I. INTRODUCTION

1.1. NEUTRAL INTEGRODIFFERENTIAL SYSTEMS

Neutral differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A neutral functional differential equation is one in which the derivatives of the past history or derivatives of functionals of the past history are involved as well as the present state of the system. Neutral differential equations occur in the study of heat conduction in materials with memory, theory of lossless transmission lines and many other physical phenomena. A large class of scientific and engineering problems modelled by partial differential equation or partial integrodifferential equations can be expressed in various forms of differential or integrodifferential equations in abstract spaces. In general, neutral integrodifferential equations serve as an abstract formulation of many partial neutral integrodifferential equations which arise in the applications of the theory of population dynamics, compartmental systems, viscoelasticity and many other fields of science. So it becomes important to study the controllability problem for such systems in Banach spaces.

The theory of functional differential equation with unbounded delay has been studied by several authors. Almost all the work deals with the Cauchy problem

\[ x'(t) = F(t, x_t), \quad t \geq \sigma, \tag{1.1} \]

\[ x_{\sigma} = \varphi, \]

where \( x_t \) represents the "history" of \( x \) at \( t \), the values \( x(t) \) belong to some finite dimensional space and \( F \) is a function usually continuous on appropriate spaces. Nevertheless this class of equations does not include partial integrodifferential equations with infinite delay which arise, for example, in the study of heat conduction in materials with memory or population dynamics for spatially distributed populations. Besides it is well known that the behavior of the first and second order abstract Cauchy problems is different in many aspects. For these reasons, there has been an increasing interest in studying equations that can be described in the form

\[ x'(t) = Ax(t) + F(t, x_t), \quad t \geq \sigma, \tag{1.2} \]
where $A$ is the infinitesimal generator of a strongly continuous semigroup of linear operators on a Banach space $X$. We call these equations as abstract retarded functional differential equations.

Similarly there exists an extensive theory for ordinary neutral functional differential equations which includes qualitative behavior of classes of such equations and applications to biological and engineering processes. However for partial neutral functional differential equations, very little is known [57,58]. Here we study the controllability of equations that can be modelled in the form

$$\frac{d}{dt}(x(t) + F(t,x_t)) = Ax(t) + G(t,x_t), \quad t \geq \sigma,$$

where the initial conditions $x_\sigma$ and $F$ and $G$ are appropriate functions. These equations will be called abstract neutral functional differential equations with unbounded delay.

Many problems arise from control systems described by an abstract retarded functional differential equation with a feed back control governed by a proportional integrodifferential law [3,61]. On the other hand some abstract retarded functional differential equations can be conveniently transformed into an abstract neutral functional differential equations. Consider the equation (1.2) with

$$F(\varphi) = \int_{-\infty}^{0} C(-\theta)\varphi(\theta)d\theta,$$

where $C$ is a strongly continuous map of continuous operators from $X$ into $X$. Assume that we can decompose $C(s) = L(s) + N(s)$ where $L$ and $N$ are also strongly continuous maps of continuous operators and further $L(s)$ is linear. We define the operator $V(t)$ by

$$V(t)x = \int_{0}^{t} L(s)xds.$$

Then (1.2) can be transformed into an abstract neutral functional differential equation

$$\frac{d}{dt}[x(t) + \int_{-\infty}^{t} V(t-s)x(s)ds] = Ax(t) + \int_{-\infty}^{t} N(t-s)x(s)ds$$

which has the form (1.3) and in some cases, depending on $V$ and $N$, it is easier to be treated than the original equation. Motivation for neutral functional differential equations can be found in [10,25,30,52-56,93].
1.2. MOTIVATION

The object of this thesis is to study the controllability of various types of nonlinear neutral integrodifferential systems. We shall motivate our study by giving the occurrence of these equations in different fields of science.

A. Theory of Lossless Transmission Line

We consider the lossless transmission line connected as shown in the Figure 1.1 where $g(v)$ is a nonlinear function of $v$ and gives the current in the indicated box in the direction shown.

