CHAPTER IV

VISCOUS FLOW OF A ROTATING THERMALLY STRATIFIED FLUID
BETWEEN TWO WAVY DISKS

4.1 Introduction

The physical importance of rotating stratified flows in Geophysics being well-known, quite a number of investigations have been done by considering rotating stratified flows contained between smooth boundaries. In view of the possible existence of irregular core-mantle interface in the terrestrial context [Moffatt (1978)], it is necessary to study these kinds of flows involving irregular boundaries. However, before attempting problems of this type, having complicated features, it is reasonable and inevitable to study certain simple and well-posed problems which provide a clear insight and embraces many of the fundamental processes. The present chapter, dealing with the motion of rotating, stratified fluid between two wavy disks, is concerned with an investigation of this kind.

Eventhough a large number of investigations have been carried out for homogeneous fluid flow between two wavy boundaries with or without rotation, only a few are available for non-homogeneous fluid. Citron (1962) has studied the slow motion of an incompressible viscous fluid between two rough circular cylinders rotating about their common axis. The
roughness was assumed small compared to the spacing between the cylinders and the solutions were explicitly obtained for the case of sinusoidal boundaries. Sastry (1976) has analysed the flow in a rotating wavy channel assuming the wavyness, in the form of sinusoidal boundaries.

The flow of an inviscid, stratified fluid in a channel with small and large deformations, was investigated by Rao and Devanathan (1972) bringing out the analogy of this flow with swirling flow in tubes with non-uniform cross-sections. Blocking phenomena and the possibility of preventing the stagnation zones in the flow field was also discussed by them. Later, Balan et al. (1975) analysed the time dependent flow of rotating and stratified fluid in geometries with non-uniform cross-sections, under Oseen approximation using Laplace transform technique for an inviscid fluid. The existence of inertial waves, blocking phenomena, back flow for certain critical values of the physical parameters and the analogy between rotating and stratified flows were studied by them.

4.2 Author's contribution

As pointed above, naturally occurring boundaries in geophysical and oceanic flows need not be smooth. We have considered the steady motion of an incompressible viscous, rotating and thermally stratified fluid when the fluid flows between two wavy disks as a simple geometric model. The corresponding problem for the flow between two smooth disks
were considered by Niimi (1971) and Balan et al (1973).

The wavyness of the boundaries is assumed to be axisymmetric and is represented by the Bessel function. Stable stratification in the fluid is attained by keeping the upper and lower disks at higher and lower temperatures respectively. The governing equations are linearized under Boussinesq approximation. The solutions are obtained using the perturbation method, up to first order, treating the amplitude of wavyness as small, for the case when the temperatures are prescribed on the disks. Exact solutions of the zero and first order equations are obtained separately. Von Kármán type similarity solutions are used to obtain the zero-order solutions of the problem. Analytical solutions of the flow field thus obtained are evaluated numerically for certain typical values of $E$, $\sigma$ and $f_R$. Variation of $E$, $\sigma$ and $f_R$ over the velocity field and temperature distribution are illustrated by means of figures 4.1 to 4.6.

The secondary effects on the flow field due to wavyness, represented by first order solutions, are also depicted graphically in figures 4.7 to 4.12. It is mainly observed from these figures that the axial velocity component is largely influenced by the wavyness of the walls. The effect of wavyness on azimuthal and radial velocity components are also significant, but is mainly confined to boundary layers. The wavyness also
affects the temperature to a considerable extent. Further, the
temperature is appreciably influenced by changes in \( f_R \) and \( \sigma \).

