II. NULL CONTROLLABILITY OF ABSTRACT NEUTRAL FUNCTIONAL INTEGRODIFFERENTIAL SYSTEMS

2.1. INTRODUCTION

Theory of neutral functional differential equations has been studied by several authors [10,12,25,26,40,55-58,84]. These types of equations occur in the study of heat conduction in materials with memory and many other physical phenomena. So it is interesting to study the controllability problem for such systems. Several papers appeared on the controllability of linear and nonlinear systems in infinite dimensional spaces [11,19,23,24,70]. Balachandran and Anandhi [7] discussed the controllability of neutral functional integrodifferential systems in abstract phase space with the help of Schauder's fixed point theorem. Recently Fu [36] studied the same problem in abstract phase space by utilizing the Sadovskii fixed point theorem. In this chapter we shall study the null controllability of neutral functional integrodifferential systems by utilizing the technique of Fu [36].

Consider the neutral functional integrodifferential system of the form

\[
\frac{d}{dt} \left[ x(t) - g \left( t, x_t, \int_0^t h(t, s, x_s) ds \right) \right] = -Ax(t) + Bu(t) + f \left( t, x_t, \int_0^t q(t, s, x_s) ds \right), \quad t \geq 0, \quad (2.1)
\]

\[
x_0 = \phi \in B,
\]

where the state variable \( x(\cdot) \) takes values in the Banach space \( X \) and the control function \( u(\cdot) \) is in \( L^2(J, U) \), a Banach space of admissible control functions with \( U \) as a Banach space and \( J = [0, a] \) and \( B \) is a bounded linear operator from \( U \) into \( X \), the unbounded linear operator \( -A \) generates an analytic semigroup and \( h : J \times J \times B \to X \), \( q : J \times J \times B \to X \), \( f, g : J \times B \times X \to X \) are appropriate functions, \( B \) is the phase space to be specified later.

2.2. PRELIMINARIES

Let \( -A : D(A) \to X \) be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator \( T(t) \) defined on a Banach space
$X$ with norm $\| \cdot \|$. Let $0 \in \rho(-A)$. Then define the fractional power $A^\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$ which is dense in $X$. Further $D(A^\alpha)$ is a Banach space under the norm

$$\|x\|_\alpha = \|A^\alpha x\|, \text{ for } x \in D(A^\alpha)$$

and is denoted by $X_\alpha$. The imbedding $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta < \alpha \leq 1$ is compact whenever the resolvent operator of $A$ is compact. For semigroup $\{T(t)\}$ the following properties will be used.

(a) there is a $M > 0$ such that $\|T(t)\| \leq M$, for all $0 \leq t \leq a$.

(b) for any $\alpha > 0$, there exists a positive constant $C_\alpha$ such that

$$\|A^\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}, 0 < t \leq a. \tag{2.2}$$

In our study we follow the axiomatic definition for the phase space $B$ introduced by Hale and Kato [46] and the terminology used in [59]. We assume that the delay $x_t : (-\infty, 0] \to X$ defined by $x_t(\theta) = x(t + \theta)$ belongs to some abstract phase space $B$ which will be a linear space of functionals mapping $(-\infty, 0]$ into $X$ endowed with the seminorm $\| \cdot \|_B$ and satisfies the following axioms:

(A1) If $x : (-\infty, a) \to X$, $a > 0$, is continuous on $[0, a]$ and $x_0 \in B$, then, for every $t \in [0, a)$, the following conditions hold:

1. $x_t$ is in $B$,
2. $\|x(t)\| \leq H\|x_t\|_B$,
3. $\|x_t\|_B \leq K(t) \sup\{\|x(s)\| : 0 \leq s \leq t\} + M(t)\|x_0\|_B$.

Here $H \geq 0$ is a constant, $K, M : [0, \infty) \to [0, \infty)$, $K$ is continuous and $M$ is locally bounded and $H, K, M$ are independent of $x(t)$.

(A2) For the function $x(\cdot)$ in (A1), $x_t$ is a $B$ valued continuous function on $[0, a]$.

(A3) The space $B$ is complete.

Throughout the thesis the symbol $B$ stands for the phase space and $K(t)$ is a continuous function.
We need the following fixed point theorem due to Sadovskii [83].

**The Sadovskii Fixed Point Theorem:** Let $P$ be a condensing operator on a Banach space $X$, that is, $P$ is continuous and takes bounded sets into bounded sets and $\alpha(P(B)) < \alpha(B)$ for every bounded set $B$ of $X$ with $\alpha(B) > 0$. If $P(H) \subset H$ for a convex, closed and bounded set $H$ of $X$, then $P$ has a fixed point on $H$.

We make the following assumptions on the system (2.1).

(H1) $g : J \times B \times X \to X$ is a continuous function and there exist constants $\beta \in (0, 1)$ and $L, L_1 > 0$ such that the function $g$ is $X_\beta$-valued and satisfies the Lipschitz condition:

$$||A^\beta g(s_1, \phi_1, \eta_1) - A^\beta g(s_2, \phi_2, \eta_2)|| \leq L \left[|s_1 - s_2| + ||\phi_1 - \phi_2||_B + ||\eta_1 - \eta_2||_X\right],$$

for $0 \leq s_1, s_2 \leq a, \phi_1, \phi_2 \in B, \eta_1, \eta_2 \in X$ and the inequality

$$||A^\beta g(t, \phi, \eta)|| \leq L_1 \left[||\phi||_B + ||\eta||_X + 1\right]$$ (2.3)

holds for $t \in [0, a], \phi \in B, \eta \in X$.

(H2) $h : J \times J \times B \to X$ is continuous and there exist constant $\beta \in (0, 1)$ and $L_2, L_3 > 0$ such that the function $h$ is $X_\beta$-valued and satisfies the Lipschitz condition:

$$||A^\beta h(t_1, s, \psi_1) - A^\beta h(t_2, s, \psi_2)|| \leq L_2 \left[|t_1 - t_2| + ||\psi_1 - \psi_2||_B\right],$$

for $0 \leq t_1, t_2 \leq a, \psi_1, \psi_2 \in B$ and the inequality

$$||A^\beta h(t, s, \psi)|| \leq L_3 \left[||\psi||_B + 1\right]$$ (2.4)

holds for $t, s \in [0, a], \psi \in B$.

