CHAPTER 6

ON A NON-MARKOVIAN TIME-DEPENDENT QUEUEING SYSTEM WITH REPEATED VACATION

Time dependent queueing system with correlated arrivals and departures has been studied by many authors, viz., Mohan [45], Chaudhry [11], Murari [56], Mohan and Murari [46], Sharma [68,69], Sharda and Indu Garg [63], etc.

Sharma [68] has discussed the time dependent queueing system with arrivals in batches of variable size and correlated departures. The arrivals and departures can take place only at the transition time marks and the inter-transition time obeys a general distribution. The departure of a unit at a time mark depends upon departure or no departure at the previous time mark. The author has derived the Laplace transform of the probability generating function for the queue length.

Sharda and Indu Garg [63] have investigated the time dependent single channel queueing system, wherein arrivals have zero-step memory and departures have random memory.

The various events take place only at the transition marks and the inter-transition times follow exponential distribution. The authors have obtained the explicit probabilities of exactly i arrivals and j departures over a time interval.
Sharma [69] has also studied the time dependent queueing system with non-Markovian type of departure mechanism. The departures at a time mark depend upon departures or no departures at the preceding two consecutive time marks. He has derived the Laplace transform of the probability generating functions of the queue length distribution.

Single channel, non-Markovian time dependent queueing system with correlated departures and server’s vacation is considered in this chapter. The Laplace transforms of the probability generating functions of the time dependent queue length distribution in the transient state are obtained for two models of this category. In the first model, an arriving unit cannot depart at the same transition time mark, whereas in the second model it can. The probability generating function of the queue length distribution in the steady state and the expected queue length are derived for the both the models. It has been shown that the steady state continuous time solution is the same as the steady state discrete time solution. Numerical values are presented for selected parameters.

In many real life situations, we come across systems, where departures depend upon departures or no departures at the preceding time marks. When people come and watch musical performance, dance, drama, circus, magic show, etc., if the programme is not upto their expectation, then people get disappointed and they try to move away from that. In these situations, the departures at subsequent time mark will be influenced by those at the preceding time.
6.1. ASSUMPTIONS AND NOTATION

Assume that various events arrivals, departures, starting of server's vacation and return of server from vacation can occur only at the transition marks $t_0, t_1, t_2, \ldots$, where the inter-transition time $t_r - t_{r-1}$ ($r=1,2,\ldots$) follow the exponential distribution with parameter $\lambda$.

The probability for the simultaneous occurrence of more than one event is not regarded as negligible. However, the probability of more than one arrival or that of more than one departure at a transition mark is zero.

Arrivals occur irrespective of the fact whether there was an arrival or no arrival at the previous transition mark. Let $p$ be the probability of an arrival and $q$ that of no arrival at a transition mark, such that $p+q=1$.

If the server finds no unit in the system, then he leaves for vacation. On returning to the system, if he finds no unit in the system and no arrivals at the transition mark, then immediately at the same transition mark, he leaves the system. Let $\theta$ be the probability for starting service after return from vacation.

Let $T_n$ be the time at which the $n$-th transition occurs and let $X_n$ be the number of customers in the system immediately after the $n$-th transition.

Define $D_n = 1$, if departure occurs at $T_n$

$= 0$, if otherwise.
If just before $T_{r+1}$, the queue length is $n > 0$, let

\[ P \left[ D_{r+1} = 1 / D_r = 1, X_r = n \right] = u_0 \]
\[ P \left[ D_{r+1} = 0 / D_r = 1, X_r = n \right] = v_0 \]
\[ P \left[ D_{r+1} = 1 / D_r = 0, D_{r+1} = 1, X_r = n \right] = u_1 \]
\[ P \left[ D_{r+1} = 0 / D_r = 0, D_{r+1} = 1, X_r = n \right] = v_1 \]
\[ P \left[ D_{r+1} = 1 / D_r = 0, D_{r+1} = 0, X_r = n \right] = u_2 \]
\[ P \left[ D_{r+1} = 0 / D_r = 0, D_{r+1} = 0, X_r = n \right] = v_2 \]

where $u_i + v_i = 1 (i = 0, 1, 2)$

Let $P_{n,i}(t)$ be the probability that at time $t$, the server is in the system, $n$ customers are in the queue and the number of transitions that occurred prior to time $t$, since the last departure, is $i (i = 0, 1, 2)$

Let $Q_{n,i}(t)$ be the probability that at time $t$, the server is in vacation, $n$ customers are in the queue and the number of transitions that occurred prior to the time $t$, since the last departure, is $i (i = 0, 1, 2)$

6.2 Model I

Assume that an arriving unit cannot depart at the same transition mark, when there is no unit in the system just before the transition of his arrival.