4.3 Basic equations

Let us consider the steady motion of an incompressible,viscous, thermally stratified fluid between two wavy disks whose
deformations are taken in the form of Bessel functions, which
are rotating with different angular velocities. The disks are
maintained at different temperatures \( T_1, T_2 \) \( (T_1 > T_2) \) and are
rotating with slightly different angular velocities \( \Omega_1 \) and \( \Omega_2 \)
\( (\Omega_1 > \Omega_2) \) about a common vertical axis. A cylindrical polar
co-ordinate system \( (r', \varphi', z') \) rotating about the \( z' \) axis
with mean angular velocity \( \Omega = (\Omega_1 + \Omega_2)/2 \) of the two disks is
chosen. The disks are represented by the equations
\[
Z' = \pm L[1 + \varepsilon J_0(\frac{r'}{L})],
\]
where \( \varepsilon \) is the coefficient of roughness
and \( \eta \) is a constant.

The governing basic equations of motion, in a rotating
frame of reference, are
\[
\rho '[(\mathbf{\tilde{q}}' \cdot \nabla')\mathbf{\tilde{q}}'] + 2(\mathbf{\nabla} \times \mathbf{\tilde{q}}') = -\mathbf{\nabla}' p' - \rho ' \mathbf{g} \hat{k} + \mu \mathbf{\nabla}'^2 \mathbf{\tilde{q}}' \quad (4.1)
\]
\[
\text{div } \mathbf{\tilde{q}}' = 0 \quad (4.2)
\]
\[
(\mathbf{\tilde{q}}' \cdot \mathbf{\nabla}') T' = \kappa \mathbf{\nabla}'^2 T' \quad (4.3)
\]

where the physical quantities involved in these equations have
their usual meanings as depicted in section 1.2.
The following non-dimensional quantities are introduced.

\[
(u,v,w) = \left( \frac{u', v', w'}{\beta \Delta L} \right)
\]

\[
\beta = \left( \frac{\eta_1 - \eta_2}{2 \Delta} \right) \text{ (small)}, \quad (r,z) = \left( \frac{r', z'}{L} \right)
\]

\[
p = \frac{p' - p_e'}{\beta \rho_e \eta^2 L^2}, \quad T = \frac{T' - T_e'}{(\eta^2 L / \kappa \beta)}
\]

\[
p_e' = p_c' - \int r z' \rho_e g dz',
\]

\[
\rho_e' = \rho_m [1 - \kappa(T_e' - T_m')]
\]

\[
T_e' = T_{eo}(r', z') + \varepsilon T_{ei}(r', z')
\]

where \(T_{eo}(r', z') = \frac{1}{2} \left( \frac{T_1' - T_2'}{2} \right) \frac{z'}{L} + \frac{T_1' + T_2'}{2} \) and \(T_{ei}(r', z') = \frac{1}{2 \sinh \eta} \left( \frac{T_1' - T_2'}{2} \right) (\sinh \eta z'/L) J_0(\eta r'/L) \)

where \((u', v', w')\) are the perturbed velocity components in the cylindrical co-ordinate system \((r', \varphi', z')\), \(p'\) the pressure, \(\rho'\) the density and \(T'\) the temperature. In the above expressions \(p_e', \rho_e'\) and \(T_e'\) denote the equilibrium values of pressure, density and temperature respectively where \(\rho_m'\) and \(T_m'\) stand for the mean density and temperature and \(p_c'\) is a constant of integration.

The governing equations are linearized under Boussinesq approximation and the motion is considered as axisymmetric. They are given by.
\begin{align*}
2v - \frac{\partial p}{\partial r} + E\mathcal{L}(u) &= 0 \tag{4.4} \\
-2u + E\mathcal{L}(v) &= 0 \tag{4.5} \\
T - \frac{\partial p}{\partial z} + E\nabla^2 w &= 0 \tag{4.6} \\
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} &= 0 \tag{4.7} \\
w &= \frac{EfR}{\sigma} \nabla^2 T \tag{4.8}
\end{align*}

where \(\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \mathcal{L} = \nabla^2 - 1/r^2\)

\(E = \frac{\mathcal{L}}{\mathcal{N}^2} \) (Ekman number) \(\omega = \frac{\mathcal{L}}{\mathcal{N}^2} \)

\(f_R = \frac{\mathcal{N}^2}{\mathcal{K}g(T_1 - T_2)} \) (internal Froude number)