This problem may be described by the following system of partial differential equations

$$ L \frac{\partial i}{\partial t} = -\frac{\partial v}{\partial x}, \quad C \frac{\partial v}{\partial t} = -\frac{\partial i}{\partial x}, \quad 0 < x < 1, \quad t > 0 $$

with the boundary conditions,

$$ E - v(0,t) - R i(0,t) = 0, \quad C_1 \frac{dv(1,t)}{dt} = i(1,t) - g(v(1,t)). $$

We now indicate how one can transform this problem into a differential equation with delays. If $s = (LC)^{-1/2}$ and $z = (LC)^{1/2}$, then the general solution of
the partial differential equations is
\[ v(x, t) = \phi(x - st) + \psi(x + st) \]
\[ i(x, t) = \frac{1}{z} [\phi(x - st) - \psi(x + st)] \]
or
\[ 2\phi(x - st) = v(x, t) + zi(x, t) \]
\[ 2\psi(x + st) = v(x, t) - zi(x, t). \]

This implies
\[ 2\phi(-st) = v\left(1, t + \frac{1}{s}\right) + zi\left(1, t + \frac{1}{s}\right) \]
\[ 2\psi(st) = v\left(1, t - \frac{1}{s}\right) - zi\left(1, t - \frac{1}{s}\right). \]

Using these expressions in the general solution and using the first boundary condition at \( t = (1/s) \), one obtains
\[ i(1, t) - Ki\left(1, t - \frac{2}{s}\right) = \alpha - \frac{1}{z} v(1, t) - \frac{K}{z} v\left(1, t - \frac{2}{s}\right) \]
where \( K = (z - R)/(z + R), \alpha = 2E/(z + R) \). Inserting the second boundary condition and letting \( u(t) = v(1, t) \), we obtain the equation
\[ \dot{u}(t) - Ku\left(t - \frac{2}{s}\right) = f(u(t), u\left(t - \frac{2}{s}\right)) \]
where \( s = \sqrt{LC} \),
\[ C_1 f(u(t), u(t - r)) = \alpha - \frac{1}{z} u(t) - \frac{K}{z} u(t - r) - g(u(t)) + Kg(u(t - r)), \]
where all constants are positive and depend on the parameters in the original equations. Also if \( R > 0 \), then \( K < 1 \).

If generalized solutions of the original partial differential equations were considered, the delay equation would require differentiating the difference \( u(t) - Ku(t - (2/s)) \) rather than each term separately; that is, one would consider the equation
\[ \frac{d}{dt} \left[ u(t) - Ku\left(t - \frac{2}{s}\right) \right] = f(u(t), u\left(t - \frac{2}{s}\right)). \]
The above example has amply illustrated the importance and frequency of occurrence of equations which depend upon past history. The diversity of the different types of equations makes it seem at first glance to be almost impossible to find a class of equations which contains all of these and is still mathematically tractable and interesting.

B. Compartamental Systems

Compartmental models are frequently used in theoretical epidemiology, physiology and population dynamics to describe the evolution of systems which can be divided into separate compartments, marking the pathways of material flow between compartments and the possible outflow into and the inflow from the environment of the system. Generally the time required for the material flow between compartments cannot be neglected. A model for such a system can be visualized as one in which compartments are connected by pipes through which material passes in definite time. Because of the time lags caused by pipes, the model equations for such systems are differential equations with deviating arguments, as opposed to the classical case where model equations are ordinary differential equations. A concrete example is the radio cardiogram where the two compartments correspond to the left and right ventricles of the heart and the pipes between them represent the pulmonary and systematic circulation. Pipes coming out from and returning into the same compartment may represent shunts and the coronary circulation (see [44]). By letting \( C_1, C_2, \ldots, C_n \) to be the compartments of a compartamental system, \( x_i(t) \) the amount of the material in compartment \( C_i \) at time \( t \) and \( C_0 \) the environment of the compartamental system and with additional assumptions Gyori and Wu [45] derived the following functional differential equation as a model of the compartamental system

