CHAPTER - 6

EXISTENCE OF SOLUTION FOR FOURTH ORDER NONLINEAR FUNCTIONAL RANDOM DIFFERENTIAL EQUATION IN BANACH SPACE.

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6.1 INTRODUCTION:

In the previous chapter we have seen the existence theory for the Third order nonlinear functional random differential equation of in Banach Space. In this chapter we prove the existence of solution for the fourth order boundary value problem of nonlinear functional random differential equation through fixed point theory. The study of existence of solution of fourth order random differential equation is done by author like V. Lakshmikantham [8, 29].

6.2 STATEMENT OF PROBLEM:

Let $R$ denote the real line and let $I = [-a, 0]$ and $I_0 = [0, b]$ be two closed and bounded intervals in $R$ for some real number $a > 0$ and $b > 0$. Let $J = I_0 \cup I$ and let $C(I_0, R)$ denote the class of real valued function define and continuously differentiable on $I_0$. Clearly $C$ is a Banach space with supremum norm in a $C(I_0, R)$ denoted by $\| x \|_C$ and define by

$$\| x \|_C = \sup_{t \in I_0} | x(t) |.$$

for given $t \in I$, define a continuous $R$-valued function $x_t : I_0 \to R$ by

$$x_t(\theta) = x(t + \theta), \ \theta \in I_0$$

Given a measurable space $(\Omega, A)$ and for a given measurable function

$\Phi : \Omega \to C^1(J, R)$. Consider the Fourth Order Nonlinear Functional Random Differential Equation (in short FNFRDE)
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\[ x^{(iv)}(t, \omega) = f(t, x(t, \omega), x'(t, \omega), x''(t, \omega), x'''(t, \omega)), \ 0 < t < 1 \]

with boundary condition
\[
x(0, \omega) = x'(1, \omega) = x''(0, \omega) = x'''(1, \omega) = 0
\]

for all \( \omega \in \Omega \) where \( f : J \times R \times R \times \Omega \to R \).

By a random solution of equation (6.2.1) we mean a measurable function.

\[ x : \Omega \to C(J, R) \cap AC^3(J, R) \] that satisfies equation in (6.2.1) on \( J \). Where \( AC^3(J, R) \) is the space of all real-valued function whose third derivative exists and is absolutely continuously differentiable on \( J \).

### 6.3 EXISTENCE RESULT:

We know that \((\Omega, A)\) be a measurable space and let \(X\) be a separable Banach space with norm \(\| . \|_C\). Let \(Q : X \to X\). We further assume that the Banach Space \(X\) is separable i.e. \(X\) has a countable dense subset and let \(\beta_X\) we denote the \(\sigma\)-algebra of all Boral subsets of \(X\). A mapping \(x : \Omega \to X\) is called measurable if for any Boral subset \(A\) of \(X\).

\[ x^{-1}(B) = \{ \omega \in \Omega \mid x(\omega) \in B \} \]

A mapping \(Q : \Omega \times X \to X\) is called random operator if \(Q(\omega, x)\) is measurable in \(\omega\) for all \(x \in X\) and denote it by \(Q(\omega) x = Q(\omega, x)\).

Let \(Q : \Omega \times X \to X\) Then \(Q\) is called continuous if for any open set \(V\) in \(X\), \(Q^{-1}(V)\) is open in \(X\). \(Q\) is called compact if \(Q(x)\) is relatively compact subset of \(X\). \(Q\) is called totally bounded if for any bounded subset \(A\) of \(X\), \(Q(A)\) is totally bounded on \(X\). Finally \(Q\) is called completely continuous if it is continuous and totally bounded on \(X\). Note that every compact operator is totally bounded, but the converse may not be true. However these two notions are equivalent on a bounded subset of \(X\) similarly a random operator \(Q : \Omega \times X \to X\) is called continuous (a compact, totally bounded and completely continuous) for each \(\omega \in \Omega\).