\(\sigma = \frac{\mathcal{L}}{\mathcal{X}} \) (Prandtl number)

\(\mathcal{K} = \) coefficient of thermal diffusion

\(g = \) acceleration due to gravity

\(K = \) coefficient of volume expansion

The rotating disks are assumed to be non-conducting and rigid. We have the following boundary conditions,

\(u = w = T = 0\) at \(z = z_1\) and \(z = z_2\)

\(v(r, z) = r\) at \(z = z_1\) and \(v(r, z) = -r\) at \(z = z_2\) \(\tag{4.9}\)

where \(z_1 = l + \varepsilon J_0(\eta r)\) \(\tag{4.10}\)

\(z_2 = -l - \varepsilon J_0(\eta r)\).
Assuming that the coefficient of roughness, $\epsilon$, is small, all the physical quantities are expanded in terms of $\epsilon$ as:

$$F(x, z) = F_0(x, z) + \epsilon F_1(x, z) + \epsilon^2 F_2(x, z) + \ldots \quad (4.11)$$

where $F$ denotes any physical quantity and the suffixes denote solutions of various orders. We will restrict ourselves up to first order only, for the future calculations. From the equations (4.4) and (4.6), $p$ is eliminated. Considering the expansions of the dependent variables in terms of $\epsilon$, as given in (4.11), and writing the equations order-wise, we have

$$2 \frac{\partial v_0}{\partial z} + E \frac{\partial}{\partial z} (\nabla^2 u_0 - u_0/r^2) - \frac{\partial T_0}{\partial r} - E \frac{\partial}{\partial r} (\nabla^2 w_0) = 0 \quad (4.12)$$

$$-2u_0 + E(\nabla^2 v_0 - v_0/r^2) = 0 \quad (4.13)$$

$$w_0 = \frac{E}{\sigma} f r \nabla^2 T_0 \quad (4.14)$$

$$\frac{\partial u_0}{\partial r} + \frac{\partial w_0}{\partial z} + \frac{u_0}{r} = 0 \quad (4.15)$$

with the boundary conditions,

$$u_0(r, l) = 0 = u_0(r, -l) \quad (4.16)$$

$$v_0(r, l) = r, \quad v_0(r, -l) = -r$$

$$w_0(r, l) = 0 = w_0(r, -l)$$

$$T_0(r, l) = 0 = T_0(r, -l)$$

and

$$2 \frac{\partial v_1}{\partial z} + E \frac{\partial}{\partial z} (\nabla^2 u_1 - u_1/r^2) - \frac{\partial T_1}{\partial r} - E \frac{\partial}{\partial r} (\nabla^2 w_1) = 0 \quad (4.17)$$

$$-2u_1 + E(\nabla^2 v_1 - v_1/r^2) = 0 \quad (4.18)$$
\[ w_1 = \frac{\text{Ef} R}{\sigma} v^2 \tau \] 

(4.19)

\[ \frac{\partial u_1}{\partial r} + \frac{\partial w_1}{\partial z} + \frac{u_1}{r} = 0 \] 

(4.20)

with the boundary conditions

\[ u_1(r,1) = -u_0'(r,1) J_0(\eta r) \]

\[ u_1(r,-1) = u_0'(r,-1) J_0(\eta r) \]

\[ v_1(r,1) = -v_0'(r,1) J_0(\eta r) \]

\[ v_1(r,-1) = v_0'(r,-1) J_0(\eta r) \]

\[ w_1(r,1) = -w_0'(r,1) J_0(\eta r) \]

\[ w_1(r,-1) = w_0'(r,-1) J_0(\eta r) \]

\[ T_1(r,1) = -T_0'(r,1) J_0(\eta r) \]

\[ T_1(r,-1) = T_0'(r,-1) J_0(\eta r) \]

