribution is k-Erlang with \( k = 2 \).
2. Batch size distribution of the arrival is geometric with mean 2.
3. Vacation time, exceptional last vacation time and repair time are exponential with parameters \( \alpha = 10, \beta = 8 \) and \( \gamma = 8 \) respectively.
4. Service capacity with minimum ‘a’ = 3 and maximum ‘b’ = 4
5. \( \pi = 0.2 \)
6. Rs. 4 - Startup cost
   Rs.0.50 - holding cost per customer
   Rs.5 - operating cost per unit time
   Rs.2 - reward per unit time due to vacation
   Rs.0.25 - repair cost per unit time
Numerical results are presented in tables 2.1, 2.2 and 2.3 for service rate \( \mu = 2.5 \) and arrival rate \( \lambda \) ranging 1.0 to 2.0, for service rate \( \mu = 3.0 \) and arrival rate \( \lambda \) ranging 1.0 to 2.5 and for service rate \( \mu = 3.5 \) and arrival rate \( \lambda \) ranging 1.0 to 3.0.

From the tables, the following observations are made:

1. When expected queue length increases, expected waiting time in the queue increases.
2. Expected queue length decreases, when the service rate increases for a particular arrival rate.
3. The total average cost per unit time with server vacation is less compared to the total average cost per unit time obtained without server vacation.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \rho )</th>
<th>( E(Q) )</th>
<th>( E(I) )</th>
<th>( E(B) )</th>
<th>( W_q )</th>
<th>Total Avg. Cost with vacation</th>
<th>Total Avg. Cost without vacation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.413</td>
<td>2.044</td>
<td>0.2305</td>
<td>15.566</td>
<td>1.022</td>
<td>3.311</td>
<td>3.3439</td>
</tr>
<tr>
<td>1.5</td>
<td>0.619</td>
<td>3.556</td>
<td>0.230</td>
<td>15.5667</td>
<td>1.185</td>
<td>5.097</td>
<td>5.1299</td>
</tr>
<tr>
<td>2.0</td>
<td>0.825</td>
<td>5.815</td>
<td>0.228</td>
<td>25.000</td>
<td>1.454</td>
<td>7.1729</td>
<td>7.1925</td>
</tr>
</tbody>
</table>
Table 2.1: Service rate $\mu = 2.5$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\rho$</th>
<th>$E(Q)$</th>
<th>$E(I)$</th>
<th>$E(B)$</th>
<th>$Wq$</th>
<th>Total Avg. Cost with vacation</th>
<th>Total Avg. Cost without vacation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.346</td>
<td>1.915</td>
<td>0.2310</td>
<td>11.723</td>
<td>0.958</td>
<td>2.9834</td>
<td>3.0287</td>
</tr>
<tr>
<td>1.5</td>
<td>0.519</td>
<td>3.357</td>
<td>0.2315</td>
<td>10.641</td>
<td>1.119</td>
<td>4.5988</td>
<td>4.6494</td>
</tr>
<tr>
<td>2.0</td>
<td>0.692</td>
<td>5.392</td>
<td>0.230</td>
<td>12.351</td>
<td>1.348</td>
<td>6.4373</td>
<td>7.4798</td>
</tr>
<tr>
<td>2.5</td>
<td>0.865</td>
<td>7.392</td>
<td>0.228</td>
<td>21.740</td>
<td>1.478</td>
<td>8.1823</td>
<td>8.2049</td>
</tr>
</tbody>
</table>

Table 2.2: Service rate $\mu = 3.0$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\rho$</th>
<th>$E(Q)$</th>
<th>$E(I)$</th>
<th>$E(B)$</th>
<th>$Wq$</th>
<th>Total Avg. Cost with vacation</th>
<th>Total Avg. Cost without vacation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.298</td>
<td>2.104</td>
<td>0.2310</td>
<td>9.620</td>
<td>1.052</td>
<td>2.9011</td>
<td>2.9578</td>
</tr>
<tr>
<td>1.5</td>
<td>0.447</td>
<td>3.215</td>
<td>0.2320</td>
<td>7.952</td>
<td>1.072</td>
<td>4.2745</td>
<td>4.3455</td>
</tr>
<tr>
<td>2.0</td>
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<td>5.084</td>
<td>0.2315</td>
<td>8.060</td>
<td>1.271</td>
<td>5.9485</td>
<td>6.0182</td>
</tr>
<tr>
<td>2.5</td>
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<td>7.109</td>
<td>0.230</td>
<td>10.109</td>
<td>1.421</td>
<td>7.6268</td>
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<tr>
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<td>0.228</td>
<td>20.502</td>
<td>1.457</td>
<td>9.0159</td>
<td>9.0416</td>
</tr>
</tbody>
</table>

Table 2.3: Service rate $\mu = 3.5$