CHAPTER- VII

NON-LINEAR SPECTRAL ANALYSIS OF RAYLEIGH-BENARD CONVECTION IN A HIGH-POROSITY MEDIUM.

7.1 INTRODUCTION

The study of convection in a fluid-saturated porous layer uniformly heated from below is of considerable geophysical interest particularly, in understanding the mechanism of generating power from geothermal regions. It has also attracted considerable interest because of its importance in geohydrology, the petroleum industry, the chemical industry and in biomechanics. Apart from these applications, the problem investigated in this Chapter is also of theoretical interest in providing a qualitative theory for the details of transition from conduction regime to convection.

The free convection problems in a fluid-saturated porous medium are usually studied using Darcy model which results in a fourth-order differential equation with six boundary conditions \{ Rudraiah et al [1980], Ruth [1980A], Nield [1977]\} to govern the onset of instability. Therefore, in the mathematical sense the problem is ill-posed and we have an over specified system. In case the fluid is made up of sparse distribution of solid particles, such difficulties will be eliminated by applying the Brinkman model. Therefore, Tam[1969], Brenner[1970], Taylor[1971] and Rudraiah and Veerabhadraiah [1977,1978,1979 ] have studied the convection problems using the Brinkman model.

To solve the non-linear problem of thermal convection usually, Galerkin technique or Stuart's power integral method is adopted. The power integral method is mathematically cumbersome and the built-in orthogonality process takes account of only the even modes and further, the results about the cross-interaction of modes are missed. Rudraiah and Srimani [1980] used an iterative technique which combines the best features of the Galerkin method and Stuart's power integral technique.

We follow Poulakakos [1986] in modelling the porous media and study the convection in this Chapter. In the work presented here, following the method adopted by Kuo [1961], a solution of the steady non-linear equation is obtained for Brinkman model by expanding the dependent variables in series of orthogonal space functions and the coefficients of these functions in power series of $\eta$. The parameter $|\eta|$ is chosen in such a way that it is always less than 1 for all finite $R$. The solution obtained in this manner is applicable for larger temperature differences.

This elegant and simple method also takes care of the cross-interactions of higher convective modes caused by non-linear effects and further the present method helps us to predict a region where a steady state solution cannot exist, for the fundamental mode is damped by higher modes.
7.2 MATHEMATICAL FORMULATION AND SOLUTION

Consider a porous layer of depth \( h \) and of infinite horizontal extent saturated with a Boussinesquian Newtonian fluid. Let \( \Delta T \) be the temperature difference between lower and upper flat fluid surfaces. The porous medium is assumed to be of high porosity and hence the fluid flow is governed by the Brinkman model. The physical configuration is shown in the Fig.(7.1). An appropriate single-phase heat transport equation is chosen with the effective heat capacity ratio \( M \) and effective thermal diffusivity \( \chi_{\text{eff}} \). Thus, the governing equations for the Rayleigh-Benard situation in a Boussinesquian, Newtonian fluid-saturated porous medium are (Poulikakos [1986])

\[
\nabla \cdot \vec{q} = 0, \tag{7.1}
\]

\[
\frac{1}{\rho} \nabla p + \alpha g T \hat{k} + \nu' \nabla^2 \vec{q} - \frac{\nu}{k} \vec{q} = 0, \tag{7.2}
\]

\[
M \frac{\partial T}{\partial t} + \vec{q} \cdot \nabla T = \chi_{\text{eff}} \nabla^2 T, \tag{7.3}
\]

where

\[
M = \frac{\varepsilon (\rho c_p)_f + (1-\varepsilon)(\rho c_p)_s}{(\rho c_p)_f}, \tag{7.4}
\]

\[
\chi_{\text{eff}} = \frac{\varepsilon k_s + (1-\varepsilon)K_f}{(\rho c_p)_f}, \tag{7.5}
\]

\( \vec{V} = (u, v, w) \) is the filter-velocity, \( \hat{k} \) is the unit vector directed vertically upwards, \( p \) is the departure of pressure from the hydrostatic pressure \( p_h(z) \), \( T \) is the departure of temperature from the horizontal average \( T_h(z) \), \( \nu' \) is the effective kinematic viscosity in the porous media, \( \nu \) is the kinematic viscosity of the viscous fluid, \( \varepsilon \) is porosity, \( K_f \) is
the thermal diffusivity of the fluid, $K_s$ is the thermal diffusivity of the solid, $\chi_{\text{eff}}$ is the effective thermal diffusivity, $\alpha$ is the coefficient of thermal expansion, $(\rho C_p)_f, (\rho C_p)_s$ are the heat capacity of the fluid and solid respectively, $g$ is the gravitational acceleration and $k$ is the permeability of the porous medium and is related to $\varepsilon$ by the Karman–Cozeny relation

$$K = \frac{d_p^2 \varepsilon^2}{150(1-\varepsilon)^2} \quad (7.6)$$

where $d_p$ is the diameter of the spherical particles of the porous medium. The local and convective accelerations have been dropped in the equation (7.2) on reasons given by Poulikakos [1986].

In this Chapter, we shall restrict our study to the simplest model of cellular convection for an infinite roll in a steady state such that $\frac{\partial(\cdot)}{\partial y} = 0, \frac{\partial(\cdot)}{\partial t} = 0$.