(H3) The function $q : J \times J \times B \to X$ is a continuous function such that there is a positive function $a \in L^1([0, a])$ such that

$$|q(t, s, x_s)| \leq a(t, s)||x_s||$$ and $\delta = \sup_{t \in J} \int_0^t a(t, s)ds$.  

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(H4) The function $f : J \times B \times X \rightarrow X$ satisfies the following conditions:

(i) For each $t \in J$, the function $f(t, \cdot, \cdot) : B \times X \rightarrow X$ is continuous and for each $(\phi, x) \in B \times X$ the function $f(\cdot, \phi, x) : J \rightarrow X$ is strongly measurable.

(ii) For each positive integer $k$, there is a positive function $\mu_k \in L^1(0, a)$ such that

$$\sup_{\|x\|, \|y\| \leq k} \|f(t, x, y)\| \leq \mu_k(t)$$

and

$$\lim \inf_{k \to \infty} \frac{1}{k} \int_0^a \mu_k(s) \, ds = \gamma < \infty.$$

(H5) The linear operator $W$ from $L^2(J; U)$ into $X$ defined by

$$Wu = \int_0^a T(a - s)Bu(s) \, ds$$

induces an invertible operator $\hat{W}$ defined on $L^2(J; U)/\ker W$ and there exists a positive constant $M_1 > 0$ such that $\|B\hat{W}^{-1}\| \leq M_1$.

**Definition 2.1.** We say that a function $x(\cdot) : (-\infty, a] \rightarrow X$ is a mild solution of the system (2.1) if $x_0 = \phi$, the restriction of $x(\cdot)$ to the interval $J$ is continuous and, for each $0 \leq t \leq a$, the function $AT(t - s)g(s, x_s, \int_0^s h(s, \tau, x_\tau) \, d\tau), s \in [0, t)$, is integrable and the following integral equation

$$x(t) = T(t)[\phi(0) - g(0, \phi, 0)] + g\left(t, x_t, \int_0^t h(t, s, x_s) \, ds\right)$$

$$+ \int_0^t AT(t - s)g\left(s, x_s, \int_0^s h(s, \tau, x_\tau) \, d\tau\right) \, ds$$

$$+ \int_0^t T(t - s)\left[Bu(s) + f\left(s, x_s, \int_0^s g(s, \tau, x_\tau) \, d\tau\right)\right] \, ds, \quad 0 \leq t \leq a$$

(2.5)

is satisfied.

**Definition 2.2.** The system (2.1) is said to be (local) null controllable on the interval $J$ if for every initial function $\phi \in \Omega(\subset B)$ there exists a control $u \in L^2(J; U)$ such that the mild solution $x(\cdot)$ of (2.1) satisfies $x(a) = 0$.

Let $y(\cdot) : (-\infty, a) \rightarrow X$ be the function defined by

$$y(t) = \begin{cases} T(t)\phi(0), & 0 \leq t \leq a, \\ \phi(t), & -\infty < t < 0, \end{cases}$$

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then \( y_0 = \phi \) and the map \( t \rightarrow y_t \) is continuous and take \( N = \sup \{ ||y_t||_B : 0 \leq t \leq a \} \).

For simplicity, let us take \( M_2 = ||A^{-\beta}|| \), \( K_a = \sup\{K(t) : 0 \leq t \leq a\} \),

\[
L^* = [M_2LK_a + \frac{C_{1-\beta}}{\beta}a^\beta LK_a][1 + aM_2L_2] 
\]

and \( M^* = [1 + aMM_1]\frac{1}{\beta}C_{1-\beta}a^\beta L_1 \times (N + 1)(1 + aM_2L_3) + aM_1M^2(||\phi(0)|| + ||g(0,\phi,0)||). \)

(H6) Further assume that \( L^* < 1 \) and

\[
(1 + aMM_1)\left[ (M_2L_1K_a + \frac{C_{1-\beta}}{\beta}a^\beta L_1K_a)(1 + aM_2L_3) + MK_a(1 + \frac{\delta}{\gamma}) \right] < 1
\]

2.3. MAIN RESULTS

Theorem 2.1. If the assumptions (H1)-(H6) are satisfied and \( \phi \in B \), then the system (2.1) is null controllable on the interval \( J \).

Proof. Using the assumption (H5) for an arbitrary function \( x(\cdot) \), define the control

\[
u(t) = -\tilde{W}^{-1}\left\{ T(a)[\phi(0) - g(0,\phi,0)] + g\left(a, x_a, \int_0^a h(a, s, x_s)ds\right) 
+ \int_0^a AT(a - s)g\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)ds
+ \int_0^a T(a - s)f\left(s, x_s, \int_0^s q(s, \tau, x_\tau)d\tau\right)ds\right\}(t).
\]

It shall be shown that when using this control the operator \( S \) defined by

\[
(Sx)(t) = T(t)[\phi(0) - g(0,\phi,0)] + g\left(t, x_t, \int_0^t h(t, s, x_s)ds\right)
+ \int_0^t AT(t - s)g\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)ds
+ \int_0^t T(t - s)\left[ Bu(s) + f\left(s, x_s, \int_0^s q(s, \tau, x_\tau)d\tau\right)\right]ds, \quad 0 \leq t \leq a,
\]
has a fixed point \( x(\cdot) \). Then \( x(\cdot) \) is a mild solution of system (2.1) and it is easy to verify that

\[
x(a) = (Sx)(a) = 0,
\]

which implies that the system is controllable.

Next we will prove that the operator \( S \) has a fixed point.

