CHAPTER - 4

EXISTANCE THEORY FOR SECOND ORDER NONLINEAR FUNCTIONAL RANDOM DIFFERENTIAL EQUATION IN BANACH ALGEBRA.

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CHAPTER - 4
EXISTENCE THEORY FOR SECOND ORDER
NONLINEAR FUNCTIONAL RANDOM DIFFERENTIAL
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4.1 INTRODUCTION:

We have seen the existence theory and extremal solution for second order nonlinear functional random differential equation in the previous chapter which is discussed in Dhage [14, 16, 37] and any others. Now in this chapter we prove the existence of solution for second order nonlinear functional random differential equation in Banach Algebra under the Carathe’odory condition.

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4.2 STATEMENT OF PROBLEM:

Let \( R \) denote the real line and Let \( I_0 = [-s, 0] \) and \( I = [0, q] \) be two closed and bounded interval in \( R \) for some \( s > 0 \) and \( q > 0 \). Let \( J = I_0 \cup I \). Let \( C(I_0, R) \) denote the space of continuous \( R \) valued function \( I_0 \). We equip the space \( C = C(I_0, R) \) with a supremum norm \( \| \cdot \|_c \) defined by

\[
\| \cdot \|_c = \sup_{t \in I_0} |x(t)|
\]

Clearly \( C \) is a Banach Space which is also a Banach Algebra with respect to this norm.

For a given \( t \in I \) define a continuous \( R \)-valued function.

\[
x_t : I_0 \rightarrow R \quad \text{by}
\]

\[
x_t(\theta) = (t + \theta), \quad \theta \in I_0
\]
Let $(\Omega, A)$ be a measurable space. Given a random variable $q : \Omega \to C$

We consider a Nonlinear Functional Random Differential Equation (in short NFRDE)

$$
\left( \frac{x(t, \omega)}{f(t, x(t, \omega), \omega)} \right)'' = g(t, x_t(\omega), \omega) \quad \text{a.e. } t \in I
$$

$$
x_0(\omega) = q_0(\omega)
$$

$$
x'_0(\omega) = q_1(\omega)
$$

for all $\omega \in \Omega$ where

$$
f : I \times R \times \Omega \to R; \quad g : I \times C \times \Omega \to R \setminus \{0\}; \quad q_1, q_2 : \Omega \to R.
$$

**Theorem : 4.2.1** : [14] : let $X$ be a Banach algebra and let $A, B, C : X \to X$ be three operator such that

(i) $A$ and $C$ are Lipschitzicians with functions $q$ and $\psi$ respectively.

(ii) $B$ is compact and continuous.

(iii) $M q(r) + \psi(r) < r$, $r > 0$ where $M = B(X) = \sup \{ Bx : x \in X \}$.

Then

(i) The operator equation $\lambda A(x/\lambda) Bx + \lambda C(x/\lambda) = x$ has a solution for $\lambda = 1$

or

(ii) The solution set $\epsilon = \{ u \in X/\lambda A(x/\lambda) B x + \lambda C(x/\lambda) = x ; 0 < \lambda < 1 \}$ is unbounded.

Before going to the main result, we state the following useful lemmas.

**Lemma : 4.2.1** : Assume that all the conditions of theorem (4.2.1) hold then map $T : X \to X$ define by $Tx = Ax Bx + Cx$ is continuous on $X$.

**Lemma : 4.2.2** : Assume that all the conditions of theorem (4.2.1) hold then the set $\text{Fix}(T) = \{ x \in X/ Ax Bx + Cx = x \}$ is compact.
Theorem : 4.2.2 : Let X be a separable Banach Algebra and let $A, B, C : \Omega \times X \to X$ be three random operators satisfying for each $\omega \in \Omega$.

a) $A(\omega)$ and $C(\omega)$ are $D -$ Lipschitzicians with D-functions $q_A(\omega)$ and $q_{AC}(\omega)$ respectively.

b) $B(\omega)$ is compact and continuous.

c) $M(\omega) q_A(\omega)(r) + q_C(\omega)(r) < r$, $r > 0$ for all $\omega \in \Omega$ where $M(\omega) = \| B(\omega)(x) \|$

d) The set $\varepsilon = \{ u \in X / \lambda(\omega) A(\omega)(u/\lambda) B(\omega) u + \lambda(\omega) C(\omega)(u/\lambda) = u \}$ is bounded for all measurable $\lambda : \Omega \to R$ with $0 < \lambda(\omega) < 1$

Then the random equation

$$A(\omega)x B(\omega)x + C(\omega)x = x \quad \text{--- (4.2.2)}$$

has a random solution.

Corollary : 4.2.1 : Let X be a separable Banach Algebra and let $A, B, C : \Omega \times X \to X$ be three random operators satisfying for each $\omega \in \Omega$.

a) $A(\omega)$ and $C(\omega)$ are D- Lipschitzicians with Lipschitz constant $\alpha(\omega)$ and $\beta(\omega)$ respectively.

b) $B(\omega)$ is compact and continuous.

c) $\alpha(\omega) M(\omega) + \beta(\omega) < 1$ for all $\omega \in \Omega$ where $M(\omega) = \| B(\omega)(x) \|$

d) The set $\varepsilon = \{ u \in X / \lambda A(\omega)(u/\lambda) B(\omega) u + \lambda C(\omega)(u/\lambda) = u \}$ is bounded for all $0 < \lambda < 1$

Then the random equation (4.2.1) has a random solution and the set of such random solution is compact.