Introducing the non-dimensional quantities

$$t' = \frac{t}{\left(\frac{h^2}{\chi_{\text{eff}}}\right)} , \quad (u',v',w') = \left(\frac{u,v,w}{\chi_{\text{eff}} h}\right) , \quad T' = \left(\frac{T}{\chi_{\text{eff}} h^2 g\varepsilon}\right) , \quad (7.7a)$$

$$\nabla' = \frac{\nabla}{\left(\frac{1}{h}\right)} , \quad p' = \frac{p}{\left(\frac{\chi_{\text{eff}}^2}{h^2}\right)} \quad (7.7b)$$

in equations (7.1) and (7.2), we get

$$-\frac{1}{p'} \nabla p' + Pr_{\text{eff}} \left\{ \Lambda \nabla^2 q' + \hat{K} - Da q' \right\} = 0 \quad (7.8)$$

and

$$q' \cdot \nabla T' = \nabla^2 T' , \quad (7.9)$$
where

\[ Pr_{\text{eff}} = \frac{\nu}{\chi_{\text{eff}}} \] (Effective Prandtl number),

\[ \Lambda = \frac{v'}{v} \] (Brinkman number),

\[ Da = \frac{h^2}{k} \] (Darcy number).

Since we shall essentially be dealing with two-dimensional convection, we introduce the non-dimensional stream function \( \Psi \) in the form

\[ u = -\frac{\partial \Psi}{\partial z}, \quad w = \frac{\partial \Psi}{\partial x} \] (7.10)

Expressing the temperature as the sum of the undisturbed mean temperature and the departure \( \Theta \) from this mean, the total non-dimensional temperature is given by

\[ T = T_0 - RaZ + \Theta, \] (7.11)

where

\[ Ra \equiv \frac{g\alpha(\Delta T')h^3}{\nu\chi_{\text{eff}}}, \] (7.12)

and \( \Delta T' \) is the dimensional temperature difference between the boundaries.

We now eliminate \( p \) by applying curl on (7.8) and introduce the non-dimensional stream function \( \Psi \) to obtain the steady state vorticity equation as

\[ \Lambda \nabla^4 \Psi - Da\nabla^2 \Psi + \frac{\partial \Theta}{\partial x} = 0 \] (7.13)
The non-dimensional steady state thermal equation after introducing the non-dimensional stream function as given in (7.10) takes the form

$$\nabla^2 \theta + Ra \frac{\partial \Psi}{\partial x} = 0,$$

(7.14)

where $H = \frac{\partial (\Psi, \theta)}{\partial (x, z)}$ is the heat advection Jacobian. (7.15)

In the case of isothermal stress-free boundaries, the vertical velocity and tangential stress vanish and the boundary conditions are given by

$$\Psi = \nabla^2 \Psi = \theta = 0.$$  

(7.16)

3 SPECTRAL EQUATIONS

All the space functions of various modes $\Psi(l, n)$ and $\theta(l, n)$ are functions of sine and cosine functions when the boundaries are free. Hence

$$\Psi(l, n) \sim \sin(l \pi x) \sin(n \pi z),$$

$$\theta(l, n) \sim \cos(l \pi x) \sin(n \pi z),$$

(7.17)

where $l$ and $n$ are integers and $a$ is the horizontal wave-number of first mode ($l=1$, $n=1$), which is geometrically unrestricted when the fluid extends to infinity in the horizontal direction. It is known that convection is set in when $R$ reaches its critical value $R_0$, a function of the wave number $a$. We choose that value for $a$ which makes $R$ minimum. The function chosen above in equation (7.17) satisfies the boundary conditions
given in the equations (7.16). We now express $\Psi$ and $\Theta$ as an infinite double Fourier expansion in the form

$$
\Psi = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \hat{\Psi}_{l,n} \sin(la\pi x)\sin(n\pi z),
$$

$$
\Theta = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \hat{\Theta}_{l,n} \cos(la\pi x)\sin(n\pi z)
$$

where $\hat{\Psi}_{l,n}$ and $\hat{\Theta}_{l,n}$ are functions of $R$ and these satisfy the boundary conditions (7.16). The representation of $\Psi$ and $\Theta$ as in the above form are introduced with a view to transforming the governing differential equations into spectral domain of the spectra of the linear case. This is justified by the proviso that the spectrum is complete and hence forms a basis for the representation of a solution of the non-linear problem. This task is more conveniently accomplished by expressing $\Psi$ and $\Theta$ in the following complex forms

$$
\Psi = -\sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \Psi_{l,n} \exp(la\pi x + nz)i,
$$

$$
\Theta = -i\pi^3 \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \Theta_{l,n} \exp(la\pi x + nz)i,
$$

where

$$
\Psi_{l,n} = -\Psi_{-l,n} = -\Psi_{l,-n} = \Psi_{-l,-n} = \frac{1}{4} \hat{\Psi}_{l,n},
$$

$$
\Theta_{l,n} = \Theta_{-l,n} = -\Theta_{l,-n} = -\Theta_{-l,-n} = \frac{1}{4\pi^3} \hat{\Theta}_{l,n},
$$

where the range of $Z$ has been extended to $-1 \leq Z \leq 1$. Furthermore, all the co-efficients $\Psi_{l,n}$ and $\Theta_{l,n}$ are real. Now let us introduce the simplified notation $\gamma$ to replace the pair
of integers $l$ and $n$. By virtue of this the summation over the components $l$ and $n$ are represented by the summation over $\gamma$. Now the task is to determine the coefficients $\Psi_\gamma$ and $\Theta_\gamma$ so that to make (7.18) satisfy the equations (7.13) and (7.14). Introducing (7.18) into the equations (7.13) and (7.14) and equating to zero the coefficients of individual components of $\exp[(lax + nx)\pi i]$, we obtain the following spectral equations

$$\Lambda \alpha_\gamma^4 \Psi_\gamma + \sigma^2 \alpha_\gamma^2 \Psi_\gamma - (la)\Theta_\gamma = 0 ,$$  \hspace{1cm} (7.20)

and

$$\alpha_\gamma^2 \Theta_\gamma - laR \Psi_\gamma = -aH_\gamma ,$$  \hspace{1cm} (7.21)