6.2.1 Governing Equations

The differential - difference equations governing the system are given below:

\[ w \cdot P_{n,0}(t) + P_{n,0}(t) = \sum_{i=0}^{2} [ p P_{n,i}(t) + q P_{n+1,i}(t) ], n \geq 1 \]  \hspace{1cm} (6.2.1)
\[ w^{-1} P_{n,1}'(t) + P_{n,2}(t) = v_0 \left[ q P_{n,0}(t) + p P_{n-1,0}(t) \right], \quad n \geq 2 \quad (6.2.2) \]

\[ w^{-1} P_{1,1}'(t) + P_{1,2}(t) = q v_0 P_{1,0}(t) + p q Q_{0,0}(t) \quad (6.2.3) \]

\[ w^{-1} P_{n,1}'(t) + P_{n,2}(t) = \sum_{i=1}^{n-1} v_i [q P_{n,i}(t) + p P_{n-1,i}(t)] + \theta p Q_{n-1,2}(t) + \theta q Q_{n,2}(t), \quad n \geq 3 \quad (6.2.4) \]

\[ w^{-1} P_{2,2}'(t) + P_{2,2}(t) = \theta q Q_{2,2}(t) + p \theta \sum_{i=1}^{2} Q_{i,0}(t) + \theta p Q_{2,2}(t) + p P_{1,1}(t) \quad (6.2.5) \]

\[ w^{-1} P_{1,1}'(t) + P_{1,2}(t) = q \sum_{i=1}^{2} v_i P_{1,1} + \theta \sum_{i=1}^{2} [q Q_{1,1}(t) + p Q_{1,0}(t)] \quad (6.2.6) \]

\[ w^{-1} Q_{0,0}'(t) + Q_{0,0}(t) = \sum_{i=0}^{2} u_i P_{i,1}(t) \quad (6.2.7) \]

\[ w^{-1} Q_{1,1}'(t) + Q_{1,1}(t) = p (1-\theta) Q_{0,0}(t) \quad (6.2.8) \]

\[ w^{-1} Q_{0,1}'(t) + Q_{0,1}(t) = q Q_{0,0}(t) \quad (6.2.9) \]

\[ w^{-1} Q_{n,2}'(t) + Q_{n,2}(t) = (1-\theta) \left[ p Q_{n-1,2}(t) + q Q_{n,2}(t) \right], \quad n > 3 \quad (6.2.10) \]

\[ w^{-1} Q_{2,2}'(t) + Q_{2,2}(t) = (1-\theta) \left[ p Q_{1,1}(t) + p Q_{1,2}(t) + q Q_{2,2}(t) \right] \quad (6.2.11) \]

\[ w^{-1} Q_{1,2}'(t) + Q_{1,2}(t) = (1-\theta) \sum_{i=1}^{2} [p Q_{0,i}(t) + q Q_{1,i}(t)] \quad (6.2.12) \]

\[ w^{-1} Q_{0,2}'(t) + Q_{0,2}(t) = q \sum_{i=1}^{2} Q_{0,i}(t) \quad (6.2.13) \]

### 6.2.2 Probability Generating Function and Expected Queue length

Assume that the system starts with a departure which makes the queue length \( N \).
Then $P_{n,0}(0) = \delta_{n,n}$

Define the probability generating functions

$$G_i(t,\alpha) = \sum_{n=1}^{\infty} P_{n,i}(t) \alpha^n, \quad 0 \leq i \leq 2$$

$$H(t,\alpha) = \sum_{n=1}^{\infty} Q_{n,i}(t) \alpha^n$$

Let $P_n^\ast(s), Q_n^\ast(s), G_i^\ast(s,\alpha)$ and $H^\ast(s,\alpha)$ be respectively the Laplace transforms of $P_{n,i}(t), Q_{n,i}(t), G_i(t,\alpha)$ and $H(t,\alpha)$

Multiplying the Laplace transform of equation (6.2.1) by $\alpha^n$ and adding for $n = 1, 2, 3, \ldots$ we get

$$(\alpha f - u_0) G_0^\ast(s,\alpha) - u_1 G_1^\ast(s,\alpha) - u_2 G_2^\ast(s,\alpha)$$

$$= \sum_{i=0}^{\infty} u_i \sum_{j=0}^{\infty} P_{j,i}^\ast(s) \alpha^n$$

$$\frac{\alpha^{n+1} f}{s+w} q\alpha$$

$$\frac{2 \sum_{i=0}^{\infty} u_i p_{j,s}^\ast(s)}{p\alpha + q}$$

$$= \sum_{i=0}^{\infty} u_i p_{j,s}^\ast(s)$$

(6.2.14)