On taking $C(\omega) = 0$ in theorem (4.2.2) we obtain

Theorem : 4.2.3 : Let X be a separable Banach Algebra and $A, B : \Omega \times X \to X$ be two random operator satisfying for each $\omega \in \Omega$.  

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a) $A(\omega)$ is Lipschitzicians with functions $q_A(\omega)$.

b) $B(\omega)$ is compact and continuous.

c) $M(\omega)q_A(\omega) < r$, $r > 0$ for all $\omega \in \Omega$ where $M(\omega) = \|B(\omega)(x)\|$

d) The set $\varepsilon = \{ u \in X/\lambda(\omega) A(\omega)(u/\lambda) B(\omega)u = u \}$ is bounded for all measurable $\lambda : \Omega \to R$ with $0 < \lambda(\omega) < 1$

Then the random equation

$$A(\omega)x B(\omega)x = x$$  \hspace{1cm} --- (4.2.3)

has a random solution.

**Corollary : 4.2.2** : Let $X$ be a separable Banach Algebra and let $A, B : \Omega \times X \to X$ be two random operator satisfying for each $\omega \in \Omega$.

a) $A(\omega)$ is $D$-Lipschitzicians with Lipschitz costant $\alpha(\omega)$.

b) $B(\omega)$ is compact and continuous.

c) $\alpha(\omega) M(\omega) < 1$ for all $\omega \in \Omega$ where $M(\omega) = \|B(\omega)(x)\|$

d) The set $\varepsilon = \{ u \in X/\lambda A(\omega)(u/\lambda) B(\omega)u = u \}$ is bounded for all

$$0 < \lambda(\omega) < 1$$

Then the random equation (4.2.2) has a random solution and the set of such random solution is compact.

In the following section we shall prove an existence of the random solution of a nonlinear functional random differential equation (4.2.1) in Banach Algebra.

**4.3 EXISTENCE THEORY FOR RANDOM SOLUTION :**

Let $M(J,R)$, $B(J,R)$, $BM(J,R)$ and $C(J,R)$ denote respectively the space of all measurable, bounded, bounded and measurable and continuous real-valued function on $J$. Notice that $C(J,R) \subset BM(J,R) \subset M(J,R)$
We shall obtain the existence of the random solution of the NFRDE (4.2.1) is the space \( X = C(J,R) \cap C(I_0,R) \cap AC(J,R) \) under some suitable condition.

Define a norm \( \| \cdot \| \) in \( X \) by
\[
\| x \| = \max_{t \in J} |x(t)| \quad (4.3.1)
\]
Clearly \( X \) is a separable Banach Algebra with this maximum norm. By \( L'(J,R) \) we denote the space of all Lebesgue integral real valued function on \( J \) equipped with a norm \( \| \cdot \|_{L'} \) given by
\[
\| x \|_{L'} = \int_{t_0}^{t_1} |x(t)| \, ds \quad (4.3.2)
\]
Now the NFRDE (4.2.1) is equivalent to the functional Random Integral equation (in short FRIE)
\[
x(t, \omega) = \begin{cases} 
  f(t, x(t, \omega), \omega)[q_0(0, \omega) + q_1(0, \omega)t + \int_0^t g(s, x_s(\omega), \omega) \, ds], & t \in I \\
  q(t, \omega) & \text{if } t \in I_0
\end{cases} \quad (4.3.3)
\]
i.e.
\[
q_1(0, \omega)t f(t, x(t, \omega), \omega) + f(t, x(t, \omega), \omega)[q_0(0, \omega) + \int_0^t g(s, x_s(\omega), \omega) \, ds], & t \in I
\]
\[
x(t, \omega) = \begin{cases} 
  q(t, \omega) & \text{if } t \in I_0
\end{cases}
\]
We need the following definition.

**Definition** : 4.3.1 : A mapping \( \beta : J \times C \times \Omega \to R \) is said to satisfy a condition of \( \omega \)-Caratheodory or simply called \( \omega \)-Caratheodory if for each \( \omega \in \Omega \)

(i) \( t \to \beta(t, x, \omega) \) is measurable for each \( x \in C \).

(ii) \( x \to \beta(t, x, \omega) \) is continuous a.e \( t \in I \)

Further a \( \omega \)-Caratheodory function \( \beta \) is called \( L_{\omega'} \)-Caratheodory if
there exist a function $h : \Omega \rightarrow L'(J, R)$ such that

$$|\beta(t, x, \omega)| \leq h(t, \omega) \quad a.e. \ t \in I$$

for all $x \in R$ and $\omega \in \Omega$

We consider the following hypothesis in the sequel.

$(H_1)$ The function $q : \Omega \rightarrow C(J, R)$ is measurable.