where

$$\alpha_\gamma^2 = a^2l^2 + n^2 ,$$

$$R = \frac{Ra}{\pi^4} ,$$

$$\sigma^2 = \frac{Da}{\pi^2} ,$$

$$H_\gamma = \sum\sum (l_1n_2 - l_2n_1)\Psi_{\gamma_1}\Theta_{\gamma_2} ,$$  \hspace{1cm} (7.22)

where $H_\gamma$ is the heat advection spectrum which represents the contribution to the $\gamma$ component by the non-linear interactions between the various wave number vector pairs $\gamma_1$ and $\gamma_2$ in the equations (7.13) and (7.14). The wave number pairs $\gamma_1$ and $\gamma_2$ satisfy the selection rule $\gamma_1 + \gamma_2 = \gamma$,

i.e., $l_1 + l_2 = l$, $n_1 + n_2 = n$.  \hspace{1cm} (7.23)

The above selection rule (7.23) leads to an important property of the advection spectra. The 'parity' of the wave vector $\gamma = l$, $n$ is defined as the parity of $l + n$; thus,
γ will be said to have even or odd parity according as \( l + n \) is even or odd. Therefore, \( l + n = (l_1 + n_1) + (l_2 + n_2) \) must be even. It follows that if \( \gamma_2 \) and \( \gamma_1 \) have opposite parity, the corresponding spectral elements can contribute only to an element (of an advection spectrum) having odd parity. While if \( \gamma_2 \) and \( \gamma_1 \) have same parity, the contribution is to an element with even parity, we adopt the known hypothesis that a solution may be constructed in which the temperature spectrum contains only elements of even parity. Hence we exclude all elements which correspond to wave the vector with odd parity, such as \((0,1), (0,3), (0,5) \ldots (1,2), (1,4), \ldots (2,3), (2,5) \ldots \) so on. The selection rule (7.23) also helps us to determine the order of magnitude of each spectral element.

The equations (7.20) and (7.21) together with the relation (7.22) form a closed system for the joint determination of \( \Psi_\gamma \) and \( \Theta_\gamma \).

The perturbation temperature \( \Theta \) defined in the equation (7.12) includes a part \( \overline{\Theta}(z) \) which is independent of \( x \). This part represents the modification of the mean temperature distribution by convection and is given by the components with \( l = 0 \) in equation (7.18). On the other hand, no mean wind is produced by convection and therefore \( \Psi_\gamma = 0 \) when \( l=0 \).

Eliminating \( \Theta_\gamma \) from (7.20) and (7.21), we obtain for \( l \neq 0 \)

\[
(R - R_\gamma)\Psi_\gamma = \frac{H_\gamma}{l}
\]

(7.24)

where

\[
R_\gamma = \frac{\alpha_\gamma^4(\Lambda\alpha_\gamma^2 + \sigma^2)}{a^2l^2} \quad \text{(Model Darcy Rayleigh number)}
\]

(7.25)

Clearly when \( \Lambda = 1 \) and \( \sigma^2 \to 0 \), this model Rayleigh number \( R_\gamma \) tends to the one given by Kuo and Platzman [1961] for the case of pure viscous flow. On the other hand, when \( \Lambda = 1, \sigma^2 \to \infty \) (Darcy flow), \( R_\gamma \) tends to the one given by Rudraiah and Balachandra Rao [1983]. However, the equations (7.20) and (7.21) do not reduce to the
corresponding ones of Kuo and Platzman [1960] and Rudraiah and Balachandra Rao [1983]. This is due to the neglect of the convective acceleration term in the momentum equation.

When \( l = 0 \), denoting \( \Theta_\gamma \) as \( \Theta_0 \) and \( H_\gamma \) as \( H_{on} \), we have from the equation (7.21)

\[
n\Theta_{0n} = -\sum_{\gamma_1} a_{l_1} \Psi_{n-\gamma_1} \Theta_{\gamma_1} \quad \text{(7.26)}
\]

For a given mode \( \gamma = (l, n) \), the critical model Rayleigh number \( (R_\gamma)_c \) and the corresponding critical wave number \( a_c \) are given by

\[
(R_\gamma)_c = \frac{1}{32A^2} \left\{ (3A_1^2 + A) - \sigma^2 \right\} \left\{ (3A_1^2 + A) + \sigma^2 \right\}, \quad \text{(7.27)}
\]

\[
a_c = \frac{1}{4A_1^2} \left\{ A - (\Lambda n^2 + \sigma^2) \right\}, \quad \text{(7.28)}
\]

where \( A = \sqrt{(\Lambda n^2 + \sigma^2)(9\Lambda n^2 + \sigma^2)} \).

The variation of \( R \) against \( a^2 \), for the modes (1,1), (3,1) and (2,2) are shown in Figs.(7.18)-(7.21). These figures clearly bring out the interactions of different modes for different values of \( \Lambda \) and \( \sigma^2 \). For instance, \( \gamma = (3,1) \) mode intersects the fundamental mode \( \gamma = (1,1) \) at \( a^2 = 0.3 \) when \( \Lambda = 5 \) and \( \sigma^2 = 10^2 \) and thus it damps the fundamental mode in the region IV of the Figs.(7.18) - (7.21).

7.4 METHOD OF SOLUTION OF THE SPECTRAL EQUATIONS

In order to solve the infinite sets of equations (7.20) and (7.21), we first find the most important terms of the advection spectra.
We expand \( \Psi_\gamma \) and the other quantities into power series of a parameter \( \eta \) of the form

\[
\Psi_{l,n}(\eta) = \Psi_{l,n,r}\eta^r + \Psi_{l,n,r+2}\eta^{r+2} + \ldots \quad ,
\]

where \( \Psi_{l,n,r+2i} \) represents the numerical co-efficient of \( \eta^{r+2i} \). The lowest power of \( \eta \) in the expansions of \( \Psi_\gamma \) and \( \theta_\gamma \) are defined as the order of magnitude. It is known that convection starts in the form \( \Psi_{11} \), it will be assumed as a first-order quantity so that \( r = 1 \) for \( \Psi_{11} \).