For each \( z \in C(J; X) \) with \( z(0) = 0 \), we denote by \( \tilde{z} \) the function defined by

\[
\tilde{z}(t) = \begin{cases} 
z(t), & 0 \leq t \leq a, \\
0, & -\infty < t < 0.
\end{cases}
\]

If \( x(\cdot) \) satisfies (2.5), we can decompose it as \( x(t) = z(t) + y(t) \), \( 0 \leq t \leq a \), which implies that \( x_t = \tilde{x}_t + y_t \) for every \( 0 \leq t \leq a \) and the function \( z(\cdot) \) satisfies

\[
z(t) = -T(t)g(0, \phi, 0) + g \left( t, \tilde{z}_t + y_t, \int_0^t h(t, s, \tilde{z}_s + y_s)ds \right) + \int_0^t AT(t - s)g \left( s, \tilde{z}_s + y_s, \int_0^s h(s, \tau, \tilde{z}_\tau + y_\tau) d\tau \right) ds + \int_0^t T(t - s)Bu(s)ds + \int_0^t T(t - s)f \left( s, \tilde{z}_s + y_s, \int_0^s q(s, \tau, \tilde{z}_\tau + y_\tau) d\tau \right) ds.
\]

Let \( P \) be the operator on \( C(J; X) \) defined by

\[
(Pz)(t) = -T(t)g(0, \phi, 0) + g \left( t, \tilde{z}_t + y_t, \int_0^t h(t, s, \tilde{z}_s + y_s)ds \right) + \int_0^t AT(t - s)g \left( s, \tilde{z}_s + y_s, \int_0^s h(s, \tau, \tilde{z}_\tau + y_\tau) d\tau \right) ds + \int_0^t T(t - s)Bu(s)ds + \int_0^t T(t - s)f \left( s, \tilde{z}_s + y_s, \int_0^s q(s, \tau, \tilde{z}_\tau + y_\tau) d\tau \right) ds.
\]

Obviously the operator \( S \) has a fixed point if and only if \( P \) has a fixed point; so we have to prove that \( P \) has a fixed point.

For each positive integer \( k \), let

\[
B_k = \{ z \in C(J; X) : z(0) = 0, \|z(t)\| \leq k, 0 \leq t \leq a \},
\]
then $B_k$ for each $k$, is a bounded, closed, convex set in $C(J;X)$. Since by (2.2) and (2.3), the following relation holds

$$\|AT(t-s)g(s,\tilde{z}_s+y_s,\int_0^s h(s,\tau,\tilde{z}_\tau+y_\tau)d\tau)\| \leq \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_1(1+aM_2L_3)[kK_a+N+1],$$

then it follows that $AT(t-s)g(s,\tilde{z}_s+y_s,\int_0^s h(s,\tau,\tilde{z}_\tau+y_\tau)d\tau)$ is integrable on $[0,t]$; so $P$ is well defined on $B_k$. We claim that there exists a positive integer $k$ such that $PB_k \subseteq B_k$. If it is not true, then, for each positive integer $k$, there is a function $z_k \in B_k$ but $Pz_k \notin B_k$, that is, $\|Pz_k(t)\| > k$ for some $t \in J$. However, on the other hand, we have

$$k \leq \|(Pz_k)(t)\|$$

$$\leq \|T(t)g(0,\phi,0) + g(t,\tilde{z}_{k,t}+y_t,\int_0^t h(t,s,\tilde{z}_{k,s}+y_s)ds)\|$$

$$+ \int_0^t |AT(t-s)g(s,\tilde{z}_{k,s}+y_s,\int_0^s h(s,\tau,\tilde{z}_{k,\tau}+y_\tau)d\tau)|ds$$

$$+ \int_0^t |T(t-s)B\tilde{W}^{-1}\{-T(a)[g(0,\phi,0)]\} - g(a,\tilde{z}_{k,a}+y_a,\int_0^a h(a,s,\tilde{z}_{k,s}+y_s)ds)\|$$

$$- \int_0^a |AT(a-\tau)g(\tau,\tilde{z}_{k,\tau}+y_\tau,\int_0^\tau h(\tau,\eta,\tilde{z}_{k,\eta}+y_\eta)d\eta)|d\tau$$

$$- \int_0^a |T(a-\tau)f(\tau,\tilde{z}_{k,\tau}+y_\tau,\int_0^\tau q(\tau,\eta,\tilde{z}_{k,\eta}+y_\eta)d\eta)|d\tau\big\{s\big\}ds$$

$$+ \int_0^t |T(t-s)f(s,\tilde{z}_{k,s}+y_s,\int_0^s q(s,\tau,\tilde{z}_{k,\tau}+y_\tau)d\tau)|ds\|$$

$$\leq M\|g(0,\phi,0)\| + \|g(t,\tilde{z}_{k,t}+y_t,\int_0^t h(t,s,\tilde{z}_{k,s}+y_s)ds)\|$$

$$+ \int_0^t |AT(t-s)g(s,\tilde{z}_{k,s}+y_s,\int_0^s h(s,\tau,\tilde{z}_{k,\tau}+y_\tau)d\tau)|ds$$

$$+ \int_0^t MM_1\{M(\|\phi(0)\| + \|g(0,\phi,0)\|)\}$$

$$+ \|g(a,\tilde{z}_{k,a}+y_a,\int_0^a h(a,s,\tilde{z}_{k,s}+y_s)ds)\|$$

$$+ \int_0^a |AT(a-\tau)g(\tau,\tilde{z}_{k,\tau}+y_\tau,\int_0^\tau h(\tau,\eta,\tilde{z}_{k,\eta}+y_\eta)d\eta)|d\tau$$

$$+ M\int_0^a \|f(\tau,\tilde{z}_{k,\tau}+y_\tau,\int_0^\tau q(\tau,\eta,\tilde{z}_{k,\eta}+y_\eta)d\eta)\|d\tau\big\{s\big\}ds$$