Multiplying the Laplace transform of equation (6.2.3) by $\alpha$ and the Laplace transform of equations (6.2.2) by $\alpha^n$ and adding the resulting equations, we obtain

$$fG_0^\ast(s,\alpha) - v_0 G_0^\ast(s,\alpha) = \frac{p\alpha^\theta}{p\alpha + q} Q_{n,0}^\ast(s)$$

(6.2.15)

Similarly from equations (6.2.4) - (6.2.6)

$$(f-v_2)G_2^\ast(s,\alpha) - v_1 G_1^\ast(s,\alpha) = \theta H^\ast(s,\alpha) + \alpha \theta Q_{n,1}^\ast(s)$$

$$\frac{p\alpha^\theta}{p\alpha + q} [Q_{n,0}^\ast(s) + Q_{n,1}^\ast(s)]$$

(6.2.16)
and from (6.2.10) - (6.2.13)

\[(f-1+\theta)(s+\alpha)(1-\theta)Q_{1,1}^*(s) + \frac{pa}{p\alpha + q}(1-\theta) \left[ Q_{0,1}^*(s) + Q_{0,2}^*(s) \right] \]

\[(6.2.17)\]

where \( f = \frac{s+w}{w(p\alpha+q)} \)

Laplace transform of equation (6.2.8) becomes

\[ Q_{1,1}^*(s) = \frac{wp(1-\theta)}{s+w} Q_{0,0}^*(s) \]  
\[(6.1.18)\]

Applying Laplace transform on equations (6.2.9) and (6.2.13) and adding both, we get

\[ Q_{0,1}^*(s) + Q_{0,2}^*(s) = \frac{qw}{s+p\omega} Q_{0,0}^*(s) \]  
\[(6.1.19)\]

Adding \( Q_{0,0}^*(s) \) on both sides of (6.2.19), we have

\[ \sum_{i=0}^{2} Q_{0,i}^*(s) = \frac{s+w}{s+p\omega} Q_{0,0}^*(s) \]  
\[(6.2.20)\]

Substituting (6.2.18) and (6.2.19) in equation (6.2.17) and simplifying we obtain

\[ H^*(s,\alpha) = \frac{p\alpha(1-\theta)w[(1-\theta)(s+p\omega)+qw]}{(f-1+\theta)(s+w)(s+p\omega)} \]  
\[(6.2.21)\]

Laplace transform of equation (6.2.7) gives

\[ \frac{s+w}{w} Q_{0,0}^*(s) = q \sum_{i=0}^{2} u, P_{1,i}^*(s) \]  
\[(6.2.22)\]
Substituting (6.2.22) in equation (6.2.14), we get

\[(\alpha f-u_0)G_0(s,\alpha)-u_1G_1(s,\alpha)-u_2G_2(s,\alpha)\]
\[= \frac{\alpha^{N+1} f}{s + w} \]
\[= \alpha f Q_{u,0}^*(s) \quad (6.2.23)\]

Adding equations (6.2.15), (6.2.16) and (6.2.23) and using (6.2.18), (6.2.19) and (6.2.21), we get

\[\frac{\alpha f-1}{f-1} G_0(s,\alpha) + G_1(s,\alpha) + G_2(s,\alpha) - \frac{1}{f-1} [ - \cdots + X] \quad (6.2.24)\]

where

\[X = \theta H^*(s,\alpha) + \alpha \theta Q_{1,1}^*(s) - \alpha f Q_{u,0}^*(s) + \sum_{i=0}^{p\alpha+q} Q_{u,i}^*(s)\]
\[= \frac{pw\alpha f(\theta-1)(f-1)}{(s+pw)(f-1+i)} \quad (s+p\alpha)(f-1+i)\]

Equation (6.2.16) can be expressed as

\[(f-v_2) G_2(s,\alpha) - v_1 G_1(s,\alpha) = X + [\alpha f - (p\alpha q)/(p\alpha+q)] Q_{u,0}^*(s) \quad (6.2.25)\]

Equations (6.2.24), (6.2.15) and (6.2.25) can be written in the matrix form as

\[AB = C \quad (6.2.26)\]
where

\[
A = \begin{bmatrix}
\alpha_{N+1} f & -1 & 1 \\
-1 & f & 0 \\
0 & -\nu_0 & f - \nu_2
\end{bmatrix}
\]

and

\[
B = [G_0*(s,\alpha), G_1*(s,\alpha), G_2*(s,\alpha)]'
\]