4.4 Solution of the problem

The stream function \( \psi \) of the meridional current is introduced to solve the equations (4.12) to (4.15). We have

\[ u_0 = \frac{\delta \psi}{\delta z}, \quad w_0 = -\frac{1}{r} \frac{\delta}{\delta r} (r \psi). \]

(4.22)

Assuming the solutions for \( \psi, \nu \) etc., in the Von Kármán type,

\[ \psi(r,z) = r\varphi(z), \quad v_0(r,z) = r\nu(z), \]

(4.23)

\[ T_0(r,z) = T_{00}(z) + r^2 T_0'(z), \] the equations (4.12) to (4.15) reduce to
\[ 2 \frac{d^2 v}{dz^2} + E \frac{d^4 \Phi}{dz^4} = 0 \]  
\[ -2 \frac{d\Phi}{dz} + E \frac{d^2 v}{dz^2} = 0 \]  
\[ \frac{E f R}{\sigma} \left[ \frac{d^2 T_{00}(z)}{dz^2} + 4 T_{01}(z) \right] = -2\Phi \]  
\[ \frac{d^2 T_{01}(z)}{dz^2} = 0 \]

Solving the equations (4.24) to (4.27) subject to the following boundary conditions

\[ \Phi = \frac{d\Phi}{dz} = T_{00} = T_{01} = 0 \text{ at } z = \pm 1 \]  
\[ V = \pm 1 \text{ at } z = \pm 1, \]  

we have the values for \( u_0, v_0, w_0 \) and \( T_0 \) as

\[ u_0 = \frac{r E}{2} \left[ \sum_{i=1}^{4} A_i \alpha_i^2 e^{\alpha_i z} \right] \]  
\[ v_0 = r \left[ \sum_{i=1}^{4} A_i e^{\alpha_i z} + A_5 \right] \]  
\[ w_0 = -E \left[ \sum_{i=1}^{4} A_i \alpha_i e^{\alpha_i z} \right] + A_8 \]  
\[ T_{00}(z) = A_6 z + A_7 - \frac{2g}{E f R} \left[ \frac{E}{2} \sum_{i=1}^{4} \frac{A_i}{\alpha_i} e^{\alpha_i z} - A_8 \frac{z^2}{4} \right] \]  
\[ T_{01}(z) = 0 \]
where the constants $A_i (i = 1,8)$ are got from the following expressions

$$\sum_{i=1}^{4} A_i \alpha_i^2 e^{\alpha_i} = 0 \quad (4.34)$$

$$\sum_{i=1}^{4} A_i \alpha_i^2 e^{-\alpha_i} = 0 \quad (4.35)$$

$$\sum_{i=1}^{4} A_i \alpha_i (\sinh \alpha_i) = 0 \quad (4.36)$$

$$\sum_{i=1}^{4} A_i \sinh \alpha_i = 1 \quad (4.37)$$

$$A_5 = 1 - \sum_{i=1}^{4} (A_i e^{\alpha_i}) \quad (4.38)$$

$$A_6 = \frac{\sigma}{\pi R} \left[ \sum_{i=1}^{4} A_i \frac{\sinh \alpha_i}{\alpha_i} \right] \quad (4.39)$$

$$A_7 = \frac{\sigma}{\pi R} \left[ \sum_{i=1}^{4} A_i \left( \frac{\cosh \alpha_i}{\alpha_i} - \frac{\alpha_i e^{\alpha_i}}{2} \right) \right] \quad (4.40)$$

$$A_8 = \frac{\sigma}{\pi} \sum_{i=1}^{4} A_i \alpha_i^2 e^{\alpha_i} \quad (4.41)$$

and $\alpha_1 = (2i/E)^{\frac{1}{2}}, \alpha_2 = (-2i/E)^{\frac{1}{2}}$

$\alpha_3 = -(2i/E)^{\frac{1}{2}}$ and $\alpha_4 = -(-2i/E)^{\frac{1}{2}} \quad (4.42)$
The first order equations (4.17) to (4.20) are solved separating the variables in the form.