$$+ M\int_0^t \|f(s,\tilde{z}_{k,s}+y_s,\int_0^s q(s,\tau,\tilde{z}_{k,\tau}+y_\tau)d\tau)\|ds.$$
Since
\[
\left\| \int_0^t A(T(t-s)g \left( s, \tilde{z}_{k,s} + y_s, \int_0^s h(s, \tau, \tilde{z}_{k,\tau} + y_\tau) d\tau \right) ds \right\| \\
\leq \left\| \int_0^t A^{1-\beta} T(t-s)A^\beta g \left( s, \tilde{z}_{k,s} + y_s, \int_0^s h(s, \tau, \tilde{z}_{k,\tau} + y_\tau) d\tau \right) ds \right\| \\
\leq \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} L_1(\|\tilde{z}_{k,s} + y_s\|) + \int_0^s h(s, \tau, \tilde{z}_{k,\tau} + y_\tau) d\tau \|x + 1) ds \\
\leq \frac{1}{\beta} C_{1-\beta} a^\beta L_1(1 + aM_2 L_3)[kK_a + N + 1],
\]

\[
\left\| g \left( t, \tilde{z}_{k,t} + y_t, \int_0^t h(t, s, \tilde{z}_{k,s} + y_s) ds \right) \right\| \\
= \left\| A^{-\beta} A^\beta g \left( t, \tilde{z}_{k,t} + y_t, \int_0^t h(t, s, \tilde{z}_{k,s} + y_s) ds \right) \right\| \\
\leq M_2 L_1(1 + aM_2 L_3)[kK_a + N + 1]
\]

and
\[
\int_0^t \| f \left( s, \tilde{z}_{k,s} + y_s, \int_0^s q(s, \tau, \tilde{z}_{k,\tau} + y_\tau) d\tau \right) \| ds \leq \int_0^a \mu_k^*(s) ds,
\]

where \( k^* = (1 + \delta)(kK_a + N) \), there holds
\[
k \leq M\|g(0, \phi, 0)\| + M_2 L_1[(1 + aM_2 L_3)(kK_a + N + 1)] \\
+ \frac{1}{\beta} C_{1-\beta} a^\beta L_1[(1 + aM_2 L_3)(kK_a + N + 1)] \\
+ aMM_1 \left\{ M(\|\phi(0)\| + \|g(0, \phi, 0)\|) \\
+ M_2 L_1(1 + aM_2 L_3)(kK_a + N + 1) + \frac{1}{\beta} C_{1-\beta} a^\beta L_1(1 + aM_2 L_3)(kK_a + N + 1) \\
+ M \int_0^a \mu_k^*(\tau) d\tau \right\} + M \int_0^a \mu_k^*(s) ds \\
\leq M^* + [1 + aMM_1](M_2 L_1 kK_a + aM_2^2 L_1 L_3 kK_a) \\
+ [1 + aMM_1] \left( \frac{1}{\beta} C_{1-\beta} a^\beta L_1(kK_a + aM_2 L_3 kK_a) \right) \\
+ [1 + aMM_1] M \int_0^a \mu_k^*(s) ds \\
\leq M^* + (1 + aMM_1)\{M_2 L_1 kK_a + aM_2^2 L_1 L_3 kK_a + \frac{1}{\beta} C_{1-\beta} a^\beta L_1 kK_a \\
+ \frac{1}{\beta} C_{1-\beta} a^\beta aL_1 M_2 L_3 kK_a + M \int_0^a \mu_k^*(s) ds\}. 
\]
Dividing on both sides by $k$ and taking the lower limit, we get

$$(1 + aM_M)(M_2L_1K_a + \frac{C_1-\beta}{\beta}a^g L_1K_a)(1 + aM_2L_3) + M K_a(1 + \delta) \gamma \geq 1.$$  

This contradicts (2.7). Hence $P B_1 \subseteq B_k$ for some positive number $k$.

Now define the operators $P_1, P_2$ on $B_k$ by

$$(P_1 z)(t) = -T(t)g(0, \phi, 0) + g \left( t, \bar{z}_t + y_t, \int_0^t h(t, s, \bar{z}_s + y_s)ds \right) + \int_0^t AT(t - s)g \left( s, \bar{z}_s + y_s, \int_0^s h(s, \tau, \bar{z}_\tau + y_\tau)d\tau \right)ds$$

and

$$(P_2 z)(t) = \int_0^t T(t - s)Bu(s)ds + \int_0^t T(t - s)f \left( s, \bar{z}_s + y_s, \int_0^s q(s, \tau, \bar{z}_\tau + y_\tau)d\tau \right)ds,$$

for $0 \leq t \leq a$ respectively. We will show that $P_1$ is a contraction mapping and $P_2$ is a compact operator.

To prove that $P_1$ is a contraction, we take $z_1, z_2 \in B_k$; then, for each $t \in J$ and by (A1 (iii)) and (H1), we have

$$\| (P_1 z_1)(t) - (P_1 z_2)(t) \| \leq \| g \left( t, \bar{z}_t + y_t, \int_0^t h(t, s, \bar{z}_s + y_s)ds \right) + \int_0^t AT(t - s)g \left( s, \bar{z}_s + y_s, \int_0^s h(s, \tau, \bar{z}_\tau + y_\tau)d\tau \right)ds \|$$

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for $0 \leq t \leq a$ respectively. We will show that $P_1$ is a contraction mapping and $P_2$ is a compact operator.
Thus

\[ ||P_1 z_1 - P_1 z_2|| \leq L^* ||z_1 - z_2|| \]

and so \( P_1 \) satisfies the contraction condition with \( L^* < 1 \).