Solution of equation (6.2.26) is given by

\[
f(f-v_2) \quad \alpha_{N+1} f \\
D(\alpha,s)G_0(s,\alpha) = \quad \alpha f-1 \\
f-1 \quad s+w \quad p\alpha q
\]

\[
- f \left[ X + (\alpha f) \frac{(p\alpha q)/(p\alpha + q)) Q_{0,0}^*(s)}{}ight] (6.2.27)
\]

\[
D(\alpha,s)G_1(s,\alpha) = \quad \alpha_{N+1} f \\
\quad \alpha f-1 \\
f-1 \quad s+w \quad p\alpha q
\]

\[
- \nu_0 \left[ X + (\alpha f) \frac{(p\alpha q)/(p\alpha + q)) Q_{0,0}^*(s)}{}ight] (6.2.28)
\]

\[
D(\alpha,s)G_2(s,\alpha) = \quad \alpha_{N+1} f \\
\quad \alpha f-1 \\
f-1 \quad s+w \quad p\alpha q
\]

\[
- \nu_0 \nu_1 \left[ X + (\alpha f) \frac{(p\alpha q)/(p\alpha + q)) Q_{0,0}^*(s)}{}ight] (6.2.29)
\]

where

\[
D(\alpha, s) = |A| = \frac{((\alpha f-1)/(f-1)) f (f-v_2) + \nu_0 (f-v_2 + \nu_1)}{}
\]
Define \( R^*(s, \alpha) = \sum_{i=0}^{2} G_i^*(s, \alpha) + H^*(s, \alpha) + \sum_{i=0}^{2} Q_{0,i}^*(s) + \alpha Q_{1,1}^*(s) \) 

(6.2.31)

Substituting the expressions of \( G_i^*(s, \alpha), H^*(s, \alpha), \sum_{i=0}^{2} Q_{0,i}^*(s) \) and \( Q_{1,1}^*(s) \) given by equations (6.2.27) - (6.2.29), (6.2.21), (6.2.20), and (6.2.18) respectively, in equation (6.2.31), we get

\[
R^*(s, \alpha) = \frac{\alpha^{N_1} f}{s+w} + \frac{f(\alpha-1)}{(s+w)(f-1)} \left[ N_1 - N_2 + N_1 - N_2 \right] + \frac{s+w}{s+pw} Q_{0,0}^*(s)
\]

(6.2.32)

where \( N_1 = \alpha \left[ p\theta(v_1-v_2)/(p\alpha+q)+f^2 \right] Q_{0,0}^*(s) \)

\[ N_2 = \left[ \alpha^{N_1} f^2 (f-v_2)/(s+w)(f-1) \right] \]

\[ N_3 = \frac{pw\alpha f^2 (1-\theta)}{(s+pw)(f-1+\theta)} \]

and \( N_4 = \frac{s\alpha}{(s+pw)(\alpha-1)} \left[ \alpha f_2 - f v_2 + v_0 (f-v_2 + v_1) \right] Q_{0,0}^*(s) \)

Since the sum of all the probabilities is equal to 1, we have

\[
\sum_{i=0}^{2} \sum_{t=0}^{\infty} P_{n,i}(t) + \sum_{t=0}^{\infty} Q_{n,2}(t) + Q_{1,1}(t) + Q_{0,0}(t) + Q_{0,0}(t) = 1 
\]

(6.2.33)

Laplace transform of equation (6.2.33) is nothing but \( R^*(s,1) = 1/s \). Hence taking \( \alpha=1 \) in equation (6.2.32) and equating the resulting equation to 1/s, the expression for \( Q_{0,0}^*(s) \) can be obtained.
Assume $R(\alpha) = \lim_{s \to 0} s R^*(s, \alpha)$ and $Q_{0,0} = \lim_{s \to 0} s Q_{0,0}(s)$. Multiplying (6.2.32) by $s$ and taking limit as $s \to 0$, we obtain the probability generating function of the queue length under steady state as

$$R(\alpha) = \begin{bmatrix} \frac{1}{p\alpha + q} \end{bmatrix}.$$

From (6.2.34), we get the expression for $Q_{0,0}$ as

$$Q_{0,0} = \frac{p[qu_2 - pv_0(u_2 + v_1)]}{p\alpha + q}(u_2 + v_1) + q(pu_1 + qu_2) + p(1-\theta)[(u_2/\theta) + p(v_2 - v_1)].$$

For the existence of steady state $Q_{0,0}$ must be greater than zero [41], and hence $qu_2 - pv_0(u_2 + v_1) > 0$.

Therefore $p < [u_2/(u_2 + v_0(v_1 + u_2))]$

Now, the expected queue length $L_q$ is given by

$$L_q = \left. \frac{d}{d\alpha} R(\alpha) \right|_{\alpha=1}.$$
Differentiating (6.2.34) with respect to $\alpha$ and putting $\alpha = 1$, we get

$$L_q = \left[pu_0 + pu_1 + q^2u_2\right] \left[q + pv_0(v_1 - v_2)\right] + \left[q(1 + u_2) - pv_0\right] \left[pu_0(v_1 - v_2) + p\left(1 - \theta\right)(u_2/\theta) + p(v_2 - v_1)\right] + \left[qu_2 - pv_0(v_1 + u_2)\right] \left[q(u_2/\theta + pv_2 - pv_1) + p\left(1 - \theta\right)u_2/\theta^2\right] (1 - \theta) \frac{Q_{0o}/D}{D^2} \tag{6.2.36}$$

where $D = qu_2 - pv_0(u_2 + v_1)$

The expressions in (6.2.34), (6.2.35) and (6.2.36) when $\theta = 1$, coincide with the results (15), (16) and (19) of Sharma [69].