$(H_2)$ The function $f : \Omega \rightarrow C(J \times R, R)$ is measurable and there exist a function $\alpha_1 : \Omega \rightarrow B(I, R)$ with bound $\|\alpha_1(\omega)\|$ satisfying for each $\omega \in \Omega$.

$$|f(t, x, \omega) - f(t, y, \omega)| \leq \alpha_1(t, \omega)|x - y| \quad a.e. \ t \in I$$

for all $x, y \in C$

$(H_3)$ The function $\omega \rightarrow g(t, x, \omega)$ is measurable for all $t \in I$ and

$(H_4)$ The function $g(t, x, \omega)$ is $L'_\omega$–Caratheodory.

$(H_5)$ There exist function $\gamma : \Omega \rightarrow L'(I, R)$ with $\gamma(t, \omega) > 0$ a.e. $I$, for all $\omega \in \Omega$ and conditions non decreasing function $\psi : [0, \infty) \rightarrow (0, \infty)$ satisfying for each $\omega \in \Omega$.

$$|g(t, x, \omega)| \leq \gamma(t, \omega)\psi(|x|) \quad a.e. \ t \in I \quad --- (4.3.4)$$

for all $x \in C$.

**Theorem :** 4.3.1: Assume that the hypothesis $(H_1) - (H_5)$ hold. Suppose further that

$$\int_{C_1(\omega)} \frac{ds}{\psi(s)} > C_2(\omega) \|r\|_{L'}$$

$$--- (4.3.5)$$

where

$$C_1(\omega) = \frac{[1 + F(\omega)]\|q(\omega)\|_{L'}}{1 - \|\alpha_1(\omega)\| [\|q(\omega)\|_{C'} + \|h(\omega)\|_{L'}]}$$

$$C_2(\omega) = \frac{F(\omega)}{1 - \|\alpha_1(\omega)\| [\|q(\omega)\|_{C'} + \|h(\omega)\|_{L'}]}$$

Then the NFRDE (4.2.1) has a random solution on $I$. 

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**Proof:** Let $X = \mathcal{C}(J, R)$ and define three mapping $A, B, C: \Omega \times X \rightarrow X$ by

$$A(\omega) x(t) = \begin{cases} f(t, x(t, \omega), \omega) & \text{if } t \in I \\ 1 & \text{if } t \in I_0 \end{cases} \quad \text{--- (4.3.6)}$$

and

$$B(\omega) x(t) = \begin{cases} q_0(0, \omega) + \int_0^t g(s, x(s, \omega), \omega) \, ds & \text{if } t \in I \\ q(t, \omega) & \text{if } t \in I_0 \end{cases} \quad \text{--- (4.3.7)}$$

and

$$C(\omega) x(t) = \begin{cases} q_1(0, \omega) f(t, x(t, \omega), \omega) & \text{if } t \in I \\ q_1(t, \omega) & \text{if } t \in I_0 \end{cases} \quad \text{--- (4.3.8)}$$

Then the FRIE (4.3.3) is transformed into the random operator equation

$$A(\omega) x(t) B(\omega) x(t) + C(\omega) x(t) = x(t, \omega) \quad \text{--- (4.3.9)}$$

for $t \in J$ and $\omega \in \Omega$.

We shall show that the operator $A(\omega), B(\omega)$ and $C(\omega)$ satisfy all the conditions of corollary (4.2.1) on $X$. This will be done in the following steps.

**Step 1:** First we show that $A(\omega)$ and $B(\omega)$ are random operator on $X$. Since the function $f(t, x, \omega)$ is measurable in $\omega$ for all $t \in I$ and $x \in R$ and since constant function is measurable on $\Omega$ the function $\omega \rightarrow A(\omega)x$ is measurable for all $x \in X$. Hence $A(\omega)$ is a random operator on $X$. Now by $(H_3)$ the function $\omega \rightarrow g(t, x, \omega)$ is measurable for all $t \in I$ and $x \in \mathcal{C}$. we know that the Riemann integral is a limit of a finite sum of measurable function, which is again measurable.
Therefore the function $\omega \rightarrow \int_0^t g(s, x_s(\omega), \omega) \, ds$ is measurable. Hence $B(\omega)$ is random operator on $X$.

Similarly it is shown that $C(\omega)$ is a random operator on $X$.

Again since the function

$$t \rightarrow A(\omega) x(t) = \begin{cases} f(t, x(t, \omega), \omega) & \text{if } t \in I \\ 1 & \text{if } t \in I_0 \end{cases}$$

$$t \rightarrow B(\omega) x(t) = \begin{cases} q_0(0, \omega) + \int_0^t g(s, x_s(\omega), \omega) \, ds & \text{if } t \in I \\ q(t, \omega) & \text{if } t \in I_0 \end{cases}$$

$$t \rightarrow C(\omega) x(t) = \begin{cases} q_1(0, \omega) f(t, x(t, \omega), \omega) & \text{if } t \in I \\ q_1(t, \omega) & \text{if } t \in I_0 \end{cases}$$

are continuous. The function $A(\omega) x(t), B(\omega) x(t)$ and $C(\omega) x(t)$ are continuous and hence bounded and measurable on $f$ for each $\omega \in \Omega$. Hence $A(\omega), B(\omega)$ and $C(\omega)$ define the random operator $A, B, C : \Omega \times X \rightarrow X$.