\[
\frac{d}{dt} \left[ x_i(t) - \int_0^\infty S_i(t - u, x_i(t - u)dG_i(u) \right] \\
= -\sum_{j=0}^n g_{ji}(t, x_i(t)) + \sum_{j=1}^n \int_0^\infty g_{ij}(t - s, x_j(t - s))dF_{ij}(s) + h_i(t), \quad (1.4)
\]

for all \( t \geq 0 \), where \( g_{ij} \) is transport function for \( j = 0, 1, \ldots, n \) and \( i = 1, 2, \ldots, n \); \( F_{ji}(t, s), \ i = 1, 2, \ldots, n, \ j = 0, 1, \ldots, n, \ t \geq 0, \ s \geq 0 \) is transit time distribution function of a pipe and \( S_i : [0, \infty) \times [0, \infty) \to R, G_i : [0, \infty) \to R \) and \( h_i : [0, \infty) \to R \) are given continuous functions.
C. Thermodynamics

(i) A very special model for one dimensional heat flow in materials with memory is the partial functional delay integrodifferential equation of the form

\[ w_t(x, t) = w_{xx}(x, t) + \int_0^t f(s, w(x, s - r))ds, \quad 0 < x < 1, \quad t > 0, \]
\[ w(0, t) = w(1, t) = 0, \quad t > 0, \]
\[ w(x, t) = \phi(x, t), \quad -r \leq t \leq 0, \]

and it can be written in the abstract form as

\[ x'(t) = Ax(t) + \int_0^t f(s, x_s)ds, \quad 0 \leq t \leq b, \quad (1.5) \]
\[ x(t) = \phi(t), \quad -r \leq t \leq 0, \]

where \( A \) is the infinitesimal generator of a strongly continuous semigroup in a Banach space \( X \), \( f \) is a continuous function and \( \phi \) is an initial function.

(ii) Consider the partial integrodifferential equation of the form

\[ w_t(x, t) = w_{xx}(x, t) + \int_0^t k(t - s, w(x, s))ds + h(x, t), \quad 0 < x < 1, \quad t > 0, \]
\[ w(0, t) = w(1, t) = 0, \quad t > 0, \]
\[ w(x, 0) = w_0(x), \quad 0 < x < 1, \]

where \( k \) and \( h \) are continuous functions.

The above equation has many physical applications and occurs in the study of dynamic behavior of heat equations and it can be abstractly formulated as

\[ x'(t) = Ax(t) + \int_0^t g(t - s, x(s))ds + f(t), \quad t \geq 0, \quad (1.6) \]
\[ x(0) = x_0 \in X, \]

where \( A \) is the infinitesimal generator of a strongly continuous semigroup in a Banach space \( X \), \( g \) is a nonlinear operator and \( f \) is a continuous function.

D. Heat Conduction

Consider the model for the heat conduction in materials with memory [66,67] which can be represented as a partial functional neutral integrodifferential equa-
tion of the form
\[
\frac{\partial}{\partial t} \left[ z(t, \xi) + \int_{-\infty}^{t} \int_{0}^{\pi} b(s - t, \eta, \xi)z(s, \eta)d\eta ds \right]
\]
\[
= \frac{\partial^2}{\partial \xi^2} z(t, \xi) + a_0(\xi)z(t, \xi) + \int_{-\infty}^{t} a(s - t)z(s, \xi)ds + a_1(t, \xi),
\]
\[t \geq 0, \ 0 \leq \xi \leq \pi,
\]
\[
z(t, 0) = z(t, \pi) = 0, \ t \geq 0,
\]
\[
z(\theta, \xi) = \phi(\theta, \xi), \ \theta \leq 0, \ 0 \leq \xi \leq \pi,
\]
where the functions \(a_0, a_1, b\) and \(\phi\) satisfy appropriate conditions.

The above equation arises from control systems described by abstract retarded functional differential equations [57] with a feedback control governed by a proportional integrodifferential law and it can be written as
\[
\frac{d}{dt}[x(t) + h(t, x_t)] = Ax(t) + f(t, x_t), \ t \geq \sigma,
\]
\[
x_\sigma = \phi,
\]
where the initial function \(\phi\) takes values in some approximated phase space and \(h, f\) are nonlinear functions.