### 6.2.3 Particular Cases

1. Taking $u_2 = u_1$ and $v_2 = v_1$, we get the time dependent queueing system where departure at a time mark depends only upon the departure or no departure at the preceding time mark. In this case, the probability generating function of the queue length becomes

$$R(\alpha) = \frac{(pu_0 + pu_1)\alpha + p(1 - \theta)u_1/\theta}{[pu_0 + pu_1 + p(1 - \theta)u_1/\theta] [q(v_1 + v_2)(p\alpha + q)]} \tag{6.2.37}$$

and the expected queue length is

$$L_q = p\left[q (pu_0 + pu_1)(u_1 + \theta v_1) + p(1 - \theta)u_1(qu_2 - pv_0)/\theta^2\right] \left[pu_0 + pu_1 + p(1 - \theta)u_1/\theta\right] [qu_2 - pv_0] \tag{6.2.38}$$
2. If we assume departures are statistically independent, then 
\[ u_2 = u_1 = u_0 \] and \[ v_2 = v_1 = v_0 \] and in this case,

\[
R(\alpha) = \frac{\theta(qu_0 - p v_0)}{(h(\alpha) - 1 + \theta)(qu_0 - p \alpha v_0)} \quad (6.2.39)
\]

and expected queue length

\[
L_q = \frac{p[qu_0 + (\theta - p) v_0]}{\theta (qu_0 - p v_0)} \quad (6.2.40)
\]

when \( \theta = 1 \), (6.2.37)-(6.2.38) and (6.2.39) - (6.2.40) coincide with the corresponding results (20)-(21) and (24)-(25) of Sharma [69].

6.2.4 Discrete Time Solution

\( P_{n,i}(j) \) is the probability that at the \( j \)-th mark the server is in the system, \( n \) customers are in the queue and the number of transitions that occurred prior to the \( j \)-th mark since the last departure is \( i \) (\( i=0,1,2 \)).

\( Q_{n,i}(j) \) denotes the probability that at the \( j \)-th mark, the server is in vacation, \( n \) customers are in the queue and the number of transitions that occurred prior to the \( j \)-th mark since the last departure is \( i \) (\( i=0,1,2 \)).

The equations describing the queueing model are:

\[
P_{n,0}(j+1) = \sum_{i=0}^{2} u_i \left[ p P_{n,i}(j) + q P_{n+1,i}(j) \right], \quad n \geq 1 \quad (6.2.41)
\]

\[
P_{n,1}(j+1) = v_0 \left[ q P_{n,0}(j) + p P_{n-1,0}(j) \right], \quad n \geq 2 \quad (6.2.42)
\]
\[ P_{n+1}(j+1) = qv_0 P_{n,0}(j) + p\theta Q_{n,0}(j) \quad (6.2.43) \]

\[ P_{n,2}(j+1) = \sum_{i=0}^{2} v_i [qP_{n,i}(j) + pP_{n-1,i}(j)] + \theta pQ_{n-1,2}(j) + \theta qQ_{n,2}(j), \quad n \geq 3 \quad (6.2.44) \]

\[ P_{n,3}(j+1) = \sum_{i=1}^{2} v_i [qP_{n,i}(j) + pP_{n-1,i}(j)] + \theta qQ_{n,3}(j) \quad (6.2.45) \]

\[ P_{n,4}(j+1) = q \sum_{i=0}^{2} v_i P_{n,i}(j) + \theta \sum_{i=1}^{2} [qQ_{n,i}(j) + pQ_{n-1,i}(j)] \quad (6.2.46) \]

\[ Q_{n,0}(j+1) = q \sum_{i=0}^{2} v_i P_{n,i}(j) \quad (6.2.47) \]

\[ Q_{n,1}(j+1) = p(1-\theta) Q_{n,0}(j) \quad (6.2.48) \]

\[ Q_{n,2}(j+1) = q Q_{n,0}(j) \quad (6.2.49) \]

\[ Q_{n,3}(j+1) = (1-\theta) [pQ_{n-1,2}(j) + qQ_{n,2}(j)], \quad n \geq 3 \quad (6.2.50) \]

\[ Q_{n,4}(j+1) = (1-\theta) [qQ_{n-1,3}(j) + pQ_{n-1,2}(j)] \quad (6.2.51) \]

\[ Q_{n,5}(j+1) = (1-\theta) \sum_{i=1}^{2} [pQ_{n,i}(j) + qQ_{n-1,i}(j)] \quad (6.2.52) \]

\[ Q_{n,6}(j+1) = q \sum_{i=1}^{2} Q_{n,i}(j) \quad (6.2.53) \]