**Step II**: Next we show that $A(\omega)$ is Lipschitzian random operator on $X$. Let $x, y \in X$.

Then by (H$_1$)

$$|A(\omega)x(t) - A(\omega)y(t)| = |f(t, x(t, \omega), \omega) - f(t, y(t, \omega), \omega)|$$

$$\leq \| \alpha_1 \| \| x(\omega) - y(\omega) \|$$

for all $t \in I$. Similarly

$$|A(\omega)x(t) - A(\omega)y(t)| = |1 - 1| = 0 \leq \| \alpha_1 \| \| x(\omega) - y(\omega) \|$$

for all $t \in I_0$. Thus
for all $t \in I$ and $\omega \in \Omega$.

Taking the maximum over $t$ in the above inequality. We obtain

$$ | A(\omega)x(t) - A(\omega)y(t) | \leq \| \alpha_1(\omega) \| \| x(\omega) - y(\omega) \| $$

This shows that $A(\omega)$ is a Lipschitzian random operator on $X$ with Lipschitz constant $\| \alpha_1(\omega) \|$.

Similarly it is shown that $C(\omega)$ is a Lipschitzian random operator on $X$ with Lipschitz constant $\| \beta_1(\omega) \|$.

**Step III**: Next we show that $B(\omega)$ is a continuous and compact random operator on $X$. Using the standard argument as in Granas et.al [20] it is shown that $B(\omega)$ is a continuous random operator on $X$. To show that $B(\omega)$ is compact. It is sufficient to show that $B(\omega)(x)$ is uniformly bounded and equi-continuous set in $X$ for each $\omega \in \Omega$. First we show that $B(\omega)(x)$ is uniformly bounded for each $\omega \in \Omega$. Let $x \in X$ be arbitrary. Thus

$$ B(\omega)x(t) = \begin{cases} 
q_0(0, \omega) + \int_0^t g(s, x_s(\omega), \omega) \, ds & \text{if } t \in I \\
q(t, \omega) & \text{if } t \in I_0
\end{cases} $$

for all $\omega \in \Omega$ Since $g$ is $L'_x(\omega)$ - Caratheodeory

We have

$$ |B(\omega)x(t)| \leq \| q_0(\omega) \|_{\mathcal{C}} + \int_0^t |g(s, x_s(\omega), \omega)| \, ds $$

$$ = \| q_0(\omega) \|_{\mathcal{C}} + \| h(\omega) \|_{L'} $$

Taking the maximum over $t$, one obtains $\| B(\omega)x \| \leq K$ for all $x \in X$ where

$$ K = \| \varphi_0(\omega) \|_{\mathcal{C}} + \| h(\omega) \|_{L'} $$

This shows that $B(\omega)(x)$ is a uniformly bounded subset of $X$ for each $\omega \in \Omega$. Secondly we show that $B(\omega)(x)$ is an equi-continuous set in $X$ for each $\omega \in \Omega$. 

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Now there are three cases

**Case I:** Let \( t, \tau \in I \) Then for any \( x \in X \) we have by (4.3.7)

\[
|B(\omega)x(t) - B(\omega)x(\tau)| \leq |\int_0^t g(s, x_s(\omega), \omega) \, ds - \int_0^\tau g(s, x_s(\omega), \omega) \, ds |
\]

\[
\leq |\int_\tau^t g(s, x_s(\omega), \omega) \, ds |
\]

\[
\leq |\int_\tau^t h(s, \omega) \, ds |
\]

\[
= |p(t, \omega) - p(\tau, \omega) |
\]

Where \( p(t) = \int_0^t h(s, \omega) \, ds \)

Now \( p \) is a continuous function on a compact interval \( I \) so it is uniformly continuous there and hence

\[
|B(\omega)x(t) - B(\omega)x(\tau)| \to 0 \text{ as } t \to \tau \text{ for each } \omega \in \Omega.
\]

**Case II:** Again let \( \tau \in I_0 \) and \( t \in I \) then we have

\[
|B(\omega)x(t) - B(\omega)x(\tau)| \leq |q_0(0, \omega) - q_0(\tau, \omega)| + \int_\tau^t g(s, x_s(\omega), \omega) \, ds
\]

\[
\leq |q_0(0, \omega) - q_0(\tau, \omega)| + |p(t, \omega) - p(\tau, \omega)|
\]

Where the function \( p \) defined above, again \( q_0 \) is a continuous on compact interval \( I_0 \)

And the function \( p \) is continuous on compact interval \( I \), so they are uniformly continuous and hence

\[
|B(\omega)x(t) - B(\omega)x(\tau)| \to 0 \text{ as } t \to \tau
\]

**Case III:** Similarly \( t, \tau \in I_0 \) Thus we have

\[
|B(\omega)x(t) - B(\omega)x(\tau)| = |q(t, \omega) - q(\tau, \omega)| \to 0 \text{ as } t \to \tau
\]

Thus in all three case we have

\[
|B(\omega)x(t) - B(\omega)x(\tau)| \to 0 \quad \text{as } t \to \tau
\]

for \( t, \tau \in I_0 \) and \( \omega \in \Omega \).
Hence $B(\omega)(x)$ is equicontinuous set in $X$ for each $\omega \in \Omega$. This further in view of Arzela Ascolli Theorem implies that $B(\omega)(x)$ is compact for each $\omega \in \Omega$. Hence $B(\omega)$ is a continuous and compact random operator on $X$