In the calculations which are presented in this Chapter, the expansions are not carried beyond the eighth power in \( \eta \) and therefore, the components \( l, n \) of any vector \( \gamma \) which we consider will not exceed the integer 9. Consequently, only one digit will be required to express \( l, n \) or \( r \) and therefore we neglect the separating commas when these integers are used as subscripts.

It may be noted that for the modes with odd \( l \) and odd \( n \), the power series expansion is odd in \( \eta \), while those modes with even \( l \) and even \( n \) are even functions of \( \eta \). Therefore in the expansions given in (7.29), the powers of \( \eta \) of two consecutive terms increase by 2.

In order to solve the infinite set of equations (7.24) and (7.26) one has to find the most important terms of advection spectral function \( H_\gamma \). That is, to find the interacting pairs \( (\gamma_1, \gamma_2) \) that contribute to the lower order term of \( H_\gamma \). The table (7.2) lists all such interacting pairs \( (\gamma_1, \gamma_2) \) that contribute to the \( \gamma \)-mode spectral function \( H_\gamma \) up to their \( \eta^7 \) or \( \eta^8 \) terms. It may be noted that when \( \gamma_2 \) and \( \gamma_1 \) are parallel, the coupling co-efficient namely \( (l_1n_2 - l_2n_1) \) vanishes; for this reason we have not entered such values in our table. With the help of this table, we express \( \theta_{0,2n}, H_\gamma \), in terms of \( \Psi_\gamma \) and \( \theta_\gamma \) and the results are listed below.

\[
\theta_{02} = a\{-\Psi_{11}(\theta_{11} - \theta_{13}) + \Psi_{13}(\theta_{11} + \theta_{15}) + \Psi_{15}\theta_{13} + 2\Psi_{22}\theta_{24} + 2\Psi_{26}\theta_{24} + 2\Psi_{24}(\theta_{22} + \theta_{26}) - 3\Psi_{31}\theta_{31} + 3\Psi_{33}(\theta_{31} + \theta_{35}) + 3\theta_{33}(\Psi_{31} + \Psi_{35}) + \ldots \quad (7.30a)
\]
\[ \theta_{04} = \frac{e}{2}(\Psi_{11}(\theta_{15} - \theta_{13}) + \theta_{11}(\Psi_{15} - \Psi_{13}) - 2\Psi_{22}\theta_{22} + \Psi_{13}\theta_{17} + \Psi_{17}\theta_{13}). \]

\[ + 2(\Psi_{22}\theta_{26} + \Psi_{26}\theta_{22}) - 3\Psi_{31}(\theta_{33} - \theta_{35}) - 3\theta_{31}(\Psi_{33} - \Psi_{35}) + ... \]  

\[(7.30b)\]

\[ \theta_{06} = \frac{e}{3}(-\Psi_{13}\theta_{13} - \Psi_{11}(\theta_{15} - \theta_{17}) - \theta_{11}(\Psi_{15} - \Psi_{17}) - 3\Psi_{33}\theta_{33} \]

\[ - 2(\Psi_{22}\theta_{24} + \Psi_{24}\theta_{22}) - 3\Psi_{31}\theta_{35} + \Psi_{35}\theta_{31}) + ... \]  

\[(7.30c)\]

\[ H_{11} = 2(\theta_{02}(\Psi_{13} - \Psi_{11}) - 2\theta_{04}(\Psi_{13} - \Psi_{15}) - 2\Psi_{22}(\theta_{13} + \theta_{31}) - 2\theta_{22}(\Psi_{13} + \Psi_{31}) \]

\[ + \Psi_{24}(\theta_{13} + \theta_{35} - 3\theta_{33} - 3\theta_{15}) + \theta_{24}(\Psi_{13} + \Psi_{35} - 3\Psi_{33} - 3\Psi_{15}) + ... \]  

\[(7.30d)\]

\[ H_{13} = 2(\Psi_{11}(\theta_{02} - 2\theta_{04} + 2\theta_{22} - \theta_{24}) + \theta_{11}(2\Psi_{22} - \Psi_{24}) + \theta_{02}\Psi_{15} + ...) \]  

\[(7.30e)\]

\[ H_{22} = 4(\theta_{02}\Psi_{24} - 2\theta_{04}\Psi_{22} - \Psi_{11}(\theta_{13} - \theta_{31}) + \theta_{11}(\Psi_{13} + \Psi_{31}) \]

\[ - \Psi_{13}(2\theta_{31} + 2\theta_{15} - \theta_{35}) + \theta_{13}(2\Psi_{15} - \Psi_{24} + \Psi_{35}) + .. \}. \]  

\[(7.30f)\]

\[ H_{31} = 2(3\theta_{02}(\Psi_{33} - \Psi_{31}) - \Psi_{11}(2\theta_{22} - \theta_{42}) + \theta_{11}(2\Psi_{22} + \Psi_{42}) + \Psi_{13}(4\theta_{22} - 5\theta_{24}) \]

\[ - 5\theta_{13}(4\Psi_{22} - 5\Psi_{24}) + ... \]  

\[(7.30g)\]

In order to obtain an asymptotic solution which is valid for a large range of values of \( R \), we choose an expansion parameter \( \eta \) of the form

\[ \eta = \sqrt{\frac{R - R_0}{R}}, \]  

\[(7.31)\]
where $R_o$ is the critical value of $R$ above which convection exists.