To prove that \( P_2 \) is compact, first we prove that \( P_2 \) is continuous on \( B_k \). Let \( \{z_n\} \subseteq B_k \) with \( z_n \to z \) in \( B_k \); then for each \( s \in J, \tilde{z}_{n,s} \to \tilde{z}_s \) and by (H4 (i)), we have

\[
\int_0^s q(s, \tau, \tilde{z}_{n,\tau} + y_\tau) d\tau \to \int_0^s q(s, \tau, \tilde{z}_\tau + y_\tau) d\tau \quad \text{as} \quad n \to \infty.
\]

Since

\[
\|f \left( s, \tilde{z}_{n,s} + y_s, \int_0^s q(s, \tau, \tilde{z}_{n,\tau} + y_\tau) d\tau \right) - f \left( s, \tilde{z}_s + y_s, \int_0^s q(s, \tau, \tilde{z}_\tau + y_\tau) d\tau \right) \| \\
\leq 2\mu_k^*(s),
\]

then, by the continuity of \( A^0g \) and the dominated convergence theorem, we have

\[
\|P_2 z_n - P_2 z\| = \sup_{0 \leq t \leq a} \left\| \int_0^t T(t-s) B[u_n(s) - u(s)] ds \\
+ \int_0^t T(t-s) [f \left( s, \tilde{z}_{n,s} + y_s, \int_0^s q(s, \tau, \tilde{z}_{n,\tau} + y_\tau) d\tau \right) \\
- f \left( s, \tilde{z}_s + y_s, \int_0^s q(s, \tau, \tilde{z}_\tau + y_\tau) d\tau \right)] ds \right\| \\
\to 0, \quad \text{as} \quad n \to \infty,
\]

that is, \( P_2 \) is continuous.

Next we prove that the family \( \{P_2 z : z \in B_k\} \) is an equicontinuous family of functions. To do this, let \( \epsilon > 0 \) be small, \( 0 < t_1 < t_2 \leq a \), then

\[
|| (P_2z)(t_1) - (P_2z)(t_2) || \\
= \left\| \int_0^{t_1} T(t_1-s) B u(s) ds + \int_0^{t_1} T(t_1-s) f \left( s, \tilde{z}_s + y_s, \int_0^s q(s, \tau, \tilde{z}_\tau + y_\tau) d\tau \right) ds \\
- \int_0^{t_2} T(t_2-s) B u(s) ds - \int_0^{t_2} T(t_2-s) f \left( s, \tilde{z}_s + y_s, \int_0^s q(s, \tau, \tilde{z}_\tau + y_\tau) d\tau \right) ds \right\| \\
\leq \int_0^{t_1-t_2} \| T(t_1-s) - T(t_2-s) \| || B || || u(s) || ds
\]

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\[
+ \int_{t_1}^{t_1-\epsilon} ||T(t_1-s) - T(t_2-s)|| ||B|| ||u(s)|| ds \\
+ \int_{t_1}^{t_2} ||T(t_2-s)|| ||B|| ||u(s)|| ds \\
+ \int_{t_1-\epsilon}^{t_1} ||T(t_1-s) - T(t_2-s)|| ||f\left(s, \tilde{z}_s + y_s, \int_{0}^{s} q(s, \tau, \tilde{\tau} + y_\tau) d\tau\right)|| ds \\
+ \int_{t_1}^{t_1-\epsilon} ||T(t_1-s) - T(t_2-s)|| ||f\left(s, \tilde{z}_s + y_s, \int_{0}^{s} q(s, \tau, \tilde{\tau} + y_\tau) d\tau\right)|| ds \\
+ \int_{t_1}^{t_2} ||T(t_2-s)|| ||f\left(s, \tilde{z}_s + y_s, \int_{0}^{s} q(s, \tau, \tilde{\tau} + y_\tau) d\tau\right)|| ds.
\]

Observe that

\[
||u(s)|| \leq \left|\bar{W} \right|^{-1} \left\{ M \left| \phi(0) - g(0, \phi, 0) \right| + \right| A^{-\beta} \left| A^{\beta} g\left(a, \tilde{z}_a + y_a, \int_{0}^{a} h(a, s, \tilde{z}_s + y_s) ds\right) \right| + \int_{0}^{a} \left| A^{1-\beta} T(a-s) \right| \left| A^{\beta} g\left(s, \tilde{z}_s + y_s, \int_{0}^{s} h(s, \tau, \tilde{\tau} + y_\tau) d\tau\right) \right| ds \\
+ \int_{0}^{a} \left| T(a-s) \right| \left| f\left(s, \tilde{z}_s + y_s, \int_{0}^{s} q(s, \tau, \tilde{\tau} + y_\tau) d\tau\right) \right| ds \right\} \\
\leq \left|\bar{W} \right|^{-1} \left\{ M \left( \left| \phi(0) \right| + \left| g(0, \phi, 0) \right| \right) + M_1 \left\{ M \left( \left| \phi(0) \right| + \left| g(0, \phi, 0) \right| \right) + M_2 L_1 (1 + a M_2 L_3) [k K_a + N + 1] + \frac{1}{\beta} C_{1-\beta} a^{\beta} L_1 (1 + a M_2 L_3) [k K_a + N + 1] \right\} \right\}
\]

Hence

\[
\left| (P_2 z)(t_1) - (P_2 z)(t_2) \right| \\
\leq \int_{0}^{t_1-\epsilon} \left| T(t_1-s) - T(t_2-s) \right| M_1 \left\{ M \left( \left| \phi(0) \right| + \left| g(0, \phi, 0) \right| \right) + M_2 L_1 (1 + a M_2 L_3) [k K_a + N + 1] + \frac{1}{\beta} C_{1-\beta} a^{\beta} L_1 (1 + a M_2 L_3) [k K_a + N + 1] \right\} \left| \mu_k(s) \right| ds \\
+ \int_{t_1-\epsilon}^{t_1} \left| T(t_1-s) - T(t_2-s) \right| M_1 \left\{ M \left( \left| \phi(0) \right| + \left| g(0, \phi, 0) \right| \right) + M_2 L_1 (1 + a M_2 L_3) [k K_a + N + 1] + \frac{1}{\beta} C_{1-\beta} a^{\beta} L_1 (1 + a M_2 L_3) [k K_a + N + 1] \right\} \left| \mu_k(s) \right| ds \\
+ \int_{t_1}^{t_2} \left| T(t_2-s) \right| M_1 \left\{ M \left( \left| \phi(0) \right| + \left| g(0, \phi, 0) \right| \right) + M_2 L_1 (1 + a M_2 L_3) [k K_a + N + 1] + \frac{1}{\beta} C_{1-\beta} a^{\beta} L_1 (1 + a M_2 L_3) [k K_a + N + 1] \right\} \left| \mu_k(s) \right| ds.
\]