\[
\begin{align*}
    u_1(r,z) &= U_1(z)r J_0(\eta r) + U_2(z) J_1(\eta r) \quad (4.43) \\
    v_1(r,z) &= V_1(z)r J_0(\eta r) + V_2(z) J_1(\eta r) \quad (4.44) \\
    w_1(r,z) &= W_1(z) J_0(\eta r) + W_2(z) r J_1(\eta r) \quad (4.45) \\
    T_1(r,z) &= T_1(z) J_0(\eta r) + T_2(z) r J_1(\eta r) \quad (4.46)
\end{align*}
\]

Substitution of (4.43) to (4.46) into (4.17) to (4.20), we obtain two sets of coupled equations, which are given by,

\[
\begin{align*}
    2DV_1 + ED(D^2 - \eta^2)U_1 - \eta T_2 - En(D^2 - \eta^2)W_2 &= 0 \quad (4.47) \\
    -2V_1 + E(D^2 - \eta^2)U_1 &= 0 \quad (4.48) \\
    W_2 &= \frac{E_1 R}{\sigma} (D^2 - \eta^2)T_2 \quad (4.49) \\
    D W_2 &= \eta U_1 \quad (4.50)
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
    U_1(1) &= - (D^2 \varphi)_z = 1, \quad U_1(-1) = (D^2 \varphi)_z = -1 \\
    V_1(1) &= - \left( \frac{dv}{dz} \right)_z = 1, \quad V_1(-1) = \left( \frac{dv}{dz} \right)_z = -1 \quad (4.51) \\
    W_2(1) &= 0 = W_2(-1), \quad T_2(1) = 0 = T_2(-1)
\end{align*}
\]

and

\[
\begin{align*}
    2DW_2 - 2EnDU_1 + ED(D^2 - \eta^2)U_2 + \eta T_1 + En(D^2 - \eta^2)W_1 + 2En^2W_2 &= 0 \quad (4.52)
\end{align*}
\]
\[ -2U_2 - 2E\eta V_1 + E(D^2 - \eta^2)V_2 = 0 \] \hspace{1cm} (4.53)

\[ (D^2 - \eta^2)T_1 = \frac{\sigma}{\beta R} W_1 - 2\eta T_2 \] \hspace{1cm} (4.54)

\[ DW_1 = -2U_1 - \eta U_2 \] \hspace{1cm} (4.55)

with the boundary conditions,

\[ U_2(l) = 0, \quad U_2(-l) = 0 \]

\[ V_2(l) = 0, \quad V_2(-l) = 0 \]

\[ W_1(l) = 2(D\varphi)_z = 1 \] \hspace{1cm} (4.56)

\[ W_1(-l) = -2(D\varphi)_z = -1 \]

\[ T_1(l) = -(DT_{oo})_z = 1 \]

\[ T_1(-l) = (DT_{oo})_z = -1 \]

where \( D = \frac{d}{dz} \)

Solving the equations (4.47) to (4.50), subject to the boundary conditions (4.51), we have

\[ U_1 = \sum_{i=1}^{3} \left( B_i e^{m_i z} + B_i e^{-m_i z} \right) \] \hspace{1cm} (4.57)

\[ V_1 = \frac{1}{2E} \sum_{i=1}^{3} \frac{1}{(m_i^2 - \eta^2)} \left( B_i e^{m_i z} + B_i e^{-m_i z} \right) + B_7 e^{\eta z} + B_8 e^{-\eta z} \] \hspace{1cm} (4.58)

\[ W_2 = \sum_{i=1}^{3} W_2(m_i) \left( B_i e^{m_i z} - B_i e^{-m_i z} \right) \] \hspace{1cm} (4.59)
\[
T_2 = \sum_{i=1}^{3} T_2(m_i) \left( B_i e^{m_i z} - B_{-i} e^{-m_i z} \right) + 2B_7 e^{\eta z} - 2B_8 e^{-\eta z} \quad (4.60)
\]