**Lemma : 6.3.1 :** Let \(B_R(0)\) and \(\overline{B}_R(0)\) be the open and closed ball centered at origin of radius \(R\) in the separable Banach space \(X\) and let \(Q : \Omega \times \overline{B}_R(0) \to X\) be a
compact and continuous random operator. Further suppose that there does not exists an \( u \in X \) with \( ||u|| = R \) such that \( Q(\omega)u = \alpha u \) for all \( \alpha \in \Omega \) where \( \alpha > 1 \). Then the random equation \( Q(\omega)x = x \) has a random solution, i.e. there is a measurable function \( \xi : \Omega \rightarrow B_R(0) \) such that \( Q(\omega)\xi(\omega) = \xi(\omega) \) for all \( \omega \in \Omega \).

**Lemma : 6.3.2 :** (Carathéodory) Let \( Q : \Omega \times X \rightarrow X \) be a mapping such that \( Q(\cdot, x) \) is measurable for all \( x \in X \) and \( Q(\omega, \cdot) \) is continuous for all \( \omega \in \Omega \). Then the map \( (\omega, x) \rightarrow Q(\omega, x) \) is jointly measurable.

We seek random solution of equation (6.2.1) in Banach space \( C(J, R) \) of continuous real valued function defined on \( J \). We equip the space \( C(J, R) \) with the supremum norm \( ||.||_C \) defined by

\[
||.||_C = \sup_{t \in J} |x(t)|.
\]

It is known that the Banach space \( C(J, R) \) is separable. By \( L^1(J, R) \) we denote the space of Lebesgue measurable real-valued function defined on \( J \). By \( ||.||_{L^1} \) we denote the usual norm in \( L^1(J, R) \) defined by

\[
||.||_{L^1} = \int_0^1 |x(t)| dt.
\]

We need the following definition in the sequel.

**Definition : 6.3.1 :** A function \( f : J \times R \times R \times \Omega \rightarrow R \) is called random Carathéodory if

i) the map \( (t, \omega) \rightarrow f(t, x, y, \omega) \) is jointly measurable for all \( (x, y) \in R^2 \), and

ii) the map \( (x, y) \rightarrow f(t, x, y, \omega) \) is continuous for almost all \( t \in J \) and \( \omega \in \Omega \).

**Definition : 6.3.2 :** A Carathéodory function \( f : J \times R \times R \times \Omega \rightarrow R \) is called random \( L^1 \)- Carathéodory if for each real number \( r > 0 \) there is a measurable and bounded function \( h_r : \Omega \rightarrow L^1(J, R) \) such that

\[
|f(t, x, y, \omega)| \leq h_r(t, \omega) \quad a.e. \ t \in J.
\]

where \( |x|, |y| \leq r \) and for all \( \omega \in \Omega \). Similarly a Carathéodory function \( f \) is called \( L^r \)- Carathéodory if there is a measurable and bounded function \( h : \Omega \rightarrow L^1(J, R) \) such that
\[ |f(t,x,y,\omega)| \leq h(t,\omega) \text{ a.e. } t \in J. \]

for all \( \omega \in \Omega \) and \((x,y) \in R^2\)

We consider the following set of hypotheses

- **(A_1)** The function \( f \) is random Carathéodory on \( J \times R \times R \times \Omega \)
- **(A_2)** There exists a measurable and bounded function \( \gamma : \Omega \rightarrow L^2(J,R) \) and a continuous and non-decreasing function \( \psi : R_+ \rightarrow (0,\infty) \) such that

\[ |f(s,x,x',x'',x''',\omega)| \leq \gamma(t,\omega)\psi(|x|) \text{ a.e. } t \in J \]

for all \( \omega \in \Omega \) and \( x \in R \)

Moreover we assume that \( \int_c^\infty \frac{dr}{\psi(r)} = \infty \) for all \( c \geq 0 \).