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\begin{align*}
+ M \int_0^a \mu_{k^*}(s) ds \right] ds \\
+ \int_0^{t_1-\epsilon} ||T(t_1-s) - T(t_2-s)|| \mu_{k^*}(s) ds \\
+ \int_{t_1-\epsilon}^{t_1} ||T(t_1-s) - T(t_2-s)|| \mu_{k^*}(s) ds + \int_{t_1}^{t_2} ||T(t_2-s)|| \mu_{k^*}(s) ds,
\end{align*}

where $\mu_{k^*}(s) \in L^1$, we see that $||(P_2z)(t_1) - (P_2z)(t_2)||$ tends to zero independent of $z \in B_k$ as $t_1 \to t_2$ with $\epsilon$ sufficiently small, since the assumption on $T(t)$ implies the continuity of $T(t)$ in $t$ in the uniform operator topology. Hence $P_2$ maps $B_k$ into an equicontinuous family of functions.

It remains to prove that $V(t) = \{(P_2z)(t) : z \in B_k\}$ is relatively compact in $X$. Let $0 < t \leq a$ be fixed, $0 < \epsilon < t$, then for $z \in B_k$, we define

\begin{align*}
(P_{2,\epsilon}z)(t) &= \int_0^{t-\epsilon} T(t-s) \left[ Bu(s) + f \left( s, \bar{z}_s + y_s, \int_0^s q(s, \tau, \bar{z}_\tau + y_\tau) d\tau \right) \right] ds \\
&= T(\epsilon) \int_0^{t-\epsilon} T(t-s-\epsilon) \left[ Bu(s) + f \left( s, \bar{z}_s + y_s, \int_0^s q(s, \tau, \bar{z}_\tau + y_\tau) d\tau \right) \right] ds.
\end{align*}

Using the estimation of $||u(s)||$ and by the assumption on $T(t)$, we prove $V_\epsilon(t) = \{(P_{2,\epsilon}z)(t) : z \in B_k\}$ is relatively compact in $X$ for every $\epsilon, 0 < \epsilon < t$. Moreover, for every $z \in B_k$, we have

\begin{align*}
|| (P_2z)(t) - (P_{2,\epsilon}z)(t) || \\
&\leq \int_0^t ||T(t-s)\left[ Bu(s) + f \left( s, \bar{z}_s + y_s, \int_0^s q(s, \tau, \bar{z}_\tau + y_\tau) d\tau \right) \right] || ds \\
&\leq \int_0^t M \left\{ M_1 \left[ M(||f(0)|| + ||g(0, \phi, 0)||) + M_2 L_1 (1 + a M_2 L_3) [k K_a + N + 1] \\
+ \frac{1}{\rho} C_1 \beta a^\beta L_1 (1 + a M_2 L_3) [k K_a + N + 1] \\
+ M \int_0^a \mu_{k^*}(s) ds \right] + \mu_{k^*}(s) \right\} ds.
\end{align*}

Therefore there are relatively compact sets arbitrarily close to the set $V(t) = \{(P_2z)(t) : z \in B_k\}$; hence the set $V(t)$ is also relatively compact in $X$.

Thus, by the Arzela-Ascoli theorem, $P_2$ is a compact operator. These arguments show that $P = P_1 + P_2$ is a condensing mapping on $B_k$ and by the Sadovskii fixed point theorem [83], there exists a fixed point $z(\cdot)$ for $P$ on $B_k$. If we define $x(t) = z(t) + y(t), -\infty < t \leq a$, then it is easy to see that $x(\cdot)$ is a mild solution of (2.1) satisfying $x_0 = \phi, x(a) = 0$. Hence the proof.
Now we discuss the local null controllability of the system (2.1). For this purpose, we impose the weaker assumptions on the system (2.1) as follows.

\begin{enumerate}
    \item[(H7)] $g : J \times \Omega \times X \to X$ is a continuous function and there exist constant $\beta \in (0,1)$ and $L > 0$ such that the function $g$ is $X_\beta$-valued and satisfies the Lipschitz condition:
    \[
    \|A^\beta g(s_1, \phi_1, \eta_1) - A^\beta g(s_2, \phi_2, \eta_2)\| \leq L[s_1 - s_2] + \|\phi_1 - \phi_2\|_B + \|\eta_1 - \eta_2\|_X,
    \]
    for $0 \leq s_1, s_2 \leq a$.
    \item[(H8)] The function $f : J \times \Omega \times X \to X$ satisfies the following conditions:
        \begin{enumerate}
            \item For each $t \in J$, the function $f(t, \cdot, \cdot) : \Omega \times X \to X$ is continuous and for each $(\phi, x) \in \Omega \times X$ the function $f(\cdot, \phi, x) : J \to X$ is strongly measurable.
            \item For each positive integer $k$, there is a positive function $\mu_k \in L^1(0, a)$ such that
                \[
                \sup_{\|x\|_B, \|y\| \leq k} \|f(t, x, y)\| \leq \mu_k(t),
                \]
                where $\Omega \subseteq B$ is an open set.
        \end{enumerate}
\end{enumerate}

**Theorem 2.2.** If the assumptions (H5),(H7) and (H8) are satisfied and $\phi \in \Omega$, then the system (2.1) is locally null controllable on the interval $J$.

**Proof.** We prove this theorem by using again the Sadovskii fixed point theorem. Let $y(\cdot), u(\cdot), S, P, P_1, P_2$ be as in the Proof of Theorem 2.1. It is enough to prove that $P$ has a fixed point which implies that $S$ has a fixed point.

Since $A^\beta g$ is continuous and $\Omega$ is open, there exist $0 < b_1 < b, r > 0$ such that $B_r(\phi) \subseteq \Omega$ and
\[
\|A^\beta(t)g(t, \psi, \eta)\| \leq C_1
\]
for some constant $C_1 \geq 0$ and $(t, \psi, \eta) \in [0, b_1] \times B_r(\phi) \times X$. As $y_0 = \phi$, we choose $0 < b_2 < b_1$ such that $\|y_t - \phi\|_B \leq r/2$ for all $0 \leq t \leq b_2$. 