where

\[
W_2(m_i) = \frac{1}{P_1} \left[ \frac{4}{E} m_i + E m_i (m_i^2 - \eta^2) - E m_i^3 + 2E \eta^4 m_i \right] \quad (4.61)
\]

\[
P_1 = \frac{\eta \sigma}{E \rho} + E \eta^5 \quad (4.62)
\]

\[
T_2(m_i) = \frac{4}{E \rho} \frac{m_i}{m_i^2 - \eta^2} + \frac{E}{\eta} m_i (m_i^2 - \eta^2) - E(m_i^2 - \eta^2) W_2(m_i) \quad (4.63)
\]

where \( m_i \) are the roots of the equation

\[
E \delta^3 - (3 \eta^2) \delta^2 + (3 \eta^4 + 4/E) \delta - \frac{\sigma \eta^2}{E \rho} - E \eta^6 = 0 \quad (4.64)
\]

The constants \( B_i, i = 1 \) to 3 and \( B_{-i}, i = 1 \) to 3 are derived from the boundary conditions (4.51).

Solving equations (4.52) to (4.55), utilizing the boundary conditions, (4.56) we have the solutions as

\[
U_2 = \sum_{i=1}^{3} \left( C_i e^{m_i z} + C_{-i} e^{-m_i z} \right) + \sum_{i=1}^{3} U_2(m_i). \quad (4.65)
\]

\[
V_2 = \sum_{i=1}^{3} V_{21}(m_i) \left( C_i e^{m_i z} + C_{-i} e^{-m_i z} \right) + \sum_{i=1}^{3} V_{23}(m_i). \quad (4.66)
\]

\[
(3_i e^{m_i z} + B_{-i} e^{-m_i z}) + z e^{\eta z} B_7 - z e^{-\eta z} B_8 + C_7 e^{\eta z} + C_8 e^{-\eta z} \quad (4.66)
\]
\[
W_1 = \sum_{i=1}^{3} W_{1i}(m_i) \left( C_i e^{m_i z} - C_i e^{-m_i z} \right) + \sum_{i=1}^{3} W_{12}(m_i) \left( B_i e^{m_i z} - B_i e^{-m_i z} \right) + \sum_{i=1}^{3} W_{13}(m_i) z \left( B_i e^{m_i z} + B_i e^{-m_i z} \right)
\]

(4.67)

\[
T_1 = \sum_{i=1}^{3} T_{1i}(m_i) \left( C_i e^{m_i z} - C_i e^{-m_i z} \right) + \sum_{i=1}^{3} T_{12}(m_i) \left( B_i e^{m_i z} - B_i e^{-m_i z} \right) + \sum_{i=1}^{3} T_{13}(m_i) z \left( B_i e^{m_i z} + B_i e^{-m_i z} \right) - 2 B_7 e^{\eta z} (z + \frac{1}{\eta}) - 2 B_8 e^{-\eta z} (z - \frac{1}{\eta})
\]

(4.68)

where

\[
U_2(m_i) = \frac{\left[ \frac{2E_0}{E_R} + 6E\eta \left( m_i^2 - \eta^2 \right)^2 \right]}{2E_{m_i} \pi \sum_{i \neq j} (m_i^2 - m_j^2)}
\]

(4.69)

\[
V_{21}(m_i) = \frac{2}{E(m_i^2 - \eta^2)}
\]

(4.70)

\[
V_{22}(m_i) = \frac{2}{E} \frac{U_2(m_i)}{(m_i^2 - \eta^2)}
\]

(4.71)

\[
V_{23}(m_i) = -\frac{4}{E} m_i \frac{U_2(m_i)}{(m_i^2 - \eta^2)^2} + 2\eta \frac{V_1(m_i)}{(m_i^2 - \eta^2)}
\]

(4.72)

\[
W_{1i}(m_i) = -\frac{1}{F_1} \left[ \frac{4}{E} m_i + E_{m_i}^5 + 3E_{m_i} \eta^4 - 3E\eta^2 m_i^3 \right]
\]