The advantage of using such an expansion parameter is to bring out the most important part of the solution in the lower order terms and thereby eliminating the oscillation of the solution.

In order to solve the equation (7.24), we expand $R$ in a power series of $\eta$. From the definition (7.31), we have

$$ R = \frac{R_0}{1-\eta^2} = R_0 \left[ 1 + \sum_{j=1}^{\infty} \eta^{2j} \right], $$

which has an infinite number of terms. So we rewrite the above equation to have finite number of terms in the form

$$ R = R_o + R_{os} \left( \eta^2 + \eta^4 + \ldots + \eta^{2s} \right), \quad (7.32) $$

where

$$ R_{os} = \frac{R_o}{1-\eta^{2s}}, \quad (7.33) $$

where the integer $s$ stands for the number of terms of the expansion and $R_{os}$ is treated as a constant, evaluated directly from (7.33) for a given $s$.

Using the expansion given in the equations (7.29) and (7.32) in (7.24), we obtain

$$ (R - R_o) \Psi_{\gamma,r+2j} = -\frac{1}{l} H_{\gamma,r+2j} + R_o \sum_{p=0}^{j-1} \Psi_{\gamma,r+2p}, \quad (7.34) $$

where $j = 0, 1, 2, \ldots$ and from the equation (7.20), we obtain
\[
\Theta_{\gamma,r+2j} = \frac{(\Delta^2_T)}{a^2} \Psi_{\gamma,r+2j}, \quad (7.35)
\]

where \( \Delta_T^2 = \alpha^2_T (\Lambda \alpha_T^2 + \sigma^2) \).

To solve the system of equations (7.34), first let \( \gamma=(1,1) \), \( r = 1 \) and \( j = 0 \) in (7.34). We thus obtain

\[
R_0 = R_{11} = \frac{\alpha^4_{11} (\Lambda \alpha_{11}^2 + \sigma^2)}{a^2} = \frac{(a^2+1)^2 \{ \Lambda (a^2+1)+\sigma^2 \}}{a^2}, \quad (7.36)
\]

and

\[
a_c^2 = \frac{1}{4\Lambda} \left\{ \sqrt{(\Lambda+\sigma^2)(9\Lambda+\sigma^2)} - (\Lambda+\sigma^2) \right\}. \quad (7.37)
\]

Clearly, when \( \Lambda=1, \sigma^2 \to 0 \), \( R_{11} = 6.75 \) and \( a^2 = \frac{1}{2} \) which are the known values for the viscous flow given by Kuo and Platzman [1960]. Similarly when \( \sigma^2 \to 0 \), we observe that \( R_c \approx 4\sigma^2 \) and \( a_c = 1 \) which are the known values given by Lapwood [1948].

To get a general idea, \( (R_{11})_c \) and \( a_c \) are numerically computed and listed in the Table-(7.1), which gives \( R_{11} = 4\sigma^2 \) and \( a_c = 1 \) when \( \sigma^2 \) is very large. Further in the case of viscous flow \( (\sigma^2=0) \), \( \Lambda \) has no effect on \( a_c \).

In the case of linear mode, the equation (7.34) takes the form

\[
H_{11,2p+1} = R_0 \sum_{j=1}^{p} \Psi_{11,2j}. \quad (7.38)
\]
The above form is used for finding the expansion co-efficient of first mode and the equation (7.34) is used for those of higher modes.

Now we shall demonstrate the method of finding solution by obtaining the first few expansion coefficients

To find $\Psi_{111}$ and $\theta_{111}$:

Putting $p = 1$ in the equation (7.38), gives

$$\Psi_{111} = \frac{1}{R_{0s}} H_{113}$$  \hspace{1cm} (7.39a)

Now from the equation (7.30d), we get

$$H_{113} = -2\theta_{022} \Psi_{111}$$

and from (7.30a),

$$\theta_{022} = \alpha \{ -\Psi_{111} \theta_{111} \}$$

but from (7.35)

$$\theta_{111} = \frac{\Delta_{11}^2}{\alpha} \Psi_{111}$$

Therefore,

$$\Psi_{111} = \sqrt{\frac{R_{0s}}{2\Delta_{11}^2}}$$

$$\theta_{022} = -\Delta_{11}^2 \Psi_{111}^2$$

To find $\Psi_{133}$ and $\theta_{133}$:

Putting $\gamma = 1, 3$ and $r = 3; j = 0$, equation (7.34) reduces to

$$(R_{13} - R_{11}) \Psi_{133} = -H_{133}$$  \hspace{1cm} (7.39b)
From the equation (7.30f), we find  \( H_{133} = -R_{0s} \Psi_{111} \)

substituting this in the equation (7.39b), we get

\[
\Psi_{133} = \frac{R_{0s}}{R_{13}-R_{11}} \Psi_{111}
\]

and from the equation (35)

\[
\theta_{133} = \frac{\Delta_{13}^2}{a} \Psi_{133}
\]

To find \( \Psi_{133} \) and \( \theta_{133} \)

We put \( p = 2 \) in the equation (38), we get

\[
\Psi_{113} + \Psi_{111} = \frac{1}{R_{0s}} H_{115}
\]

(7.39c)

Now from (7.30d),

\[
H_{115} = \theta_{022} (\Psi_{133} - \Psi_{113}) + \theta_{024} (-\Psi_{111})
\]

which on simplification gives

\[
H_{115} = R_{0s} \{3\Psi_{113} - [2 + \frac{\Delta_{13}^2}{\Delta_{11}^2}] \Psi_{133}\}
\]

and

\[
\theta_{024} = -\frac{R_{0s}}{2} (1 + \frac{R_{0s}}{R_{13}-R_{11}}) \]