**Step IV :** Here

\[
M(\omega) = \| B(\omega)(x) \| \\
= \sup \{ \| B(\omega)(x) \| : x \in X \} \\
= \sup_{x \in X} \{ max_{t \in J} |B(\omega)x(t)| \} \\
\leq \| q(\omega) \|_c + \sup_{x \in X} \{ max_{t \in J} \int_0^t g(s, x_s(\omega), \omega) \, ds \} \\
= \| q(\omega) \|_c + \| h(\omega) \|_{L^r}
\]

Therefore

\[
\| \alpha_1(\omega) \| M(\omega) + \| \beta_1(\omega) \| = \| \alpha_1(\omega) \| [\| q(\omega) \|_c + \| h(\omega) \|_{L^r}] + \| \beta_1(\omega) \|_c
\]

for all $\omega \in \Omega$

**Step V :** Finally we show that condition (d) of corollary (4.2.1) is satisfied. Let $u \in E$ be arbitrary. Then we have for all $\omega \in \Omega$.

\[
\lambda u(t, \omega) = A(\omega)u(t)B(\omega)u(t) + C(\omega)u(t)
\]

\[
= \begin{cases} \\
q_1(0, \omega) t f(t, u(t, \omega), \omega) + f(t, u(t, \omega), \omega)[ q_0(0, \omega) \\
+ \int_0^t g(s, u_s(\omega), \omega) \, ds ] & , \quad t \in I \\
q(t, \omega) & , \quad t \in I_0
\end{cases}
\]

for some real number $\lambda > 1$

Therefore

\[
|u(t, \omega)| < \lambda^{-1} q(t, \omega) + \lambda^{-1} [q_1(0, \omega) t f(t, u(t, \omega), \omega) + \\
f(t, u(t, \omega), \omega)[ q_0(0, \omega) + \int_0^t g(s, u_s(\omega), \omega) \, ds ]
\]
\[
\leq \| q(\omega) \|_C + \lambda^{-1} |f(t,u(t,\omega),\omega)| + f(t,u(t,\omega),\omega)|[ \quad q_0(0,\omega) + \int_0^t h(s,\omega) ds ]
\]

\[
\leq C_1(\omega) + C_2(\omega) + \int_0^t \gamma(s,\omega) \Psi(\| u_s(\omega) \|_C) ds \tag{4.3.10}
\]

Where

\[
C_1(\omega) = \frac{1 + F(\omega) \| q(\omega) \|_C + F(\omega)}{1 - \| a_1(\omega) \| \| q(\omega) \|_C + \| h(\omega) \|_{L^1} - \| b_1(\omega) \|_C}
\]

and

\[
C_2(\omega) = \frac{F(\omega)}{1 - \| a_1(\omega) \| \| q(\omega) \|_C + \| h(\omega) \|_{L^1}}
\]

Let \( m(t,\omega) = \sup_{t \in J} |u(t,\omega)| \). Then one has \( |u(t,\omega)| \leq m(t) \) and \( \| u_t(\omega) \|_C \leq m(t,\omega) \) for all \( t \in I \) and \( \omega \in \Omega \). Then there is a \( t^* \in J \) such that \( m(t,\omega) = |u(t^*,\omega)| \) for all \( \omega \in \Omega \).

Hence from inequality (4.3.10) it follows that

\[
m(t,\omega) = |u(t^*,\omega)|
\]

\[
\leq C_1(\omega) + C_2(\omega) \int_0^t \gamma(s,\omega) \Psi(\| u_s(\omega) \|_C) ds
\]

\[
\leq C_1(\omega) + C_2(\omega) \int_0^t \gamma(s,\omega) \Psi(m(s,\omega)) ds
\]

Put \( w(t,\omega) = C_1(\omega) + C_2(\omega) \int_0^t \gamma(s,\omega) \Psi(m(s,\omega)) ds \)

\[
w'(t,\omega) = C_2(\omega) \gamma(t,\omega) \Psi(m(t,\omega))
\]

\[
w(0,\omega) = C_1(\omega)
\]

which further implies that

\[
w'(t,\omega) \leq C_2(\omega) \gamma(t,\omega) \Psi(m(t,\omega))
\]

\[
w(0,\omega) = C_1(\omega)
\]

OR
\[
\frac{w'(t, \omega)}{\psi(m(t, \omega))} \leq C_2(\omega) \gamma(t, \omega)
\]

\[w(0, \omega) = C_1(\omega)\]

Integrating from 0 to \(t\) yield that

\[
\int_0^t \frac{w'(t, \omega)}{\psi(m(t, \omega))} \, ds \leq C_2(\omega) \int_0^t \gamma(s, \omega) \, ds
\]

By changing the variable formula we get

\[
\int_{C_1(\omega)} \frac{w(t, \omega)}{\psi(s)} \, ds \leq C_2(\omega) \int_0^t \gamma(s, \omega) \, ds
\]

\[\leq C_2(\omega) \int_0^a \gamma(s, \omega) \, ds
\]

\[= C_2(\omega) \| \gamma(\omega) \|_{L'}
\]

\[< \int_{C_1(\omega)}^{\infty} \frac{ds}{\psi(s)}
\]

Now by an application of mean value theorem yield that there is a constant \(M > 0\) such that \(w(t, \omega) \leq m\) for all \(t \in I\) and \(\omega \in \Omega\).