(4.73)
\[
W_{12}(m_i) = \frac{1}{\eta} \left[ 4E_1 m_i (m_i^2 - \eta^2) - 4E_2 m_i^3 + 2E_3 m_i^2 - U_2(m_i) \right]
\]
\[
\left( \frac{4}{E_1} + 3E_2 \right) - 5E_1 m_i U_2(m_i) + 9E_1^2 m_i^2 U_2(m_i) - 4E_2.
\]
\[
(m_i^2 - \eta^2) W_2(m_i)]
\] (4.74)

\[
W_{13}(m_i) = \frac{1}{\eta} \left[ 3E_1^2 m_i^3 U_2(m_i) - (4/E_1 + 3E_2) U_2(m_i)m_i
\right]
\]
\[
- E_3 m_i^5 U_2(m_i)]
\] (4.75)

\[
T_{11}(m_i) = \frac{1}{\eta} \left[ -2m_i V_{21}(m_i) - E_1 (m_i^3 - \eta^2 m_i) - E_2 W_{11}(m_i)
\right]
\]
\[
(m_i^2 - \eta^2)]
\] (4.76)

\[
T_{12}(m_i) = \frac{1}{\eta} \left[ -2 V_{22}(m_i) - 2m_i V_{23}(m_i) + 2E_1 m_i - 3E_2 m_i^2 U_2(m_i)
\right]
\]
\[
+ E_1^2 U_2(m_i) - E_2 W_{12}(m_i) (m_i^2 - \eta^2) - 2E_2 m_i.
\]
\[
W_{13}(m_i) - 2E_2 W_2(m_i)]
\] (4.77)

\[
T_{13}(m_i) = \frac{1}{\eta} \left[ -2m_i V_{22}(m_i) - E_3 m_i^3 U_2(m_i) + E_2 m_i^2 U_2(m_i)
\right]
\]
\[
- E_2 m_i^2 W_{13}(m_i) + E_3 W_{13}(m_i)]
\] (4.78)

where the constants \( C_i \) and \( C_i^2 \), \( i = 1 \) to 3 are determined from the boundary conditions (4.56).
4.5 Results and discussion

The solutions obtained above are displayed graphically in the figures 4.1 to 4.12, by assigning typical values to the parameters and variation of $E$, $\sigma$ and $f_R$ on the flow have been examined. Figures 4.1 to 4.6 show the velocity and temperature fields calculated up to first order. To have a clear idea of the secondary flow caused by the deformation in the boundaries, the first order flow quantities are depicted separately through the figures 4.7 to 4.12. Because the boundaries are disposed symmetrically, all the profiles for axial velocity component and for temperature are symmetric while the profiles for radial and azimuthal velocity components are antisymmetric.

In figure 4.1, the radial velocity profiles for various Ekman numbers are drawn at the radial distances $r = 1$ and $r = 3$. From the profile corresponding to $E = 0.01$, it is clear that the variation of this velocity component is mainly confined to the boundaries within a distance of order $L/3$, indicating the formation of Ekman layer. For $E = 0.05$ and $0.1$, the radial velocity component is significant in the interior, and vanishes at the mid plane. The curves at $r = 3$ are magnified and are similar to those at $r = 1$ and exhibit Von-Kármán similarity. The azimuthal velocity distribution, shown in figure 4.2 for various $E$, behave in a manner similar to the radial velocity profiles.
The curves corresponding to the variation of axial velocity component with respect to Ekman number are depicted in figure 4.3. These curves at \( r = 1 \) and \( r = 3 \) are almost identical, because the radial dependence of the axial velocity is only through first and higher terms whose contributions are small. It is further noted that this particular component of velocity changes substantially with respect to \( E \). For small values of \( E \), i.e., \( E = 0.01, 0.05 \), the profiles are flattened while for \( E = 0.1 \), they become bell shaped.