(7.39d)

using this equation (7.39c) becomes
It may be noted that the co-efficients $\Psi_{11, r}$ and $\theta_{02, r+1}$ are determined simultaneously. The higher order co-efficients can be obtained in the same manner. Here we list some of the spectral co-efficients which are required for the further proceedings in the process of obtaining solution.

\[
\Psi_{113} = \frac{1}{2} \left( 1 + \frac{R_{0s}}{R_{13} - R_{11}} \left( 2 + \frac{\Delta^2_{13}}{\Delta^2_{11}} \right) \right) \Psi_{111}.
\]

\[
\theta_{024} = -\frac{R_{0s}}{2} \left\{ 1 + \frac{R_{0s}}{R_{13} + R_{11}} \right\}
\]

\[
\theta_{044} = \frac{\left( \Delta^2_{11} + \Delta^2_{13} \right)}{2} \Psi_{111} \Psi_{133}
\]

\[
\Psi_{133} = \frac{2}{R_{13} - R_{11}} \Psi_{111}^2 \theta_{022}
\]

\[
\Psi_{224} = \frac{2(\Delta^2_{13} - \Delta^2_{11})}{R_{22} - R_{11}} \Psi_{111} \Psi_{133}
\]

\[
\Psi_{244} = \frac{\Delta^2_{13} - \Delta^2_{11}}{a(R_{24} - R_{11})} \Psi_{111} \Psi_{133}
\]

\[
\Psi_{135} = \frac{1}{R_{13} - R_{11}} \left[ 2R_{0s} \Psi_{133} - 2 \left( \Psi_{111}(\theta_{024} - 2\theta_{044} + 2\theta_{224} - \theta_{244}) + \Psi_{113}\theta_{022} \right) + \theta_{111}(2\Psi_{224} - \Psi_{244}) \right]
\]

\[
\Psi_{155} = \frac{1}{R_{15} - R_{11}} \left[ \Psi_{111}(4\theta_{044} + 6\theta_{244}) + 2\Psi_{133}\theta_{022} + 6\theta_{111}\Psi_{244} \right]
\]

\[
\Psi_{315} = \frac{4}{3(R_{31} - R_{11})} \left[ \Psi_{111}\theta_{224} - \Psi_{224}\theta_{111} \right]
\]
\[ \Psi_{335} = \frac{2}{R_{33}-R_{11}} \left[ \Psi_{111} \theta_{244} - \theta_{111} \Psi_{244} \right] \]

\[ \Psi_{226} = \frac{-2}{R_{22}-R_{11}} \left[ \Psi_{111} (\theta_{315} - \theta_{135}) + \theta_{111} (\Psi_{135} + \Psi_{315}) + \Psi_{133} \theta_{113} - \theta_{133} \Psi_{113} \right] + \Psi_{244} \theta_{022} - R_{02} \Psi_{224}/2 \]

\[ \Psi_{355} = \frac{2(\Psi_{111} \theta_{244} - \Psi_{244} \theta_{111})}{3(R_{35}-R_{11})} \]

\[ \Psi_{246} = \frac{R_{02} \Psi_{244}}{R_{24}-R_{11}} \left[ \Psi_{111} (\theta_{135} - 3 \theta_{155} + 3 \theta_{335} - \theta_{355}) + \theta_{133} \Psi_{113} - \Psi_{133} \theta_{113} + \theta_{111} (3 \Psi_{155} + 3 \Psi_{335} - \Psi_{355} - \Psi_{135}) + 2 \theta_{022} \Psi_{224} \right] \]

\[ \Psi_{117} = \frac{1}{2 \theta_{022} - R_{02}} \left\{ \begin{array}{l}
2[2 \theta_{224} (\Psi_{135} + \Psi_{315}) - 2 \Psi_{224} (\theta_{135} + \theta_{315}) + \theta_{022} \Psi_{137} + \\
\theta_{024} (\Psi_{135} - \Psi_{115}) + \theta_{026} (\Psi_{133} - \Psi_{113}) - 2 \theta_{044} (\Psi_{135} - \Psi_{155}) + \\
\Psi_{133} \theta_{046} - 2 \Psi_{226} \theta_{133} - 2 \Psi_{133} \theta_{226} + \Psi_{246} \theta_{133} + \Psi_{133} \theta_{246} + \\
\theta_{244} (\Psi_{135} + \Psi_{355} - 3 \Psi_{335} - 3 \Psi_{155}) + \Psi_{244} (\theta_{135} + \theta_{335} - 3 \theta_{335} - 3 \theta_{155}) \right\}
\]

\[ \Psi_{266} = \frac{2}{R_{26}-R_{11}} \left[ \Psi_{111} (2 \theta_{355} + \theta_{155}) - \theta_{111} (\Psi_{155} - 2 \Psi_{355}) + \Psi_{244} \theta_{022} \right] \]

\[ \Psi_{426} = \frac{1}{2(R_{42}-R_{11})} \left[ \Psi_{111} (\theta_{315} + 3 \theta_{335}) + \theta_{111} (\Psi_{315} + 3 \Psi_{335}) \right] \]

\[ \Psi_{317} = \frac{R_{02} \Psi_{315}}{R_{31}-R_{11}} \left\{ \begin{array}{l}
3 \theta_{022} (\Psi_{335} - \Psi_{315}) - \Psi_{111} (2 \theta_{226} - \theta_{426}) - 2 \theta_{224} \Psi_{113} + 2 \Psi_{224} \theta_{113} + \\
\Psi_{133} (4 \theta_{224} - 5 \theta_{244}) + \theta_{111} (2 \Psi_{226} + \Psi_{426}) - \theta_{133} (4 \Psi_{224} - 5 \Psi_{244}) \right\} \]
After knowing these expansions in terms of \( \Lambda, \sigma^2 \) and \( q \), it is a simple task to find the spectral coefficients \( \Psi_\gamma \) and \( \Theta_\gamma \) using the equation (7.29).