### 6.4 EXISTENCE THEORY:

**Theorem : 6.4.1** : Assume that the hypothesis \((A_1) - (A_2)\) hold. Suppose that there exist a real number \( R > 0 \) such that

\[ R > r_1 \|\gamma(t)\|_{L^1} \psi(r) \]

for all \( \omega \in \Omega \) where \( r_1 = \max_{t \in J} r(t), r(t) \text{ is in the greens function} \)

Then the equation (6.2.1) has a random solution defined on \( J \)

**Proof :** Set \( X = C(J,R) \) and define a mapping \( Q : \Omega \times X \rightarrow X \) by

\[ Q(\omega)x(t) = \int_0^1 G(t,s)f(s,x(s,\omega),x'(s,\omega),x''(s,\omega)x'''(s,\omega),\omega)ds \]

for all \( t \in J, \omega \in \Omega \).

Then the solution of (6.2.1) has a fixed point of operator \( Q \).

Define a closed ball \( \overline{B_R}(0) \) in \( X \) centered at origin 0 of radius \( R \), Where the real number \( R \) satisfies the inequality (6.1). We show that \( Q \) satisfies all the condition of lemma (6.3.1) on \( \overline{B_R}(0) \).

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First we show that \( Q \) is random operator in \( \overline{B}_R(0) \) since \( f(t, x, x', x'', x''', \omega) \) is random Carathéodory and \( x(t, \omega) \) is measurable; the map \( \omega \rightarrow f(t, x, x', x'', x''', \omega) \) is measurable.

Similarly the product \( G(t, s) f (s, x(s, \omega), x'(s, \omega), x''(s, \omega), x'''(s, \omega), \omega) \) of continuous and measurable function is again measurable. Further the integral is a limit of a finite sum of measurable function therefore the map.

\[
\omega \rightarrow \int_0^1 G(t, s) f(s, x(s, \omega), x'(s, \omega), x''(s, \omega), x'''(s, \omega), \omega) ds = Q(\omega)x(t)
\]

is measurable. As a result \( Q \) is random operator on \( \Omega \times \overline{B}_R(0) \) into \( X \).

i.e. \( Q : \Omega \times \overline{B}_R(0) \rightarrow X \)

Next we show that the random operator \( Q(\omega) \) is continuous on \( \Omega \times \overline{B}_R(0) \). Let \( x_n \) be a sequence of point in \( \Omega \times \overline{B}_R(0) \) converging to the point \( x \) in \( \Omega \times \overline{B}_R(0) \). Then it is sufficient to prove that

\[
\lim_{n \rightarrow \infty} Q(\omega) x_n(t) = Q(\omega) X(t) \quad \text{for all } t \in J, \omega \in \Omega
\]

By the dominated convergence theorem we obtain

\[
\lim_{n \rightarrow \infty} Q(\omega) x_n(t) = \lim_{n \rightarrow \infty} \int_0^1 G(t, s) f(s, x_n(s, \omega), x_n'(s, \omega), x_n''(s, \omega), x_n'''(s, \omega), \omega) ds
\]

\[
= \int_0^1 G(t, s) \lim_{n \rightarrow \infty} f(s, x_n(s, \omega), x_n'(s, \omega), x_n''(s, \omega), x_n'''(s, \omega), \omega) ds
\]

\[
= \int_0^1 G(t, s) f(s, x(s, \omega), x'(s, \omega), x''(s, \omega), x'''(s, \omega), \omega) ds
\]

\[
= Q(\omega) x(t)
\]

for all \( t \in J, \omega \in \Omega \). This show that \( Q(\omega) \) is a continuous random operator on \( \overline{B}_R(0) \).

Now we show that \( Q(\omega) \) is compact random operator on \( \overline{B}_R(0) \). We should prove that \( Q(\omega) \overline{B}_R(0) \) is uniformly bounded and equicontinuous set in \( X \) for each \( \omega \in \Omega \). Since the map \( \omega \rightarrow \gamma(t, \omega) \) is bounded and \( L^2(J, R) \subset L^1(J, R) \), by
hypothesis (A₂), there is a constant c such that $||\gamma(\omega)||_L^1 \leq c$ for all $\omega \in \Omega$. Let $\omega \in \Omega$ be fixed then for any $x : \Omega \rightarrow \overline{B}_R(0)$ one has.