While we find that the velocity of the fluid does not change appreciably as the internal Froude number \( f_R \) changes, the temperature considerably changes with \( f_R \). Negative values of \( T \) indicate reduction in the temperature from its equilibrium value. It is found that the reduction in temperature decreases when either \( f_R \) or \( E \) increases, while the increase in \( \sigma \) leads to decrease in temperature, as seen from figures 4.4 to 4.6. Coming to first order flow field, it is revealed from figures 4.7 and 4.8 that the perturbations for \( E = 0.01 \) in radial and azimuthal components are large near the boundaries, but become negligible far from them. When \( E = 0.05 \) and \( 0.1 \), they are significant throughout the region between the boundaries.

From figure 4.9, which shows the behaviour of first order axial velocity component for different \( E \), it is interesting to note the wide ranging character of these profiles with appreciable changes in this velocity component. When \( E = 0.01 \),
at \( r = 1 \), the profile runs straight parallel to the axis in the interior and then rapidly shoots up to nearly four times of its core value, near the boundaries. For \( E = 0.05 \) and \( C_1 \), the curves have a minimum at the mid plane forming troughs. When \( E = 0.05 \), the minimum value becomes negative at the mid plane. The axial velocity component increases to a maximum near the boundaries for all \( E \).

At the section \( r = 3 \), the axial velocity suffers sudden change in sign for small values of \( E \) i.e., \( E = 0.01 \) and \( E = 0.05 \). Thus it is inferred that the effect of roughness largely affects the flow in the axial direction as we move in the radial direction.

First order temperature also elucidates wide changes as the value of \( E \) changes from \( 0.01 \) to \( 0.1 \) as disclosed from the figure 4.10. The decrease in temperature increases with decreasing value of \( E \). But the temperature increases as \( f_R \) increases, as seen from figure 4.11. Again from figure 4.12, it is observed that as the Prandtl number increases from 0.044 to 0.7, large changes occur in temperature. Finally, we notice that the first order temperature is reduced to a significant level at a radial distance \( r = 3 \) compared to its values at \( r = 1 \).

In conclusion, we find that roughness of the boundaries induces a small velocity field which modifies the
flow of rotating stratified fluid between two plane smooth boundaries, significantly. For $E = 0.01$, the changes in velocity are mostly confined near the boundaries. The perturbed axial velocity is significant and is still dominant as we move in the radial direction. When the temperatures are prescribed on the disks, the effect of wavyness is to reduce the temperature of the fluid as we move along the radial direction.
REFERENCES


Citron, S.J. 1962 Slow viscous flow between rotating concentric infinite cylinders with axial roughness. Trans. ASME. 29, 188.


FIG. 4.1 RADIAL VELOCITY PROFILES FOR THE VARIATION OF E
FIG. 4.2 AZIMUTHAL VELOCITY PROFILES FOR THE VARIATION OF $E$
FIG. 4.3 AXIAL VELOCITY PROFILES FOR THE VARIATION OF $E$
FIG. 4.4 TEMPERATURE PROFILES FOR THE VARIATION OF E
FIG. 4.5 TEMPERATURE PROFILES FOR THE VARIATION OF $f_R$
FIG. 4.6 TEMPERATURE PROFILES FOR THE VARIATION OF $\sigma$

- $\sigma = 0.044$
- $\sigma = 0.09$
- $\sigma = 0.7$

Graph showing temperature profiles with $E = 0.01$ & $fR = 0.2$.
FIG. 4.8 FIRST ORDER AZIMUTHAL VELOCITY COMPONENT
FOR THE VARIATION OF E
FIG. 4.9 FIRST ORDER AXIAL VELOCITY COMPONENT FOR THE VARIATION OF E
FIG. 4.10 FIRST ORDER TEMPERATURE FOR THE VARIATION OF $E$
FIG. 4.11  FIRST ORDER TEMPERATURE FOR THE VARIATION OF $f_R$
FIG. 4.12  FIRST ORDER TEMPERATURE FOR THE VARIATION OF $\sigma$