CHAPTER VI
VI. CONTROLLABILITY OF NEUTRAL INTEGRODIFFERENTIAL SYSTEMS WITH TIME VARYING DELAYS

6.1. INTRODUCTION

Several authors have studied the existence problem for various types of differential and neutral differential equations in Banach spaces [5, 29, 34, 41, 47, 50, 51, 60, 65, 72-74, 81, 89, 90]. Using the method of semigroups, existence and uniqueness of mild, strong and classical solutions of semilinear evolution equations have been discussed by Pazy [78]. In recent years the theory of semigroups of bounded linear operators has been applied to a large class of nonlinear integrodifferential equations in Banach spaces. Arino et al [4] studied the existence results for initial value problem for neutral functional differential equations. Mazouzi and Tator [68] and Ntouyas [73] proved the global existence theorem for neutral functional integrodifferential equations. Fu and Ezzinbi [40] discussed the existence of solutions for neutral functional differential evolution equations with nonlocal conditions whereas Fu [38] investigated the same problem for neutral evolution equations with nondense domain. For more results about partial neutral functional differential equations we refer to Adimy [1-2], Datko [27] and Hernandez [55].

Balachandran and Sakthivel [12] and Dauer and Balachandran [28] studied the existence of solutions for neutral functional integrodifferential equations in Banach spaces. Recently Balachandran et al [15] discussed the existence of solutions of nonlinear neutral integrodifferential equations with the help of the Schaefer fixed point theorem. In this chapter we discuss the existence of a mild solution and the controllability of nonlinear neutral integrodifferential systems. Further we prove the existence of mild solutions of a nonlinear neutral time varying multiple delay differential equation and study the controllability problem for the same equation. The results are obtained by using the Schaefer fixed point theorem.
6.2. DELAY INTEGRODIFFERENTIAL EQUATION

6.2.1. Preliminaries

Consider the neutral integrodifferential equation of the form

\[
\frac{d}{dt}[x(t) - f(t, x(t), x(\delta(t)))] = Ax(t) + F\left(t, x(\delta_1(t)), \int_0^t g\left(t, s, x(\delta_2(s))\right) ds \right),
\]

\[
\int_0^t k\left(s, \tau, x(\delta_3(\tau))\right) d\tau ds, \quad t \in [0, b],
\]

(6.1)

\[x(0) = x_0,\]

where \(A\) is the infinitesimal generator of a compact analytic semigroup of bounded linear operator \(T(t), t > 0, \) in a Banach space \(X, \) \(F : J \times X \times X \to X, \) \(g : J \times J \times X \to X, \) \(f : J \times X \times X \to X, \) and \(\delta_1, \delta_2, \delta_3 \in C(J, J)\) are given functions such that \(0 < \delta_i(t) \leq t, \) for \(i = 1, 2, 3\) and \(\delta(0) = 0.\) Here \(J = [0, b].\)

**Definition 6.1.** A function \(x(\cdot)\) is called a mild solution of the system (6.1) if \(x(0) = x_0,\) the restriction of \(x(\cdot)\) to the interval \(J,\) is continuous and, for each \(0 \leq t \leq b,\) the function \(AT(t - s)f(s, x(s), x(\delta(s))), s \in [0, t],\) is integrable and the following integral equation

\[
x(t) = T(t)[x_0 - f(0, x_0, x_0)] + f(t, x(t), x(\delta(t)))
\]

\[
+ \int_0^t AT(t - s)f(s, x(s), x(\delta(s))) ds + \int_0^t T(t - s)\phi(s, x) ds,
\]

where

\[
\phi(s, x) = F\left(s, x(\delta_1(s)), \int_0^s g\left(s, \tau, x(\delta_2(\tau))\right) ds, \int_0^\tau k(\tau, \eta, x(\delta_3(\eta))) d\eta\right) d\tau,
\]

is satisfied.

We need the following fixed point theorem due to Schaefer [87]:

**The Schaefer Fixed Point Theorem:** Let \(E\) be a normed linear space. Let \(F : E \to E\) be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set and let

\[
\zeta(F) = \{x \in E : x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.
\]
Then either $\zeta(F)$ is unbounded or $F$ has a fixed point.