This further implies that \(|u(t, \omega)| \leq M\) for all \(t \in I\) and \(\omega \in \Omega\).

Hence the set \(\varepsilon\) is bounded and condition (d) of corollary (4.2.1) yield

Hence the random operator equation (4.3.9) and consequently by the FRDE (4.2.1) has a random solution. This completes the proof.

### 4.4 \textbf{EXISTENCE OF EXTREMAL RANDOM SOLUTION :}

#### 4.4.1 : \textbf{MONOTON TECHNIQUE :}

A non-empty and closed cone \(P\) in a Banach Space \(X\) is called cone if

(i) \(P + P \subseteq P\)

(ii) \(\lambda P \subseteq P\) for \(\lambda \in \mathbb{R}\) and

(iii) \(\{-P\} \cap P = 0\), where \(\{0\}\) is a zero element of \(X\).
The details of cones and their properties may be found in Deimling [44] and Lakshmikantham [8] etc. We introduce an order relation ‘≤’ in $X$ as follows. Let $x, y \in X$. Then we have

$$x \leq y \iff y - x \in P.$$  \hspace{1cm} \text{--- (4.4.1)}

The Banach Algebra $X$ together with the order relation $\leq$ is called an order Banach Algebra and usually denoted by $(X, \| \cdot \|, \leq)$ or simply $(X, P)$. A cone $P$ in $X$ is called normal if the norm $\| \cdot \|$ is semi monotone on $P$ i.e. if $a, b \in P$ with $a \leq b$ then there exist a constant $R > 0$ such that $\| a \| \leq R \| b \|$. Let $u, v \in X$ be such that $u \leq v$. Then the interval $[u, v]$ is a set in $X$ defined by

$$[u, v] = \{ x \in X / u(\omega) \leq x \leq v(\omega) \}$$

$$= \cap_{\omega \in \Omega} [u(\omega), v(\omega)]$$ \hspace{1cm} \text{--- (4.4.2)}

**Definition : 4.4.1:** A mapping $T : \Omega \times X \to X$ is said to be monotone non-decreasing if $T(\omega)x \leq T(\omega)y$ for all $x, y \in X$ for which $x \leq y$. Again the random operator $T : \Omega \times X \to X$ is called positive if $\text{rang} (T(\omega)) \subseteq P$ for each $\omega \in \Omega$. A well-known fixed point theorem for non-decreasing mapping in an ordered Banach algebra is given bellow.

**Theorem : 4.4.1:** Let $A, B, C : \Omega \times X \to X$ be two positive and monotone non-decreasing random operators satisfying for each $\omega \in \Omega$,

(i) $A(\omega)$ and $C(\omega)$ are Liptchitzians with Liptchitz constants $\alpha(\omega)$ and $\beta(\omega)$ respectively.

(ii) $B(\omega)$ is completely continuous, and

(iii) There two measurable function $a, b : \Omega \times X \to X$ such that

$$a(\omega) \leq A(\omega)a B(\omega)a + C(\omega)a \text{ and } A(\omega)b B(\omega)b + C(\omega)b \leq b(\omega).$$

Further if the cone $P$ in $X$ is positive and normal, then the random equation $A(\omega)x B(\omega)x = x$ has a minimal random solution $x_*$ and maximal random solution $x^*$ in $[a, b]$

Moreover,
Where
\[ x'_{n+1}(\omega) = A(\omega)x_n B(\omega)x_n + C(\omega)x_n, \quad n \geq 0 \]\nand \[ y'_{n+1}(\omega) = A(\omega)y_n B(\omega)y_n + C(\omega)y_n, \quad n \geq 0 \]

**4.4.2 : EXTREMAL RANDOM SOLUTION** : First we can prove the extremal random solution of the NFRDE (4.2.1) in \( C(J, R) \) of all bounded and measurable real-valued functions on \( J \) and then we shall obtain the existence theorems for extremal random solution to the NFRDE (4.2.1) on \( J \).

Define the order relation \( \leq \) in \( BM(J, R) \) by a cone \( P \) in \( C(J, R) \) defined by
\[ P = \{ x \in C(J, R) / x \geq 0 \} \quad \cdots (4.4.3) \]

**Definition : 4.4.2** : A random solution \( x_M \) to the nonlinear functional random differential equation (4.2.1) is said to be maximal if for any other random solution \( x \), we have \( x(t, \omega) \leq x_M(t, \omega) \quad \forall t \in J \) and \( \omega \in \Omega \). A minimal solution of equation (4.2.1) can be defined in a similar way.

We need the following definition in sequel.