Let \( P_{n,i} \) and \( Q_{n,i} \) be the steady state probabilities corresponding to \( P_{n,i}(j) \) and \( Q_{n,i}(j) \) respectively. Then,

\[ P_{n,i} = \lim_{j \to \infty} P_{n,i}(j) \quad \text{and} \quad Q_{n,i} = \lim_{j \to \infty} Q_{n,i}(j) \]
Define the probability generating functions
\[ G_j(\alpha) = \sum_{n=1}^{\infty} P_{n,j} \alpha^n \] and
\[ H(\alpha) = \sum_{n=1}^{\infty} Q_{n,2} \alpha^n \]

Multiplying the steady state of equation of (6.2.41) by \( \alpha^n \) and adding for \( n = 1, 2, \ldots \) we get
\[
(\alpha h(\alpha) - u_0) G_0(\alpha) - u_1 G_1(\alpha) - u_2 G_2(\alpha) = - \left[ q \alpha h(\alpha) \sum_{i=0}^{2} u_i p_1 i \right]
\]
\[(6.2.54)\]

Multiplying the steady state equation of (6.2.42) by \( \alpha^n \) for successive values of \( n = 2, 3, \ldots \) and adding them to the steady state equation of (6.2.43) multiplied by \( \alpha \), we obtain
\[- v_0 G_0(\alpha) + h(\alpha) G_1(\alpha) = p\theta \alpha h(\alpha) Q_{0,0} \]
\[(6.2.55)\]

Similarly from the steady state equations corresponding to equations (6.2.44) - (6.2.53), we get
\[
(h(\alpha) - v_2) G_0(\alpha) - v_1 G_1(\alpha) = \theta H(\alpha) + \theta \alpha Q_{1,1} + p\theta \alpha h(\alpha)[Q_{0,1} + Q_{0,2}] \]
\[(6.2.56)\]

and
\[
(h(\alpha) - 1 + \theta) H(\alpha) = \alpha (1 - \theta) Q_{1,1} + p\alpha (1 - \theta) h(\alpha)[Q_{0,1} + Q_{0,2}] \]
\[(6.2.57)\]

Using equations (6.2.48),(6.2.49) and (6.2.53), we get
\[
Q_{1,1} = p(1 - \theta) Q_{0,0} \]
\[
Q_{0,1} + Q_{0,2} = (q/p) Q_{0,0} \]
\[
Q_{0,0} + Q_{0,1} + Q_{0,2} = (1/p) Q_{0,0} \]
Substituting the expression of $Q_{i1}$ and $Q_{i2}$, the equation (6.2.57) gives,

$$H(\alpha) = \frac{(1-\theta)\alpha}{h(\alpha)-1+\theta} \left[ p(1-\theta) + qh(\alpha) \right] Q_{0,0}$$  \hspace{1cm} (6.2.58)

and equation (6.2.56) becomes

$$\alpha \theta h(\alpha) \left[ p(1-\theta) + qh(\alpha) \right]$$

$$h(\alpha) \left( h(\alpha) - v_2 \right) G_2(\alpha) - v_1 G_1(\alpha) = \frac{Q_{0,0}}{h(\alpha) - 1 + \theta} \hspace{1cm} (6.2.59)$$

Adding (6.2.54), (6.2.55), (6.2.59) and using (6.2.47) we get $(\alpha h(\alpha) - 1)$

$$G_0(\alpha) + (h(\alpha)-1) G_1(\alpha) + (h(\alpha)-1) G_2(\alpha) + \left( v_1 h(\alpha) - Q_{11}(\alpha) \right) Q_{0,0}$$

$$= \frac{\alpha h(\alpha)(1-\theta)}{(h(\alpha)-1+\theta)} Q_{0,0} \hspace{1cm} (6.2.60)$$

Dividing by $h(\alpha)-1$, equation (6.2.60) becomes

$$qG_0(\alpha) - p G_1(\alpha) - p G_2(\alpha) = \frac{p \alpha h(\alpha)(1-\theta) Q_{0,0}}{h(\alpha) - 1 + \theta} \hspace{1cm} (6.2.61)$$