E. Population Dynamics

Consider the following population model with time delay proposed by Rey and Mackey [82]
\[
\frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x}[xu(x, t)] = \mu u(\alpha x, t - \tau)(1 - u(\alpha x, t - \tau)), t > 0
\]
\[
u(x, t) = \phi(x, t), \ -\tau \leq t \leq 0, \ x \in [0, 1]
\]
where \(u(x, t)\) is the population density of cells with respect to \(x\) at time \(t\) and \(\mu, \alpha, \tau\) are parameters satisfying \(\mu > 0, 0 < \alpha < 1, \tau > 0\). The variable \(x\) has value in \([0, 1]\) and may relate to the intracellular haemoglobin content of individual cells. The term \(\frac{\partial}{\partial x}[g(x)u(x, t)]\) assumes that all cells have maturation rate \(g(x) = x\). The time delay \(\tau\) and maturity displacement \(\alpha\) arise when we assume that all cells divide at exactly the same age.

The abstract form of the above equation is the following semilinear functional differential equation in the Banach space \(X = C([0, 1]):\)
\[
\frac{du(t)}{dt} = Atu(t) + F(u_t), \ t \geq 0,
\]
\[ u(t) = \phi(t), \quad -\tau \leq t \leq 0, \quad (1.9) \]

where \( A_T \) is defined by

\[
D(A_T) = \{ w \in C([0, 1]; xw(x) \in C^1([0, 1])) \},
\]

\[
(A_T)(x) = -(xw(x))'
\]

and \( F: C([-\tau, 0]; X) \to X \) is the mapping given by

\[ F(w)(x) = \mu w(\alpha x, -\tau)(1 - w(\alpha x, -\tau)). \]

Here \( A_T \) is the generator of a \( C_0 \) semigroup of bounded operators in \( X \) and \( F \) is locally Lipschitz continuous. The existence, regularity, invariance and asymptotic behaviors of solutions are studied by Dyson et al [32].

F. Viscoelasticity

Abstract functional differential equations of neutral type also originate in the theory of viscoelastic materials. In Desch et al [31], it was illustrated that the equation

\[
\dot{u}(t) = A_T[u(t) + \int_{-\infty}^{t} F(t - s)u(s)ds] + \int_{-\infty}^{t} K(t - s)u(s)ds, \quad t \geq 0
\]

\[ (1.10) \]

can be regarded as the abstract formulation of the models proposed and studied respectively by Coleman and Gurtin [21], Gurtin and Pipkin [43] and Miller [71] for the heat flow in a rigid, isotropic, homogeneous material, the model of anti-plane strain for an isotropic viscoelastic material in the elastic case, and a model in thermo viscoelasticity studied by Leugering [63]. We can show that if \( F \) is differentiable, then the above equation can be transformed into the abstract neutral functional differential equation with infinite delay

\[
\frac{d}{dt} \left[ u(t) + \int_{-\infty}^{t} F(t - s)u(s)ds \right] = A_T[u(t) + \int_{-\infty}^{t} F(t - s)u(s)ds] + F(0)u(t) + \int_{-\infty}^{t} [K(t - s) + \dot{F}(t - s)]u(s)ds
\]

\[
\frac{d}{dt} \left[ u(t) + \int_{-\infty}^{t} F(t - s)u(s)ds \right] = A_T[u(t) + \int_{-\infty}^{t} F(t - s)u(s)ds] + F(0)u(t) + \int_{-\infty}^{t} [K(t - s) + \dot{F}(t - s)]u(s)ds
\]
1.3. PROBLEM AND METHOD

Theory of semigroups of bounded linear operators developed quite rapidly since the discovery of the generation theorem by Hille and Yosida in 1948. By now, it is an extensive mathematical subject with substantial applications to many fields of analysis. The theory of semigroups of bounded linear operators is closely related to the solution of differential equations in Banach spaces [42]. In recent years, this theory has been applied to a large class of integrodifferential equations in Banach spaces. Using the method of semigroups, existence and uniqueness of mild, strong and classical solutions of evolution equations have been discussed by Pazy [78]. Many problems in the fields of ordinary and partial differential equations can be recast as integral equations. Several existence and uniqueness results can be derived from the corresponding results of integral equations. Such results can be obtained by applying the suitable fixed-point theorems.