**Definition : 4.4.3** : A measurable function \( a : \Omega \to C(J, R) \) is called a lower random solution of FRDE (4.2.1) if for each \( \omega \in \Omega \)
\[
\{ \begin{align*}
a''(t, \omega) &\leq f(t, a_t(\omega), \omega), \quad a.e. \ t \in J \\
a(0, \omega) &\leq q_0(\omega) ; \\
a'(0, \omega) &\leq q_1(\omega) \quad t \in I_0 \quad \text{for all} \ \omega \in \Omega.
\end{align*} \quad \cdots (4.4.4)
\]

again measurable function \( b : \Omega \to C(J, R) \) is called a upper random solution of FRDE (4.2.1) if for each \( \omega \in \Omega \)
\[ b''(t, \omega) \geq f(t, b_t(\omega), \omega), \text{ a.e. } t \in J \] 
\[ b(0, \omega) \geq q_0(\omega) \quad : \]
\[ b'(0, \omega) \geq q_1(\omega) \quad t \in J \quad \text{for all } \omega \in \Omega. \]

Note that the measurable function \( P : x \to C(J, R) \) is a random solution to equation (4.2.1) and (4.2.2) if it is a lower as well as upper random solution of the corresponding nonlinear functional random differential equation on \( J \).

We consider the following hypothesis.

(H₆) The function \( f(t, x, \omega) \) is \( L_\omega \)-Caratheodory.

(H₇) \( f \) and \( g \) define the function \( f : I \times R \times \Omega \to R - \{0\} \) and \( g : I \times C \times \Omega \to R - \{0\} \),

(H₈) The function \( f(t, x, \omega) \) and \( g(t, x, \omega) \) are monotone non-decreasing in \( X, t \in J \) for all \( \omega \in \Omega \).

(A₉) The NFRDE (4.2.1) has a lower random solution \( a \) and an upper random solution \( b \) with \( a \leq b \).

**Remark : 4.4.1 :** Suppose the hypothesis (H₆) - (H₉) holds and define \( h : J \times \Omega \to R \) by

\[ h(t, \omega) = |f(t, a_t(\omega), \omega)| + |f(t, b_t(\omega), \omega)| \quad t \in J \quad \text{for all } \omega \in \Omega \]

Then the function \( \omega \to h(t, \omega) \) is a measurable and \( t \to h(t, \omega) \) is a Lebesgue Integrable. Further for all \( x \in J \)

\[ |f(t, x, \omega)| \leq h(t, \omega) \quad \text{a.e. } t \in J \quad \text{and all } x \in [a, b]. \]

**Theorem : 4.4.2 :** Assume that the hypothesis (H₁) - (H₉) holds. If \( \|\alpha(\omega)\|_{L^1} < 1 \) for each \( \omega \in \Omega \), then equation (4.2.1) has a minimal random solution \( x_* \) and a maximal solution \( x^* \) in \( [a, b] \). Moreover

\[ x_* (\omega) = \lim_n x_n (\omega) \quad \text{and} \quad x^*(\omega) = \lim_n y_n (\omega) \]

--- (4.4.6)

And a sequence \( \{x_n\} \) and \( \{y_n\} \) are defined by
\[
x_{n+1}(t, \omega) = \begin{cases} 
    f(t, x_n(t, \omega), \omega)[q_0(0, \omega) + q_1(0, \omega)t + \int_0^t g(s, x_n(s, \omega), \omega)\,ds], & t \in I \\
    q(t, \omega) & \text{if } t \in I_0
    
\end{cases} \tag{4.4.7}
\]
for all \( n \geq 0 \) with \( x_0(\omega) = \alpha(\omega) \); and

\[
y_{n+1}(t, \omega) = \begin{cases} 
    f(t, y_n(t, \omega), \omega)[q_0(0, \omega) + q_1(0, \omega)t + \int_0^t g(s, y_n(s, \omega), \omega)\,ds], & t \in I \\
    q(t, \omega) & \text{if } t \in I_0
    
\end{cases} \tag{4.4.8}
\]
For all \( n \geq 0 \) with \( y_0(\omega) = b(\omega) \).

**Proof:** Let \( X = C(J, R) \). Define a operator \( A(\omega), B(\omega), C(\omega): \Omega \times X \rightarrow X \) by (4.4.2), (4.4.3), (4.4.4) respectively.

We shall prove that the operator \( A(\omega), B(\omega) \) and \( C(\omega) \) satisfies all the condition of theorem (4.4.1). It is shown, as in the theorem (4.4.1), that \( A(\omega) \) and \( C(\omega) \) is Lipschitz and \( B(\omega) \) completely continuous random operators on \( X \) respectively. Here the Lipschitz constant of \( A(\omega) \) is \( \|\alpha(\omega)\| \) and \( C(\omega) \) is \( \|\beta(\omega)\| \) and \( M(\omega) = \|B(\omega)[a, b]\| \) on \( X \). Then by (H_8) \( A(\omega), B(\omega) \) and \( C(\omega) \) are positive random operator on \( X \).

To prove they are monotone non-decreasing

Let \( x, y \in X \) be such that \( x \leq y \). Then for each \( \omega \in \Omega \)

\[
A(\omega)\, x(t) = \begin{cases} 
    f(t, x(t, \omega), \omega) & \text{if } t \in I \\
    1 & \text{if } t \in I_0
    
\end{cases} \leq \begin{cases} 
    f(t, y(t, \omega), \omega) & \text{if } t \in I \\
    1 & \text{if } t \in I_0
    
\end{cases} = A(\omega)\, y(t)
\]
for all $t \in I$ and $A(\omega) x(t) = 1 = A(\omega) y(t)$ for all $t \in I_0$. Hence $A(\omega)x \leq A(\omega)y$ for each $\omega \in \Omega$.