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Let \( r = \frac{r}{2Kb_2} \) and \( \epsilon > 0 \); then from the continuity of the functions \( g, K(t) \) and \( t \to y_t \), the compactness of the set \( \{ g(t, y_t, \int_0^t h(t, s, y_s)ds) : 0 \leq t \leq b_2 \} \) and the absolute continuity of Lebesgue integral, it follows that there exists a constant \( a, 0 < a < b_2 \) for which the followings hold:

\[
\| y_t - \phi \|_B \leq \epsilon, \tag{2.8}
\]

\[
\|(T(t) + I)g(t, y_t, \int_0^t h(t, s, y_s)ds)\| \leq \epsilon, \tag{2.9}
\]

\[
(aMM_1 + 1)[M_2LK_a(1 + M_2L_2)] \leq 1, \tag{2.10}
\]

\[
(aMM_1 + 1)\{M_2LM(a + \epsilon) + aMM_2L_2L_3(N + 1) + \frac{1}{\beta}C_{1-\beta}a^\beta C_1 + \epsilon\} \leq \frac{1}{3}[1 - (aMM_1 + 1)\{M_2LK_a(1 + M_2L_2)\}]\rho, \tag{2.11}
\]

\[
(aMM_1 + 1)M \int_0^a \mu_{k^*}(s)ds \leq \frac{1}{3}[1 - (aMM_1 + 1) \times \{M_2LK_a(1 + M_2L_2)\}]\rho, \tag{2.12}
\]

\[
aMM_1(M||\phi(0)||) \leq \frac{1}{3}[1 - (aMM_1 + 1)\{M_2LK_a(1 + M_2L_2)\}]\rho, \tag{2.13}
\]

\[
[M_2LK_a + \frac{C_{1-\beta}}{\beta}a^\beta LK_a][1 + aM_2L_2] < 1, \tag{2.14}
\]

for all \( 0 \leq t \leq a \).

Define the set

\[
S(\rho) = \{ z \in C(J; X) : z(0) = 0, \|z(t)\| \leq \rho, 0 \leq t \leq a \}.
\]

Then \( S(\rho) \) is also a non-empty bounded, closed and convex subset of \( C(J; X) \) and \( P \) is well defined on \( S(\rho) \). We will show that \( P \) maps \( S(\rho) \) into \( S(\rho) \). In fact, for \( z \in S(\rho) \), we have

\[
(P_1z)(t) = -T(t)g(0, \phi, 0) + g\left(t, \bar{z}_t + y_t, \int_0^t h(t, s, \bar{z}_s + y_s)ds\right) + \int_0^t AT(t-s)g\left(s, \bar{z}_s + y_s, \int_0^s h(s, \tau, \bar{z}_\tau + y_\tau)\tau \right)ds
\]

\[
= A^{-\beta}T(t)\left[-A^\beta g(0, \phi, 0) - A^\beta g\left(t, y_t, \int_0^t h(t, s, y_s)ds\right)\right] + (T(t) + I)g\left(t, y_t, \int_0^t h(t, s, y_s)ds\right)
\]

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\[-A^{-\beta}[A^\beta g \left(t, y_t, \int_0^t h(t, s, y_s)ds\right)]
\quad + \int_0^t A^{1-\beta} T(t - s) A^\beta g \left(s, \bar{z}_s + y_s, \int_0^s h(s, \tau, \bar{z}_\tau + y_\tau) d\tau\right) ds.\]

Then from conditions (H7) and (2.8) and (2.9), it yields that

\[\|Pz(t)\| \leq \|A^{-\beta}\|\|T(t)\|L[t + \|y_t - \phi\|_B + \|\int_0^t h(t, s, y_s)ds\|_X]
\quad + \|T(t) + I\|g \left(t, y_t, \int_0^t h(t, s, y_s)ds\right)\|
\quad + \|A^{-\beta}\|L[\|\bar{z}_t\|_B + \|\int_0^t h(t, s, \bar{z}_s + y_s)ds - \int_0^t h(t, s, y_s)ds\|_X]
\quad + \int_0^t \frac{C_{1-\beta}}{(t - s)^{1-\beta}} C_1 ds
\leq M_2 M L[a + \epsilon + aM_2 L_3 (N + 1)] + \epsilon + M_2 L[\rho K_a + M_2 L_2 \rho K_a]
\quad + \frac{1}{\beta} C_{1-\beta} a^\beta C_1
\leq M_2 L [M(a + \epsilon + aM_2 L_3 (N + 1)) + \rho K_a + M_2 L_2 \rho K_a]
\quad + \frac{1}{\beta} C_{1-\beta} a^\beta C_1 + \epsilon.\]

Thus by (2.11) to (2.13), we derive that

\[\|Pz(t)\| = \| - T(t)g(0, \phi, 0) + g \left(t, \bar{z}_{k,t} + y_t, \int_0^t h(t, s, \bar{z}_{k,s} + y_s)ds\right)
\quad + \int_0^t A T(t - s) g \left(s, \bar{z}_{k,s} + y_s, \int_0^s h(s, \tau, \bar{z}_{k,\tau} + y_\tau) d\tau\right) ds
\quad + \int_0^t T(t - s) BW^{-1}\left\{-T(a)[\phi(0) - g(0, \phi, 0)]\right\}
\quad - g \left(a, \bar{z}_{k,a} + y_a, \int_0^a h(a, s, \bar{z}_{k,s} + y_s)ds\right)
\quad - \int_0^a A T(a - \tau) g \left(\tau, \bar{z}_{k,\tau} + y_\tau, \int_0^\tau h(\tau, \eta, \bar{z}_{k,\eta} + y_\eta) d\eta\right) d\tau
\quad - \int_0^a T(a - \tau)f \left(\tau, \bar{z}_{k,\tau} + y_\tau, \int_0^\tau q(\tau, \eta, \bar{z}_{k,\eta} + y_\eta) d\eta\right) d\tau\{s)ds
\quad + \int_0^t T(t - s)f \left(s, \bar{z}_{k,s} + y_s, \int_0^s q(s, \tau, \bar{z}_{k,\tau} + y_\tau) d\tau\right) ds\|
\leq (aM M_1 + 1)\{M_2 L [M(a + \epsilon + aM_2 L_3 N) + \rho K_a + M_2 L_2 \rho K_a]
\quad + \frac{1}{\beta} C_{1-\beta} a^\beta C_1 + \epsilon\} + (aM M_1 + 1)M \int_0^a \mu_{k^*}(s)ds
\[+ aMM_1(M\|\phi(0))\]
\[\leq [1 - (aMM_1 + 1)[M_2LK_a(1 + M_2L_2)]]\]
\[+ (aMM_1 + 1)[M_2L_2K_a(1 + M_2L_2)]\]
\[\leq \rho.\]