The higher order co-efficients can be obtained in the same manner. Although the solution can be developed for arbitrary values of the cell-scale but the calculations becomes more tedious. We shall therefore, restrict the higher-order expansions and subsequent developments to the critical wave number of linear theory.

The values of the spectral coefficients \( \Psi_\gamma \) and \( \Theta_\gamma \) have been computed for \( \Lambda = 0.5, 1, 3; \sigma^2 = 0, 10, 10^2 \) and for different values of \( R \), up to \( R = 8R_0 \). It has been observed that within this range the solutions for \( \Psi_{11} \) and \( \Theta_{11} \) converge rapidly.

The values of the various spectral functions \( \Psi_\gamma \) and \( \Theta_\gamma \) as given by their respective fourth approximations have been plotted in Figs.(7.2)-(7.13) for different values of \( \frac{R}{R_0} \). In these figures some of the spectral co-efficients are multiplied by some proper constants making them to have the same order of magnitude. From these figures it has been observed that when \( \Lambda \) is increased with fixed \( \sigma^2 \), the first order spectral coefficients \( \Psi_{11} \) increases its value and the higher order spectral coefficients \( \Psi_{13}, \Psi_{15}, \Psi_{22}, \Psi_{31}, \Psi_{33} \) decrease. We also observe that \( \Theta_{02}, \Theta_{04}, \Theta_{06}, \Theta_{08} \) decrease and \( \Theta_{11}, \Theta_{13}, \Theta_{15}, \Theta_{22}, \Theta_{31} \) increase when either Brinkman's ratio \( \Lambda \) or \( \sigma^2 \) is increased.
7.5 CONVECTIVE HEAT TRANSPORT AND MEAN TEMPERATURE DISTRIBUTION

In the results obtainable from the non-linear solution, the dependence of the rate of heat transfer upon the imposed temperature difference is of primary interest and it can be expressed by the functional relation between the Nusselt number $N$, or the heat transport ratio $S$, and the Rayleigh number $Ra$.

The Nusselt number is the ratio of the actual heat transport rate to the rate at which heat would be transported by conduction alone for the given temperature difference. Thus,

$$
N = -\frac{\left(\frac{\partial \bar{T}}{\partial z}\right)_{z=0}}{\left(\frac{\Delta T}{h}\right)} = -\frac{1}{Ra} \left(\frac{\partial \bar{T}}{\partial z}\right)_{z=0},
$$

(7.40)

where $\bar{T}$ is the horizontally averaged temperature.

The heat transport ratio $S$ is defined as the ratio of the actual heat transport to the rate at which heat would be transported conductively at the critical Rayleigh number. It is given by

$$
S = \frac{Ra}{(Ra)_c} N = -\frac{1}{(Ra)_c} \left(\frac{\partial \bar{T}}{\partial z}\right)_{z=0}.
$$

(7.41)

According to equations (7.12) and (7.17), $\bar{T}$ is given by

$$
\bar{T} = T_0 - Raz + \sum_{n=1}^{\infty} \hat{\theta}_{0,2n} \sin(2\pi nz)
$$

(7.42)

Substituting $\hat{\theta}_{0,2n}$ from (7.19), we obtain

$$
\hat{\theta} \equiv \frac{\bar{T} - T_0}{Ra}
$$
The mean temperature profiles for different values of $\Lambda$ and $\sigma^2$ are shown in Figs.(7.22) - (7.25) to study the effect of Brinkman's number and permeability. Each of these profiles has a point of inflexion at $Z = 0.5$, the midway between the boundaries. The effect of convective heat transport has been clearly brought out by these figures and particularly when $R > 2R_0$, a region of isothermal stratification is produced by convection in the middle of the layer. Now the heat transport ratio $S$ is given by

$$S = \frac{R}{R_0} - \frac{4}{R_0} \sum_{n=1}^{\infty} n \theta_0,2n$$

The variation of $S$ with respect to $\frac{R}{R_0}$ for different $\Lambda$ and $\sigma$ are plotted in the Figs.(7.34) - (7.37). We observe that $S$ increases with $\Lambda$, $\sigma^2$ and $\frac{R}{R_0}$. It also be observed that the eighth order expansion $S^{(4)}$ is much higher than the second order expansion $S^{(1)}$ when $\frac{R}{R_0}$ is very large whereas the fourth and sixth order expansions (namely $S^{(2)}$ and $S^{(3)}$) are closer to the eighth order expansion $S^{(4)}$.

### 7.6 STREAMLINE PATTERN

In this section an analysis of the streamline pattern is discussed using the spectral method. Since the stream function $\Psi$ is a sine series in $x$ and therefore, antisymmetric with respect to $x = 0$, it is sufficient to consider only a half cell. Hence we restrict our attention to the region $0 \leq x \leq \frac{\pi}{\alpha}$ and $0 \leq Z \leq \pi$.