$$|Q(\omega)x_n(t)| \leq \int_0^1 G(t,s) |f(s,x(s,\omega),x'(s,\omega),x''(s,\omega),x'''(s,\omega)\omega)| \, ds$$

$$\leq \int_0^1 G(t,s) \gamma(s,\omega)\psi(|x(s,\omega)|) \, ds$$

$$\leq \gamma_1 c \psi(r) = K$$

for all $t \in J$ and each $\omega \in \Omega$. Where $K = \gamma_1 c \psi(R)$. This shows that $Q(\omega)\overline{B}_R(0)$ is uniformly bounded subset of $X$ for each $\omega \in \Omega$. Next we show that $Q(\omega)\overline{B}_R(0)$ is equi-continuous set in $X$.

for any $x \in \overline{B}_R(0)$, $t_1, t_2 \in J$ we have

$$|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \leq \int_0^1 |G(t_1,s) - G(t_2,s)|\gamma(s,\omega)\psi(|x(s,\omega)|) \, ds$$

$$\leq \int_0^1 |G(t_1,s) - G(t_2,s)| \gamma(s,\omega)\psi(r) \, ds$$

By Hölder inequality

$$|Q(\omega)x(t_1) - Q(\omega)x(t_2)|$$

$$\leq (\int_0^1 |G(t_1,s) - G(t_2,s)|^2 \, ds)^{1/2} (\int_0^1 |\gamma(s,\omega)|^2 \, ds)^{1/2} (R) .$$

Hence for all $t_1, t_2 \in J$

$$|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \rightarrow 0 \text{ as } t_1 \rightarrow t_2$$

Uniformly for all $x \in \overline{B}_R(0)$. Therefore $Q(\omega)\overline{B}_R(0)$ is an equi-continuous set in $X$, then we know it is compact by Arzela-Ascoli theorem for each $\omega \in \Omega$. Consequently, $Q(\omega)$ is a completely continuous random operator on $\overline{B}_R(0)$.

Finally, we suppose there exists such an element $u$ in $X$ with $||u|| = R$ satisfying $Q(\omega)u(t) = \alpha u(t,\omega)$ for some $\omega \in \Omega$, where $\alpha > 1$. Now for this $\omega \in \Omega$ we have

$$|u(t,\omega)| \leq \frac{1}{\alpha} |Q(\omega)u(t)|$$
\[
\leq \int_0^1 G(t,s) f(s, u(s, \omega), u'(s, \omega), u''(s, \omega), u'''(s, \omega), \omega) \, ds
\]
\[
\leq r_1 \int_0^1 \gamma(s, \omega) \psi |u(s, \omega)| \, ds
\]
\[
\leq r_1 \| \gamma(t) \|_{L^1} \psi \| u(\omega) \| \quad \text{for all } t \in J
\]

Taking supremum over \( t \) in the above inequality yields.

\[
R = \| u(\omega) \| \leq r_1 \| \gamma(t) \|_{L^1} \psi(r)
\]

for some \( \omega \in \Omega \). This is contradicts to the condition of statement. This all the condition of Lemma (6.3.1) are satisfied. Hence the random equation \( Q(\omega) x(t) = x(t, \omega) \) has a random solution in \( \overline{B}_R(0) \) i.e. there is a measurable function \( \xi : \Omega \rightarrow \overline{B}_R(0) \) such that \( Q(\omega) \xi(t) = \xi(t, \omega) \) for all \( t \in J, \omega \in \Omega \). As a result, the random equation (6.2.1) has a random solution defined on \( J \). This completes the proof.

### 6.5 EXTREMAL RANDOM SOLUTIONS:

It is sometime desirable to know the realistic behavior of random solution of a given dynamical system. Therefore we prove the existence of extremal positive random solution of equation (6.2.1) defines on \( \Omega \times J \).