Of all the methods, the fixed point method is the most effective one to study the controllability of nonlinear systems. In this method the problem is transformed to a fixed point problem for an appropriate nonlinear operator in a function space. An essential part of this approach is to guarantee the existence of an invariant subset for this operator. Moreover, by using the fixed point theorems, one can obtain controllability conditions in the Banach space of continuous functions.

Fixed point method is used to prove the existence theorems for integrodifferential equations and studying the controllability problem for integrodifferential systems. Due to its importance, several researchers have extended the technique to infinite dimensional spaces and studied the existence and controllability problems for nonlinear evolution equations in abstract spaces. Schauder’s fixed point theorem is helpful in asserting the existence of solutions of integrodifferential equations. The Banach fixed point theorem is an important source of existence and uniqueness theorem in different branches of analysis. Recently, the Schaefer fixed point theorem is also used to prove the existence of solutions of various types of differential equations. Further they are applied to investigate the controllability problem for nonlinear systems in Banach spaces.
1.4. CONTROLLABILITY OF NEUTRAL INTEGRODIFFERENTIAL SYSTEMS

Controllability is one of the most important qualitative behaviour of a dynamical system. The problem of controllability is concerned with the question of existence of a control function which steers the solution of the system from its initial state to a final state in finite time. The concept of controllability plays a major role in finite dimensional control theory so it is natural to try to generalize this to infinite dimensions. The controllability problem for nonlinear dynamical systems can be studied by different methods depending on the type of nonlinearity in the state equation. For continuous time-invariant linear systems in finite-dimensional space, the concepts of controllability and reachability are equivalent and they are independent of the time. But in infinite dimensional spaces, the situation is more complex and many different types of controllability and reachability have been studied in the literature.


Many control systems arising from realistic models heavily depend on infinite delay [52,58]; so there is an increasing interest to study the controllability of partial functional differential and integrodifferential systems with infinite delay. By using the Schaefer's fixed point theorem Wang and Wang [91] studied the neutral functional differential systems with infinite delay by where as Balachandran et al [14] discussed the same problem for neutral functional integrodifferential systems in Banach spaces. Balachandran and Anandhi [8] obtained the control-
lability results for neutral functional integrodifferential infinite delay systems in Banach spaces with the help of Nussbaum's fixed point theorem. Recently Liu [64] discussed the controllability of nonlinear neutral evolution integrodifferential systems with infinite delay with the help of resolvent operators and the Schaefer fixed point theorem.

The present work deals with the study of controllability of various types of neutral integrodifferential systems in Banach spaces by using the analytic semigroup theory and the fixed point theorems due to Nussbaum [75], Sadovskii [83] and Schaefer [87].

1.5. CONTRIBUTIONS OF THE AUTHOR

In the light of the above, the author has obtained some significant results on the following topics:

1. Null controllability of neutral functional integrodifferential systems.

2. Null controllability of neutral evolution integrodifferential systems.

3. Controllability of neutral functional evolution integrodifferential systems with infinite delay.

4. Controllability of neutral evolution integrodifferential systems.

5. Controllability of neutral integrodifferential systems with time varying delays.

6. Boundary controllability of neutral integrodifferential systems.

The rest of the thesis contains a detailed account of the above topics.

The present thesis generalise the results of [36] (in chapter 2), [37,38] (in chapter 3), [8,36,91] (in chapter 4), [64,85,86] (in chapter 5), [14,40] (in chapter 6) and [9] (in chapter 7). Further it covers a larger class of dynamical systems represented by neutral integrodifferential equations when compared with others work.