Let $A$ be the infinitesimal generator of a compact analytic semigroup $T(t)$ on the Banach space $X$. The operator $(-A)^{\alpha}$ can be defined for $0 \leq \alpha \leq 1$ as the inverse of the bounded linear operator

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) dt$$

and $(-A)^{\alpha}$ is a closed linear invertible operator with domain $D((-A)^{\alpha})$, dense in $X$. The closedness of $(-A)^{\alpha}$ implies that $D((-A)^{\alpha})$ endowed with the graph norm of $(-A)^{\alpha}$, that is, $\|\|x\|\| = \|x\| + \|(-A)^{\alpha} x\|$, is a Banach space. Since $(-A)^{\alpha}$ is invertible, its graph norm $\|\|\|$ is equivalent to the norm $\|x\|_\alpha = \|(-A)^{\alpha} x\|$. Thus $D((-A)^{\alpha})$, equipped with the norm $\|\|_\alpha$, is a Banach space which we denote by $X_\alpha$. From this definition it is clear that $0 < \alpha < \beta$ implies $X_\alpha \supset X_\beta$ and that the imbedding of $X_\beta$ in $X_\alpha$ is compact whenever the resolvent operator of $A$ is compact. For more results on fractional powers of operators, one can refer [78].

Assume that the following conditions hold:

(H1) $A$ is the infinitesimal generator of an analytic semigroup of bounded linear operator $T(t)$ in $X$ such that $\|T(t)\| \leq K_1$, for some $K_1 > 0$ and for any $\alpha > 0$, there exists a positive constant $K_2(\alpha) > 0$ such that $\|(-A)^{\alpha} T(t)\| \leq K_2 t^{-\alpha}$.

(H2) For each $(t, s) \in J \times J$, the function $k(t, s, \cdot) : X \to X$ is continuous and for each $x \in X$ the function $k(\cdot, s, x) : J \to X$ is strongly measurable.

(H3) For each $(t, s) \in J \times J$, the function $g(t, s, \cdot, \cdot) : X \times X \to X$ is continuous and for each $(x, y) \in X \times X$ the function $g(\cdot, s, x, y) : J \times J \to X$ is strongly measurable.

(H4) For each $t \in J$, the function $F(t, \cdot, \cdot) : X \times X \to X$ is continuous and for each $(x, y) \in X \times X$ the function $F(\cdot, x, y) : J \to X$ is strongly measurable.

(H5) For each positive integer $k$, there exists $\mu_k \in L^1(0, b)$ such that

$$\sup_{\|x\|, \|y\| \leq k} \|F(t, x, y)\| \leq \mu_k(t).$$
(H6) The function \( f : J \times X \times X \to X \) is completely continuous and for any bounded set \( Q \) in \( C(J,X) \), the set \( \{ t \to f(t,x(t),x(\delta(t))) : x \in Q \} \) is equicontinuous in \( C(J,X) \).

(H7) There exist \( \beta \in (0,1) \) and a constant \( b_1 > 0 \) such that
\[
\|(-A)^\beta f(t,x,y)\| \leq b_1, \quad t \in J, \quad (x,y) \in X.
\]

(H8) There exists an integrable function \( m : [0,b] \to [0,\infty) \) such that
\[
\|k(t,s,x)\| \leq m(t,s)\Omega_0(\|x\|), \quad 0 \leq s \leq t \leq b, \quad x \in X,
\]
where \( \Omega_0 : [0,\infty) \to (0,\infty) \) is a continuous nondecreasing function.

(H9) There exists an integrable function \( m_1 : [0,b] \to [0,\infty) \) such that
\[
\|g(t,s,x,y)\| \leq m_1(t,s)\Omega_1(\|x\| + \|y\|), \quad t,s \in J, \quad x,y \in X,
\]
where \( \Omega_1 : [0,\infty) \to (0,\infty) \) is a continuous nondecreasing function.