We introduce an order relation \( \leq \) in \( C(J,R) \) with the help of cone \( K \) defined by

\[
K = \{ x \in C(J,R) : x(t) \geq 0 \text{ on } J \}
\]

Let \( x, y \in X \) then \( x \leq y \) if and only if \( y - x \in K \). Thus we have \( x \leq y \iff x(t) \leq y(t) \) for all \( t \in J \). It is known that the cone \( K \) is normal in \( C(J,R) \). For any function \( a, b : \Omega \rightarrow C(J,R) \) we define a random interval \([a, b]\) in \( C(J,R) \) by

\[
[a, b] = \{ x \in C(J,R) : a(\omega) \leq x \leq b(\omega) \ \forall \omega \in \Omega \} = \bigcap_{\omega \in \Omega} [a(\omega), b(\omega)]
\]

**Definition 6.5.1:** A operator \( Q : \Omega \times X \rightarrow X \) is called non-decreasing if \( Q(\omega) x \leq Q(\omega)y \) for all \( \omega \in \Omega \) and for all \( x, y \in X \) for which \( x \leq y \)

We use the following random fixed point theorem.
Theorem 6.5.1: Let \((\Omega, A)\) be a measurable space and let \([a, b]\) be a random order integral in the separable Banach space \(X\). Let \(Q : \Omega \times [a, b] \rightarrow [a, b]\) be a completely continuous and non-decreasing random operator. Then \(Q\) has a minimal fixed point \(x_\ast\) and maximal random fixed point \(y^\ast\) in \([a, b]\). Moreover the sequence \(\{Q(\omega)x_n\}\) with \(x_0 = a\) and \(\{Q(\omega)y_n\}\) with \(y_0 = b\) converge to \(x_\ast\) and \(y^\ast\).

Definition 6.5.2: A measurable function \(\alpha : \Omega \rightarrow C(J, R)\) is called a lower random solution of equation (6.2.1) if

\[
\alpha(t, \omega) \leq f(t, \alpha(t, \omega), \alpha'(t, \omega), \alpha''(t, \omega), \alpha'''(t, \omega), \omega) \quad a.e. \ t \in J
\]

\[
\alpha(0, \omega) \leq 0; \alpha'(0, \omega) \leq 0; \alpha''(0, \omega) \leq 0; \alpha'''(0, \omega) \leq 0 \quad \text{for all } t \in J \text{ and } \omega \in \Omega.
\]

Similarly a measurable function \(\beta : \Omega \rightarrow C(I, R)\) is called a upper random solution of equation (6.2.1) if

\[
\beta(t, \omega) \geq f(t, \beta(t, \omega), \beta'(t, \omega), \beta''(t, \omega), \beta'''(t, \omega), \omega) \quad a.e. \ t \in J
\]

\[
\beta(0, \omega) \geq 0; \beta'(0, \omega) \geq 0; \beta''(0, \omega) \geq 0; \beta'''(0, \omega) \geq 0 \quad \text{for all } t \in J \text{ and } \omega \in \Omega.
\]

Definition 6.5.3: A random solution of (6.2.1) is called maximal if for all random solution of (6.2.1), one has \(x(t, \omega)\) for all \(t \in J\) and \(\omega \in \Omega\).

\((A_3)\) The (6.2.1) has lower random solution \(a\) and an upper random solution \(b\) with \(a \leq b\) on \(J\).

\((A_4)\) The function \(h : J \times \Omega \rightarrow R_+\) defined by

\[
h(t, \omega) = |f(t, a(t, \omega), \omega)| + |f(t, b(t, \omega), \omega)| \text{ is Lebegue integrable in } t \text{ for all } \omega \in \Omega.
\]

Hypothesis \((A_3)\) holds in particular when there exist measurable function \(u, v : \Omega \rightarrow C(J, R)\) such that for each \(\omega \in \Omega\),

\[
u(t, \omega) \leq f(t, x, y, \omega) \text{ for all } t \in J \text{ and } x, y \in R.
\]

In this case the lower and upper random solution of equation (6.2.1) are given by

\[
a(t, \omega) = \int_0^1 G(t, s) u(s, \omega) \, ds
\]
and \( b(t, \omega) = \int_0^1 G(t, s) \nu(s, \omega) \, ds \) respectively.

If \( f \) is \( L^1 \)-Carathéodory on \( R \times \Omega \) then \( (A_4) \) remains valid.