Consider an arbitrary point $A$ with co-ordinates $(x, Z)$ and let $B$ designate the image point with co-ordinate $(\frac{\pi}{\alpha} - x, \pi - Z)$. According to the equation (7.18)

$$\Psi(A) = -\sum_{\gamma} \psi_{\gamma}\exp\{i\pi(lax + nz)\}$$
where $\gamma = (l, n)$; similarly

$$\Psi(B) = -\sum_{\gamma} \Psi_{\gamma} (-1)^{\ell_\gamma n} \exp \left[ i\pi (lax + nz) \right]$$

Since we consider only the even-parity components $(l, n)$, $(l+n)$ is even so that

$$\Psi(B) = -\sum_{\gamma} \Psi_{\gamma} \exp \left[ -i\pi (lax + nz) \right]$$

Since $\Psi_{-\gamma} = \Psi_{\gamma}$, we have

$$\Psi(B) = \Psi(A).$$

This means that the streamline pattern is radially symmetric with respect to the centre of the region. Expanding $\Psi$, given by (7.18), in terms of powers of $r|$, we get

$$\Psi = 4(\Psi_{11} \sin z + \Psi_{13} \sin 3z) \sin(ax) + (\Psi_{22} \sin 2z + \Psi_{24} \sin 4z) \sin 2a \quad (7.44)$$

The stream function presented in (7.18) is evaluated for different values of $\Delta, \sigma^2, \frac{R}{R_0}$ when $\psi = 1$ and the profiles are shown in the Figs. (7.26) - (7.29). These figures convey that for the given $\Lambda, \sigma^2$ and $\psi$, the streamline pattern corresponding to $R/R_0 = 2$ is almost circular and when $R/R_0$ increases the streamlines tend to be a square by pushing the volume transport closer to the boundaries. Similar observations have been made when $\Lambda$ is increased for the fixed values of the other parameters. When $\psi$ is increased, it is observed that the volume transports are pushed towards the centre. The streamline patterns corresponding to $R/R_0 = 2$ and for various values of other parameters have been shown in the Figs. (7.30) - (7.33).
<table>
<thead>
<tr>
<th>Values of $\sigma^2$</th>
<th>Values of $(R_{11})_c$</th>
<th>Values of $ac^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Lambda=0.5$</td>
<td>$\Lambda=1.5$</td>
</tr>
<tr>
<td>0</td>
<td>3.375</td>
<td>10.125</td>
</tr>
<tr>
<td>1</td>
<td>7.6369</td>
<td>14.50429</td>
</tr>
<tr>
<td>10</td>
<td>43.91624</td>
<td>51.42276</td>
</tr>
<tr>
<td>$10^2$</td>
<td>403.9902</td>
<td>411.91504</td>
</tr>
<tr>
<td>$10^3$</td>
<td>4003.999</td>
<td>4011.99105</td>
</tr>
<tr>
<td>$10^4$</td>
<td>40003.999</td>
<td>40011.9991</td>
</tr>
<tr>
<td>$10^5$</td>
<td>4.0x10^5</td>
<td>4.0001x10^5</td>
</tr>
</tbody>
</table>

Table - 7.1: Values of $(R_{11})_c$ and $ac^2$ for different values of $\Lambda$ and $\sigma^2$. 
Table 7.2: Interacting pairs $\gamma_1$ and $\gamma_2$ that contribute to the $\gamma$ mode $H_\gamma$ and values of $r.$
### Table 7.3: Values of $a^2$ at which the curves of $R_{11}$ and $R_{31}$ intersect.

<table>
<thead>
<tr>
<th>$\sigma^2$</th>
<th>$\Lambda$</th>
<th>0</th>
<th>1</th>
<th>10</th>
<th>$10^2$</th>
<th>$10^3$</th>
<th>$10^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.15608</td>
<td>0.2782815</td>
<td>0.3249956</td>
<td>0.3324503</td>
<td>0.3332445</td>
<td>0.3333324</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.15608</td>
<td>0.235419</td>
<td>0.3050233</td>
<td>0.3307188</td>
<td>0.3330672</td>
<td>0.333307</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.15608</td>
<td>0.2157982</td>
<td>0.2946303</td>
<td>0.3290314</td>
<td>0.3328021</td>
<td>0.3333289</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.15608</td>
<td>0.2</td>
<td>0.2858435</td>
<td>0.326579</td>
<td>0.332626</td>
<td>0.3333262</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.15608</td>
<td>0.193501</td>
<td>0.2782815</td>
<td>0.3249938</td>
<td>0.3324504</td>
<td>0.3333244</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.15608</td>
<td>0.1713352</td>
<td>0.2354193</td>
<td>0.3109928</td>
<td>0.3307188</td>
<td>0.3333067</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.15608</td>
<td>0.1656889</td>
<td>0.2157966</td>
<td>0.2995957</td>
<td>0.3290314</td>
<td>0.3332889</td>
<td></td>
</tr>
</tbody>
</table>
Fig. (7.1) : Schematic illustration of a porous layer
Figs. (7.2) - (7.5): Variation of Spectral function \( \psi \), as a function of \( (K/K_0) \).
Figs. (7.6) - (7.9): Variation of Spectral function $\Theta$, as a function of ($R/R_0$).
Figs. (7.10) - (7.13): Variation of Spectral function $\theta_{a,2n}$ as a function of $(R/R_0)$.
Figs. (7.14) - (7.17) : Variation of $R_{ui}$ as a function of $a^2$ for different values of $\Lambda$. 
Figs. (7.18) - (7.21): Variation of $R_\phi$ as a function of $\alpha^2$ for different values of $\Lambda$ and $\sigma^2$. 
Figs. (7.22) - (7.25): Distribution of the mean temperature for different values of 
\((R/R_0)\) .
Figs. (7.26) - (7.29): Streamline patterns for different values of \( \left( \frac{R}{R_0} \right) \).
Figs. (7.30) - (7.33): Streamline patterns for different values of $\psi$. 

\[ \psi = 2 \times 10 \]

\[ \psi = 2 \times 100 \]

\[ \psi = 2 \times 1 \]

\[ \psi = 2 \times 10 \]
Figs. (7.34) - (7.37) : Plot of $S$ Vs $(R / R_0)$

$\Lambda = 1$
$\sigma = 0$

$\Lambda = 1$
$\sigma = 10$

$\Lambda = 5$
$\sigma = 0$

$\frac{\Lambda}{2} = 0$
$\sigma = 0$