(H10) There exists an integrable function \( m_2 : [0,b] \to [0,\infty) \) such that
\[
\|F(s,x,y)\| \leq m_2(s)\Omega_2(\|x\| + \|y\|), \quad s \in J, \quad x,y \in X,
\]
where \( \Omega_2 : [0,\infty) \to (0,\infty) \) is a continuous nondecreasing function.

(H11) The function
\[
p(t) = \max\{K_1m_2(t),m_1(t,t) + \int_0^t \frac{\partial m_1(t,s)}{\partial t} ds, m(t,t) + \int_0^t \frac{\partial m(t,s)}{\partial t} ds\}
\]
satisfies the inequality
\[
\int_0^b p(s) ds \leq c \int_0^\infty \frac{ds}{\Omega_2(s) + \Omega_1(s) + \Omega_0(s)},
\]
where \( c = K_1[\|x_0\| + M_0b_1] + M_0b_1 + \frac{K_2b_1}{\beta} \) and \( \|(-A)^{-\beta}\| = M_0 \).

6.2.2. Main Result

**Theorem 6.1.** If the assumptions (H1)-(H11) are satisfied, then the problem (6.1) has a mild solution on \( J \).
Proof. To prove the existence of mild solution of (6.1), we have to apply Schaefer's theorem for the following operator equation

$$x(t) = \lambda \Psi x(t), \quad 0 < \lambda < 1,$$

where $\Psi : Z \to Z$ is defined as

$$\Psi x(t) = T(t)[x_0 - f(0, x_0, x_0)] + f(t, x(t), x(\delta(t))) + \int_0^t A T(t - s) f(s, x(s), x(\delta(s))) ds + \int_0^t T(t - s) \phi(s, x) ds.$$

Then we have

$$\|x(t)\| \leq K_1[\|x_0\| + M_0 b_1] + M_0 b_1 + \frac{K_2 b_1 b^2}{\beta}$$

$$+ K_1 \int_0^t m_2(s) \Omega_2 \left(\|x(s)\| + \int_0^s m_1(s, \tau) \Omega_1(\|x(\tau)\| + \int_0^\tau m(\tau, \eta) \Omega_0(\|x(\eta)\|) d\eta) d\tau\right) ds.$$

Denoting the right hand side of above the inequality as $r(t)$, then

$$\|x(t)\| \leq r(t) \text{ and } r(0) = c = K_1[\|x_0\| + M_0 b_1] + M_0 b_1 + \frac{K_2 b_1 b^2}{\beta}$$

$$r'(t) = K_1 m_2(t) \Omega_2 \left(r(t) + \int_0^t m_1(t, s) \Omega_1\left(r(s) + \int_0^s m(s, \tau) \Omega_0(r(\tau)) d\tau\right) ds\right).$$

Let

$$\omega(t) = r(t) + \int_0^t m_1(t, s) \Omega_1\left(r(s) + \int_0^s m(s, \tau) \Omega_0(r(\tau)) d\tau\right) ds.$$

Then

$$\omega'(t) = r'(t) + m_1(t, t) \Omega_1\left(r(t) + \int_0^t m(t, s) \Omega_0(r(s)) ds\right)$$

$$+ \int_0^t \frac{\partial m_1(t, s)}{\partial t} \Omega_1\left(r(s) + \int_0^s m(s, \tau) \Omega_0(r(\tau)) d\tau\right) ds$$

$$\leq K_1 m_2(t) \Omega_2(\omega(t)) + m_1(t, t) \Omega_1\left(\omega(t) + \int_0^t m(t, s) \Omega_0(\omega(s)) ds\right)$$

$$+ \int_0^t \frac{\partial m_1(t, s)}{\partial t} \Omega_1(\omega(s) + \int_0^s m(s, \tau) \Omega_0(\omega(\tau)) d\tau) ds.$$
Let

\[ \omega_0(t) = \omega(t) + \int_0^t m(t,s) \Omega_0(\omega(s)) ds. \]