Solving equations (6.2.55), (6.2.59) and (6.2.61) we get expressions for $G_0(\alpha), G_1(\alpha)$ and $G_2(\alpha)$ from the matrix equation

$$B = EF$$  \hspace{1cm} (6.2.62)

where $B = (G_0(\alpha), G_1(\alpha), G_2(\alpha))'$

$$D(\alpha) E = \begin{vmatrix}
    h(\alpha)[h(\alpha) - v_2] & p[h(\alpha) - v_2 + v_1] & ph(\alpha) \\
    v_0(h(\alpha) - v_2) & q[h(\alpha) - v_2] & pv_0 \\
    v_0v_1 & qv_1 & qh(\alpha) - pv_0
\end{vmatrix}$$
Define the generating function

\[
F = \begin{bmatrix}
\alpha \theta h(\alpha)(1-\theta) \\
h(\alpha)-1+\theta \\
p(\alpha)\theta h(\alpha) Q_{\alpha \nu} \\
\alpha \theta h(\alpha) [p(1-\theta) + qh(\alpha)] \\
\frac{h(\alpha)-1}{\alpha} + \theta 
\end{bmatrix} Q_{\alpha 0}
\]

and

\[
D(\alpha) = q h(\alpha) - (p v_0 + q v_2) h(\alpha) + p v_0 (v_2 - v_i)
\]

Substituting the expressions of \( G_{1,0} \), obtained by using (6.2.62), \( H(\alpha) \) given by (6.2.58), \( Q_{1,1} = p(1-\theta) Q_{0,0} \) and \( \sum_{i=0}^{\nu} Q_{0,i} = (1/p) Q_{0,0} \) in (6.2.63) we obtain

\[
r(\alpha) = \{ (p u_0 + p q u_i + q^2 u_2) h(\alpha) + p u_0 (v_i - v) \\
+ (1-\theta) p \alpha \theta h(\alpha) [ \frac{h(\alpha) u_2}{\alpha} + p (v_2 - v_i)] \} \frac{Q_{0,0}}{p D(\alpha)}
\]

Using the normalised equation \( R(1) = 1 \), we obtain

\[
Q_{u,v} = \frac{p [q u_2 - p v_0 (u_2 + v_i)]}{p u_0 (v_i + u_2) + q (p u_i + q u_2) + (1-\theta) [(u_2/\theta) + p (v_2 - v)]}
\]

The result (6.2.64) - (6.2.65) agree with (6.3.34) - (6.2.35). This shows that the continuous time solution is same as the discrete time solution.
6.3 MODEL 2

In this model the system of MODEL-1 is considered with the relaxation of the restriction that an arriving unit to the empty system at the transition mark $t_{r}$, cannot depart at the same transition mark.

Assume that just before a time mark $t_{r}$, the system is empty and also assume that an arrival of a unit takes place at $t_{r}$. Let the probability of departure at $t_{r}$ be $a$ and that of no departure at $t_{r}$ be $b$.

The governing equations are (6.2.1)-(6.2.2), (6.2.4)-(6.2.5), (6.2.9), (6.2.11), (6.2.13) and

\begin{align}
&\sum_{i=1}^{2} [qP_{i1}(t) + pP_{i1}(t)] + \theta p \sum_{i=1}^{2} Q_{i1}(t) + \theta qQ_{22}(t) \\
&\sum_{i=1}^{2} P_{1i}(t) + P_{1i}(t) = qv_{0} P_{10}(t) + p0 b Q_{00}(t) \quad (6.3.2) \\
&\sum_{i=0}^{2} Q_{i0}(t) + Q_{00}(t) = q \sum_{i=0}^{2} u_{1} P_{1i}(t) + pa \sum_{i=0}^{2} Q_{i0}(t) \quad (6.3.3) \\
&\sum_{i=0}^{2} P_{1i}(t) + Q_{1i}(t) = pb(1-\theta) Q_{00}(t) \quad (6.3.4) \\
&\sum_{i=0}^{2} Q_{1i}(t) + Q_{1i}(t) = (1-\theta) \sum_{i=0}^{2} [qQ_{1i}(t) + pbQ_{0i}(t)] \quad (6.3.5)
\end{align}

By proceeding as in model 1, we get the matrix equation

\[ AB = Z \quad (6.3.6) \]

where the matrices $A$ and $B$ are as in section 6.2.1 of this chapter and
Laplace transform of the probability generating function of the queue length distribution is given by

\[
R(s, \alpha) = \alpha^{N+1} f \frac{s+w}{(s+w)(f-1)} + \frac{f(s+pw)}{s+pw} \frac{M_1}{M_2} - \frac{Q_{0,0}^*(s) + f(s+pw) D(s, \alpha)}{f-1}.
\]  