**Theorem 6.5.2**: Assume that \((A_1) - (A_4)\) holds; then (6.2.1) has a minimal random solution \( x_*(\omega) \) and maximal random solution \( y^*(\omega) \) defined on \( J \) moreover,

\[
x_*(t, \omega) = \lim_{n \to \infty} x_n(t, \omega),
\]

\[
y^*(t, \omega) = \lim_{n \to \infty} y_n(t, \omega) \quad \text{for all } t \in J \text{ and } \omega \in \Omega
\]

where the random sequences \( \{ x_n(\omega) \} \) and \( \{ y_n(\omega) \} \) are given by

\[
x_{n+1}(t, \omega) = \int_0^1 G(t, s) \left| f(s, x_n(s, \omega), x_n'(s, \omega), x_n''(s, \omega), x_n'''(s, \omega), \omega) \right| \, ds
\]

for all \( n \geq 0 \) with \( x_0 = a \) and

\[
y_{n+1}(t, \omega) = \int_0^1 G(t, s) \left| f(s, y_n(s, \omega), y_n'(s, \omega), y_n''(s, \omega), y_n'''(s, \omega), \omega) \right| \, ds
\]

for all \( n \geq 0 \) with \( y_0 = b \) and for all \( t \in J, \omega \in \Omega \).

**Proof**: We set \( X = C(J, R) \) and define an operator \( Q : \Omega \times [\alpha, \beta] \to X \) by Lemma [6.3.2]. It can be shown as in the proof of theorem (6.4.1) that \( Q \) is a random operator on \( \Omega \times [a, b] \). We show that it is non-decreasing random operator on \([a, b]\).

Let \( x, y : \Omega \to [a, b] \) be arbitrary such that \( x \leq y \) on \( \Omega \). Then

\[
Q(\omega)x(t) \leq \int_0^1 G(t, s) f(s, x(s, \omega), x'(s, \omega), x''(s, \omega), x'''(s, \omega), \omega) \, ds
\]

\[
\leq \int_0^1 G(t, s) f(s, y(s, \omega), y'(s, \omega), y''(s, \omega), y'''(s, \omega), \omega) \, ds
\]

\[
\leq Q(\omega)y(t)
\]

for all \( t \in J \) and \( \omega \in \Omega \). As a result \( Q(\omega)x \leq Q(\omega)y \) for all \( \omega \in \Omega \) and that \( Q \) is non-decreasing random operator on \([a, b]\).

Second by the hypothesis \((A_3)\)

\[
a(t, \omega) \leq Q(\omega)a(t)
\]

\[
= \int_0^1 G(t, s) f(s, a(s, \omega), a'(s, \omega), a''(s, \omega), a'''(s, \omega), \omega) \, ds
\]
\[
\leq \int_0^1 G(t, s) f(s, x(s, \omega), x'(s, \omega), x''(s, \omega), x'''(s, \omega), \omega) \, ds \\
= Q(\omega) x(t) \\
\leq Q(\omega) b(t) \\
= \int_0^1 G(t, s) f(s, b(s, \omega), b'(s, \omega), b''(s, \omega), b'''(s, \omega), \omega) \, ds \\
\leq b(t, \omega)
\]

for all \( t \in J \) and \( \omega \in \Omega \). As a result \( Q \) defines a random operator \( Q : \Omega \times [a, b] \rightarrow [a, b] \).

Next since \((A_4)\) holds, the hypothesis \((A_2)\) is satisfied with \( \gamma(t, \omega) = h(t, \omega) \) for all \((t, \omega) \in J \times \Omega \) and \( \psi(r) = 1 \) for all real number \( r \geq 0 \). Now it can be shown as in the proof of Theorem (6.4.1) that the random operator \( Q(\omega) \) is completely continuous on \([a, b]\) into itself. Thus the random operator \( Q(\omega) \) satisfies all the conditions of theorem (6.4.1) and so the random operator equation \( Q(\omega)x = x(\omega) \) has a least and greatest random solution in \([a, b]\). Consequently the equation (6.2.1) has a minimal and a maximal random solution defined on \( J \). This completes the proof.

\[\cdots\]