Then

\[ \omega'(t) \leq K_1 m_2(\omega_0(t)) + m_1(t, t) \Omega_1(\omega_0(t)) + \int_0^t \frac{\partial m_1(t,s)}{\partial t} \Omega_1(\omega_0(s)) ds \]
\[ + m(t, t) \Omega_0(\omega(t)) + \int_0^t \frac{\partial m(t,s)}{\partial t} \Omega_0(\omega(s)) ds \]
\[ \leq K_1 m_2(\omega_0(t)) + m_1(t, t) \Omega_1(\omega_0(t)) + \int_0^t \frac{\partial m_1(t,s)}{\partial t} ds \Omega_1(\omega_0(t)) \]
\[ + m(t, t) \Omega_0(\omega(t)) + \int_0^t \frac{\partial m(t,s)}{\partial t} ds \Omega_0(\omega(t)) \]
\[ \leq p(t) \left[ \Omega_2(\omega_0(t)) + \Omega_1(\omega_0(t)) + \Omega_0(\omega_0(t)) \right]. \]

This implies

\[ \int_{\omega_0(t)}^{\omega_0(0)} \frac{ds}{\Omega_2(s) + \Omega_1(s) + \Omega_0(s)} \leq \int_0^b p(s) ds \leq \int_0^\infty \frac{ds}{\Omega_2(s) + \Omega_1(s) + \Omega_0(s)}. \] (6.3)

Inequality (6.3) implies that there is a constant \( K \), such that \( r(t) \leq K, t \in J \).

Hence \( \|x(t)\| \leq K, t \in J \).

We shall now prove that the operator \( \Psi : Z \to Z \) is a completely continuous operator. Let \( B_k = \{ x \in Z : \|x(t)\| \leq k \} \) for some \( k > 1 \). We first show that \( \Psi \) maps \( B_k \) into an equicontinuous family.

Let \( y \in B_k \) and \( t_1, t_2 \in [0, b] \). Then, if \( 0 < t_1 < t_2 < b \),

\[ \| (\Psi x)(t_1) - (\Psi x)(t_2) \| \leq \| (T(t_1) - T(t_2))[x_0 - f(0, x_0, x_0)] \]
\[ + \| f(t_1, x(t_1), x(\delta(t_1))) - f(t_2, x(t_2), x(\delta(t_2))) \| \]
\[ + \| \int_{t_1}^{t_2} A[T(t_1 - s) - T(t_2 - s)] f(s, x(s), x(\delta(s))) ds \| \]
\[ + \| \int_{t_1}^{t_2} A[T(t_2 - s) f(s, x(s), x(\delta(s))) ds \| \]
\[ + \| \int_{t_1}^{t_2} (T(t_1 - s) - T(t_2 - s)) \phi(s, x) ds \| \]
\[ + \| \int_{t_1}^{t_2} T(t_2 - s) \phi(s, x) ds \| \]
\[ \leq \| (T(t_1) - T(t_2))[x_0 - f(0, x_0, x_0)] \| \]
\[
\begin{align*}
+ \|f(t_1, x(t_1), x(\delta(t_1))) - f(t_2, x(t_2), x(\delta(t_2)))\| \\
\frac{C\beta b_1(t_1 - t_2)^\gamma}{\beta - \gamma} b^{\beta - \gamma} + K_2 b_1 (t_2 - t_1)^\alpha \\
+ \int_0^{t_1} \|T(t_1 - s) - T(t_2 - s)\|\mu_{k^*}(s)\,ds \\
+ \int_{t_1}^{t_2} \|T(t_2 - s)\|\mu_{k^*}(s)\,ds,
\end{align*}
\]

where \( C' > 0 \) and \( k^* \geq k + |p(t)|\Omega_1(r + |p(s)|\Omega_0(r)) \). The right hand side is independent of \( x \in B_k \) and tends to zero as \( t_1 \to t_2 \), since \( f \) is completely continuous and the compactness of \( T(t) \), for \( t > 0 \) implies continuity in the uniform operator topology. Thus \( \Psi \) maps \( B_k \) into an equicontinuous family of functions.

Next we show that \( \overline{\Psi B_k} \) is compact. Since we have shown \( \Psi B_k \) is equicontinuous, by the Arzela-Ascoli theorem, it suffices to show that \( \Psi \) maps \( B_k \) into a precompact set in \( X \).