Similarly

$$B(\omega) x(t) = \begin{cases} 
q_0(0, \omega) + \int_0^t g(s, x_s(\omega), \omega) \, ds & \text{if } t \in I \\
q(t, \omega) & \text{if } t \in I_0 
\end{cases}$$

$$\leq \begin{cases} 
q_0(0, \omega) + \int_0^t g(s, y_s(\omega), \omega) \, ds & \text{if } t \in I \\
q(t, \omega) & \text{if } t \in I_0 
\end{cases}$$

$$= B(\omega) y(t)$$

for all $t \in I$ and $B(\omega) x(t) = q(t, \omega) = B(\omega) y(t)$ for all $t \in I_0$. Hence $B(\omega)x \leq B(\omega)y$ for each $\omega \in \Omega$.

And

$$C(\omega) x(t) = \begin{cases} 
q_1(0, \omega) t f(t, x(t, \omega), \omega) & \text{if } t \in I \\
q_1(t, \omega) & \text{if } t \in I_0 
\end{cases}$$

$$\leq \begin{cases} 
q_1(0, \omega) t f(t, y(t, \omega), \omega) & \text{if } t \in I \\
q_1(t, \omega) & \text{if } t \in I_0 
\end{cases}$$

$$= C(\omega) y(t)$$

for all $t \in I$ and $C(\omega) x(t) = q_1(t, \omega) = C(\omega) y(t)$ for all $t \in I_0$. Hence $C(\omega)x \leq C(\omega)y$ for each $\omega \in \Omega$. Hence $A(\omega), B(\omega)$ and $C(\omega)$ are monotone non-decreasing operator on $X$. 

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Next we show that the condition in theorem (4.2.3) is satisfied. Since $a$ is lower random solution of NFRDE (4.2.1), we have

$$a(t, \omega) = \begin{cases} f(t, x(t, \omega), \omega) [ q_0(0, \omega) + q_1(0, \omega) t + \int_0^t g(s, x_s(\omega), \omega) \, ds ] & \text{if } t \in I \\ q(t, \omega) & \text{if } t \in I_0 \end{cases}$$

for all $t \in I$ and $\omega \in \Omega$.

Similarly $a(t, \omega) \leq q(t, \omega) = A(\omega) a(t), B(\omega) a(t) + c(\omega) a(t)$ for all $t \in I$ and $\omega \in \Omega$. In consequence $a(\omega) \leq A(\omega) a(t), B(\omega) a(t) + c(\omega) a(t)$ for all $t \in I$ and $\omega \in \Omega$. Similarly, since $b$ is an upper random solution of the NFRDE (4.2.1), it is prove that $A(\omega) b(t), B(\omega) b(t) + c(\omega) b(t) \leq b(\omega)$ for all $\omega \in \Omega$.

Now,

$$M(\omega) = \| B(\omega) x \|
= \sup \{ \| B(\omega) x \| / x \in X \}
= \sup_{x \in X} \{ \max_{t \in J} | B(\omega) x(t) | \}
= \| q_0(\omega) \| c + \sup_{x \in X} \{ \max_{t \in J} \int_0^t g(s, x_s(\omega), \omega) \, ds \}
= \| q_0(\omega) \| c + \max_{t \in J} \int_0^t h(s, \omega) \, ds
= \| q_0(\omega) \| c + \| h(\omega) \|_{L^1}.$$

And $\alpha(\omega) M(\omega) + \beta(\omega) = \| \alpha(\omega) \| \| h(\omega) \|_{L^1} + \| \beta(\omega) \| < 1$ for each $\omega \in \Omega$, where $h$ is a function as given in remark (4.4.1). Hence by theorem (4.4.1) the NFRDE (4.2.1) has a minimal random solution $x_*$ and maximal random solution $x^*$ in $[a, b]$. Moreover

$$x_*(\omega) = \lim_n x_n(\omega) \quad \text{and} \quad x^*(\omega) = \lim_n y_n(\omega)$$

Where $x_{n+1}(t, \omega) = A(\omega) x_n(t) + B(\omega) x_n(t) + c(\omega) x_n(t)$.
\[ x_{n+1}(t, \omega) = \begin{cases} f(t, x_n(t, \omega), \omega) \left[ q_0(0, \omega) + q_1(0, \omega) t + \int_0^t g(s, x_n(s, \omega), \omega) \, ds \right], & t \in I \\ q(t, \omega) & \text{if } t \notin I_0 \end{cases} \]

for all \( n \geq 0 \) with \( x_0(t, \omega) = a(t, \omega) \); and

\[ y_{n+1}(t, \omega) = A(\omega) y_n(\omega) B(\omega) y_n + c(\omega) y_n \]

\[ y_{n+1}(t, \omega) = \begin{cases} f(t, y_n(t, \omega), \omega) \left[ q_0(0, \omega) + q_1(0, \omega) t + \int_0^t g(s, y_n(s, \omega), \omega) \, ds \right], & t \in I \\ q(t, \omega) & \text{if } t \notin I_0 \end{cases} \]

--- (4.4.9)

for all \( n \geq 0 \) with \( y_0(t, \omega) = b(t, \omega) \).

\[
\cdots
\]