(6.3.7)

\[
M_1 = \frac{p \alpha b(v_i - v_j)}{\alpha + q}
\]

\[
M_2 = \frac{\alpha f^2}{s+pw} + \frac{Q_{0,0}^*(s)}{s+pw}
\]

\[
M_3 = \frac{\alpha^{N+1} f}{s+w}
\]

\[
M_4 = \frac{p \alpha b w_i u_j (1-\theta)}{(s+pw)(f-1+\theta)}
\]

\[
M_5 = \frac{s \alpha f^{-2} f v_i + v_0 (f v_i + v_j)}{(s+pw) (\alpha-1)}
\]

\[
D(s, \alpha) = \frac{\alpha f^{-1} f^{-2} f v_i + v_0 (f^{-2} f v_i + v_i)}{f^{-1}}
\]

136
By using the equation $R(s,1) = 1/s$ the unknown probability $Q_{0,0}(s)$ can be determined. Multiplying (6.3.7) by $s$ and taking limit as $s \to 0$, we get the steady state probability generating function $R(\alpha)$ as

$$R(\alpha) = Q_{0,0} \left\{ a\alpha h(\alpha) [p(v_i-v_j) - h(\alpha)] + (pu_0 + pu_1 + qu_2)h(\alpha) \right\} + pu_0(v_i-v_j) + a\alpha h(\alpha)(1-\theta)[h(\alpha)u_2/(h(\alpha) - 1 + \theta) + pv_2 - pv_1]/(D(\alpha)p)$$  

(6.3.8)

where $D(\alpha) = qh^2(\alpha) - (pv_0 + qv_2)h(\alpha) + pv_0(v_2-v_1)$

Using the fact $R(1) = 1$, the expression for $Q_{0,0}$ can be obtained as

$$Q_{0,0} = \frac{p[qu_2 - pv_0(v_1+u_2)]}{a[p(v_i-v_j)] + pu_0(v_i+u_2) + q(qu_1 + qu_2) + pb(1-\theta)((u_2/\theta) + pv_2 - pv_1]}$$

Differentiating (6.3.8) with respect to $\alpha$ and taking $\alpha = 1$, we get the expected queue length

$$L_q = Q_{0,0}[aX_1 + b(1-\theta)X_2 + X_3] / D^2$$

where

$$X_1 = [qu_2 - pv_0(v_1+u_2)] [q(pv_2-pv_1-1) + p(2pq - p(pv_0 + qv_2)) + pv_2 - pv_1 - 1]$$

$$X_2 = [q(1+u_2) - pv_0] [pu_2 / \theta + p^2(v_i - v_j)] + [qu_2 - pv_0(v_1+u_2)][q(u_2/\theta + pv_2 - pv_1) + p(1-\theta)u_2/\theta^2]$$

$$X_3 = [pu_0 + pqu_1 + qu_2] [q - pv_0(v_i - v_j)] + pu_0 (v_i - v_j) [q(1+u_2) - pv_0]$$

and $D = qu_2 - pv_2 (v_i+u_2)$

6.4 NUMERICAL RESULTS

In table 6.1 the numerical results for the expected queue length $L_q$, are given for selected parameters. As it is expected, $L_q$ increases, if $p$ increases and $L_q$ decreases, if $\theta$ and $a$ increase.
TABLE 6.1

Expected number of customers in the queue when $u_0 = 0.9, u_1 = 0.5, u_2 = 0.1$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$L_q$</th>
<th>$p = 0.7$, $a = 0.1$</th>
<th>$p = 0.7$, $q = 0.4$</th>
<th>$q = 0.4$, $a = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$a$</td>
<td>$L_q$</td>
<td>$p$</td>
</tr>
<tr>
<td>0.1</td>
<td>9.2025</td>
<td>0.1</td>
<td>4.3059</td>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
<td>5.8551</td>
<td>0.2</td>
<td>4.2768</td>
<td>0.2</td>
</tr>
<tr>
<td>0.3</td>
<td>4.8022</td>
<td>0.3</td>
<td>4.24</td>
<td>0.3</td>
</tr>
<tr>
<td>0.4</td>
<td>4.3059</td>
<td>0.4</td>
<td>4.1918</td>
<td>0.4</td>
</tr>
<tr>
<td>0.5</td>
<td>4.0238</td>
<td>0.5</td>
<td>4.1262</td>
<td>0.5</td>
</tr>
<tr>
<td>0.6</td>
<td>3.8443</td>
<td>0.6</td>
<td>4.0316</td>
<td>0.6</td>
</tr>
<tr>
<td>0.7</td>
<td>3.7205</td>
<td>0.7</td>
<td>3.8832</td>
<td>0.7</td>
</tr>
<tr>
<td>0.8</td>
<td>3.6299</td>
<td>0.8</td>
<td>3.6169</td>
<td>0.8</td>
</tr>
<tr>
<td>0.9</td>
<td>3.5603</td>
<td>0.9</td>
<td>2.9998</td>
<td></td>
</tr>
</tbody>
</table>