Let \( 0 < t \leq b \) be fixed and let \( \epsilon \) be a real number satisfying \( 0 < \epsilon < t \). For \( x \in B_k \), we define

\[
\begin{align*}
(\Psi_x(t)) &= T(t)[x_0 - f(0, x_0, x_0)] + f(t, x(t), x(\delta(t))) \\
&\quad + \int_0^{t-\epsilon} AT(t - s)f(s, x(s), x(\delta(s)))\,ds \\
&\quad + \int_0^{t-\epsilon} T(t - s)\phi(s, x)\,ds \\
= T(t)[x_0 - f(0, x_0, x_0)] + f(t, x(t), x(\delta(t))) \\
&\quad + T(\epsilon) \int_0^{t-\epsilon} AT(t - s - \epsilon)f(s, x(s), x(\delta(s)))\,ds \\
&\quad + T(\epsilon) \int_0^{t-\epsilon} T(t - s - \epsilon)\phi(s, x)\,ds.
\end{align*}
\]

Since \( T(t) \) is compact, the set \( \{\Psi_x(t) : x \in B_k\} \) is precompact in \( X \) for every \( \epsilon, 0 < \epsilon < t \). Moreover, for every \( x \in B_k \), we have

\[
\begin{align*}
\|&(\Psi x)(t) - (\Psi_x\epsilon)(t)\| \\
\leq \int_t^{t-\epsilon} \|AT(t - s)f(s, x(s), x(\delta(s)))\|\,ds + \int_0^{t-\epsilon} \|T(t - s)\phi(s, x)\|\,ds \\
\leq \int_t^{t-\epsilon} \|AT(t - s)f(s, x(s), x(\delta(s)))\|\,ds + \int_t^{t-\epsilon} \|T(t - s)\|\mu_{k^*}(s)\,ds.
\end{align*}
\]

Therefore there are precompact sets arbitrarily close to the set \( \{(\Psi x)(t) : x \in B_k\} \).

Hence the set \( \{(\Psi x)(t) : x \in B_k\} \) is precompact in \( X \).
It remains to show that $\Psi : Z \to Z$ is continuous. Let $\{x_n\}_{n=0}^\infty \subseteq Z$ with $x_n \to x$ in $Z$. Then there is an integer $q$ such that $\|x_n(t)\| \leq q$ for all $n$ and $t \in J$, so $x_n \in B_k$ and $x \in B_k$. By (H4),

$$F\left(t, x_n(\delta_1(t)), \int_0^t g\left(t, s, x_n(\delta_2(s)), \int_0^s k(s, \tau, x_n(\delta_3(\tau)))d\tau \right)ds\right)$$

$$\to F\left(t, x(\delta_1(t)), \int_0^t g\left(t, s, x(\delta_2(s)), \int_0^s k(s, \tau, x(\delta_3(\tau)))d\tau \right)ds\right)$$

for each $t \in J$ and since

$$\left\| F\left(t, x_n(\delta_1(t)), \int_0^t g\left(t, s, x_n(\delta_2(s)), \int_0^s k(s, \tau, x_n(\delta_3(\tau)))d\tau \right)ds\right) - F\left(t, x(\delta_1(t)), \int_0^t g\left(t, s, x(\delta_2(s)), \int_0^s k(s, \tau, x(\delta_3(\tau)))d\tau \right)ds\right) \right\| \leq 2\mu_q(t),$$

we have, by the dominated convergence theorem,

$$\|\Psi x_n - \Psi x\| = \sup_{t \in J} \left\|[f(t, x_n(t), x_n(\delta(t))) - f(t, x(t), x(\delta(t)))]ight.$$\n
$$+ \int_0^t AT(t - s)[f(s, x_n(s), x_n(\delta(s))) - f(s, x(s), x(\delta(s)))]ds$$