The remaining part of the proof is similar to that of Theorem 2.1 and hence it is omitted.

2.4. EXAMPLE

Consider the following partial functional differential system
\[
\frac{\partial}{\partial t} \left[ z(t, x) + \int_0^\pi b(y, x)z(t \sin t, y)dy \right] = \frac{\partial^2}{\partial x^2} z(t, x) + \mu(t, x) + \int_0^t h(t, s, z_s)ds, \quad 0 \leq t \leq a, \quad 0 \leq x \leq \pi \tag{2.15}
\]
\[z(t, 0) = z(t, \pi) = 0,
\]
\[z(0, x) = z_0(x), \quad 0 \leq x \leq \pi,
\]
where \(a \leq \pi, z_0(x) \in X = L^2([0, \pi]).\)

To write system (2.15) in the form of (2.1), let \(X = L^2([0, \pi])\) and \(A\) be defined by
\[Af = -f''
\]
with the domain
\[D(A) = \{f(\cdot) \in X : f, f' \text{ absolutely continuous, } f'' \in X, f(0) = f(\pi) = 0\}
\]
Then \(A\) generates a strongly continuous semigroup \(T(t)\) which is analytic and self-adjoint. Furthermore, \(A\) has a discrete spectrum, the eigen values are \(n^2, n \in N,\) with the corresponding normalized eigenvectors \(z_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx).\) Then the following properties hold:

(a) If \(f \in D(A),\) then
\[Af = \sum_{n=1}^\infty n^2 < f, z_n > z_n.
\]
(b) For each \( f \in X \), \( A^{-1/2}f = \sum_{n=1}^{\infty} \frac{1}{n} \langle f, z_n \rangle z_n \). In particular, \( \|A^{-1/2}\| = 1 \).

(c) The operator \( A^{1/2} \) is given by

\[
A^{1/2}f = \sum_{n=1}^{\infty} n \langle f, z_n \rangle z_n
\]

on the space \( D(A^{1/2}) = \{f(\cdot) \in X, \sum_{n=1}^{\infty} \langle f, z_n \rangle z_n \in X \} \).

Assume that the following conditions hold:

(i) The function \( b \) is measurable and

\[
\int_{0}^{\pi} \int_{0}^{\pi} b^2(y, x) \, dy \, dx < \infty.
\]

(ii) The function \( \frac{\partial}{\partial x} b(y, x) \) is measurable, \( b(y, 0) = b(y, \pi) = 0 \) and let

\[
N_1 = \left[ \int_{0}^{\pi} \int_{0}^{\pi} \left( \frac{\partial}{\partial x} b(y, x) \right)^2 \, dy \, dx \right]^{1/2} < \infty.
\]

(iii) For the function \( h : [0, a] \times [0, a] \times B \to X \), the following conditions are satisfied:

(iii.a) For each \( t, s \in [0, a] \), \( h(t, s, \cdot) \) is continuous.

(iii.b) For each \( z \in X \), \( h(\cdot, \cdot, z_s) \) is measurable.

(iii.c) There are positive functions \( h_1, h_2 \in L^1([0, a]) \) such that

\[
|h(t, s, \phi)| \leq h_1(t)|\phi| + h_2(s)
\]

for all \((t, s, \phi) \in [0, a] \times [0, a] \times B\).

We define \( f, g : J \times B \times X \to X \) by \( f(t, z_t, z) = Z_1(z), g(t, z_t, z)(x) = \int_{0}^{t} h(t, s, z_s) \, ds \) respectively, where

\[
Z_1(z)(x) = \int_{0}^{\pi} b(y, x) z(y) \, dy.
\]
From (i), it is clear that $Z_1$ is a bounded linear operator on $X$. Furthermore, $Z_1(z) \in D[A^{1/2}]$ and $\|A^{1/2}Z_1\| \leq N_1$. In fact from the definition of $Z_1$ and (ii), it follows that

$$< Z_1(z), z_n > = \int_0^\pi z_n(x) \left[ \int_0^\pi b(y, x)z(y)dy \right] dx$$

$$= \frac{1}{n} \left( \frac{2}{\pi} \right)^{1/2} < Z(z), \cos(nx) >,$$

where $Z$ is defined by

$$Z(z)(x) = \int_0^\pi \frac{\partial}{\partial x} b(y, x)z(y)dy$$

From (ii), we know that $Z : X \to X$ is a bounded linear operator with $\|Z\| \leq N_1$. Hence $\|A^{1/2}Z_1(z)\| = \|Z(z)\|$ which implies the assertion.

Let $u : J \to U \subset X$ be defined by

$$(u(t))(x) = \mu(x, t), \quad x \in [0, \pi],$$

where $\mu : [0, \pi] \times J \to [0, \pi]$ is continuous.

Assume that there exists a bounded inverse operator $W^{-1} : X \to L^2(J, U)/\ker W$ such that

$$Wu = \int_0^a T(a - s)Bu(s)ds$$

Hence, from Theorem 2.1, the system (2.15) is null controllable on $J$.

**Remark:** The results established in this chapter generalise the result of [36].

\[\star\star\star\star\\]