$$+ \int_0^t T(t - s)[\phi(s, x_n) - \phi(s, x)]ds\right\|$$

$$\leq \|f(t, x_n(t), x_n(\delta(t))) - f(t, x(t), x(\delta(t)))\|$$

$$+ \int_0^t \|AT(t - s)\|\|f(s, x_n(s), x_n(\delta(s))) - f(s, x(s), x(\delta(s)))\|ds$$

$$+ \int_0^t \|T(t - s)\|\|\phi(s, x_n) - \phi(s, x)\|ds$$

$$\to 0 \text{ as } n \to \infty.$$

Thus $\Psi$ is continuous. This completes the proof that $\Psi$ is completely continuous. Finally, the set $\zeta(\Psi) = \{x \in Z : x = \lambda\Psi x \lambda \in (0, 1)\}$ is bounded. Consequently, by Schaefer's theorem, the operator $\Psi$ has a fixed point in $Z$. This means that any fixed point of $\Psi$ is a mild solution of (6.1) on $J$ satisfying $(\Psi x)(t) = x(t)$.

6.2.3. Example

Consider the following example

$$\frac{\partial}{\partial t} [z(t, y) + \int_{-h}^t \int_0^\pi b(s - t, \eta, y)z(\eta, y)d\eta ds]$$

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\[ z(t, y) = z(t, \pi) = 0, \quad t \geq 0, \]
\[ z(0, y) = z_0(y), \quad 0 \leq y \leq \pi, \]
where \( h > 0 \) and \( b \) is a measurable function. Take \( X = L^2[0, \pi] \) and define \( A : X \to X \) by \( A\omega = \omega'' \) with domain
\[ D(A) = \{\omega \in X : \omega, \omega' \text{ are absolutely continuous}, \ \omega'' \in X, \omega(0) = \omega(\pi) = 0\}. \]
\( D(A) \) is dense in \( X \) and \( A \) is a closed operator.

Then
\[ A\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n, \ \omega \in D(A), \]
where \( \omega_n(y) = \sqrt{\frac{2}{\pi}} \sin ny, \ n = 1, 2, 3 \cdots \) is the orthogonal set of eigenvectors of \( A \). Also \( A \) is the infinitesimal generator of an analytic semigroup \( T(t), t \geq 0, \) in \( X \) and is given by
\[ T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}(\omega, \omega_n)\omega_n, \ \omega \in X, \]
where \( \lambda_n = n^2, \ n = 1, 2, 3, \cdots \) are the eigenvalues of \( A \).

Let
\[ f(t, y, z) = \int_{-h}^{t} \int_0^{\pi} b(s - t, \eta, y)z(\eta, y) d\eta ds \]
and
\[ F(t, x(t), \int_0^{t} g(t, s, x(s), \int_0^{s} k(s, \tau, x(\tau)) d\tau) ds) \]
\[ = \frac{x^2(t) \sin x(t)}{(1 + t)(1 + t^2)} + \int_0^{t} \left[ \frac{x(s)}{(1 + t)(1 + t^2)^2(1 + s)^2} \right. \]
\[ + \left. \frac{1}{(1 + t)(1 + t^2)} \int_0^{s} \frac{x(\tau)}{(1 + s)(1 + \tau)} e^{x(\tau)} d\tau \right] ds. \]

Then
\[ \|F(t, x, v)\| = \left\| \frac{1}{(1 + t)(1 + t^2)} (x^2 \sin x + v) \right\| \]
\[ \leq \frac{1}{1 + t^2} \|x\|^2 + \frac{1}{1 + t} \|v\|, \]

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where we have set \( v = \int_0^t g(t, s, x(s), \int_0^s k(s, \tau, x(\tau)) d\tau) \, ds \).

Next, if \( w = \int_0^s k(s, \tau, x(\tau)) d\tau \), then

\[
\| g(t, s, x, w) \| = \frac{x}{(1 + t)(1 + t^2)^2(1 + s)^2} + \frac{w}{(1 + t)(1 + t^2)}
\]

\[
\leq \frac{1}{(1 + t^2)(1 + s)} \| x \| + \frac{1}{(1 + t^2)(1 + t)} \| w \| .
\]

Finally we have

\[
\| k(s, \tau, x) \| = \frac{x e^x}{(1 + s)(1 + \tau)}
\]

\[
\leq \frac{1}{(1 + s)(1 + \tau)} \| x \| e^{\| x \|} .
\]

Since all the conditions of Theorem 6.1 are satisfied, the equation (6.4) has a mild solution on \([0, b] \).

6.2.4. Application

As an Application of Theorem 6.1, we shall consider the equation (6.1) with a control parameter as

\[
x(0) = x_0,
\]

where \( B \) is a bounded linear operator from a Banach space \( U \) into \( X \) and \( u \in L^2(J, U) \). Let \( X_r = \{ x \in X : \| x \| \leq r \} \) for some \( r > 0 \) and \( Z_r = C^1(J, X_r) \).

Here the mild solution of the equation (6.5) is given by

\[
x(t) = T(t)[x_0 - f(0, x_0, x_0)] + f(t, x(t), x(\delta(t)))
\]

\[
+ \int_0^t A T(t - s)f(s, x(s), x(\delta(s))) \, ds + \int_0^t T(t - s)[Bu(s) + \phi(s, x)] \, ds .
\]
Definition 6.2. The system (6.5) is said to be locally controllable on the interval \( J \) if for every \( x_0, x_1 \in Y, \) a subset of \( X \), there exists a control \( u \in L^2(J, U) \) such that the solution \( x(\cdot) \) of (6.5) satisfies \( x(b) = x_1 \).

Controllability of nonlinear systems of various types in Banach spaces has been studied by several authors by means of fixed point principles [11]. Recently Balachandran and Anandhi [7] and Fu [36] investigated the controllability problem for neutral systems. To establish the controllability result for the system (6.5), we need the following additional hypotheses.

(H12) The linear operator \( W : L^2(J, U) \rightarrow X \) defined by
\[
Wu = \int_0^b T(b - s)Bu(s)ds
\]
induces an invertible operator \( \tilde{W} \) defined on \( L^2(J, U) \backslash \text{ker} W \) and there exists a positive constant \( K_3 > 0 \) such that \( \|B\tilde{W}^{-1}\| \leq K_3 \).

(H13) The function
\[
p(t) = \max\{K_1m_2(t), m_1(t, t) + \int_0^t \frac{\partial m_1(t, s)}{\partial t}ds, \mu(t, t) + \int_0^t \frac{\partial \mu(t, s)}{\partial t}ds\}
\]
satisfies the inequality
\[
\int_0^b p(s)ds \leq \int_c^{\infty} \frac{ds}{\Omega_2(s) + \Omega_1(s) + \Omega_0(s)},
\]
where \( c = K_1[\|x_0\| + M_0b_1] + M_0b_1 + \frac{K_2b^2}{\beta} + K_1Nb \) and \( N = K_3[\|x_1\| + K_1(\|x_0\| + M_0b_1) + M_0b_1 + \frac{K_2b^2}{\beta} + \int_0^b \Omega_2(r + \int_0^r m_1(s, r)\Omega_1(s + \int_0^s m(t, \eta)\Omega_0(\tau)\eta d\eta) d\tau)ds]. \)

Theorem 6.2. If the hypotheses (H1)-(H13) are satisfied, then the system (6.5) is controllable.

Proof. Using the hypotheses (H12) for an arbitrary function \( x(\cdot) \), define the control
\[
u(t) = \tilde{W}^{-1}\left\{x_1 - T(b)[x_0 - f(0, x_0, x_0)] - f(b, x(b), x(\delta(b)))
- \int_0^b AT(b - s)f(s, x(s), x(\delta(s)))ds - \int_0^b T(b - s)\phi(s, x)ds\right\}(t).
\]
We shall show that when using this control the operator $ \Psi : Z_T \to Z_T$ defined by

$$\Psi(x)(t) = T(t)\left[x_0 - f(0, x_0, x_0)\right] + f(t, x(t), x(\delta(t))) + \int_0^1 AT(t - s)f(s, x(s), x(\delta(s)))ds + \int_0^1 T(t - s)[Bu(s) + \phi(s, x)]ds$$

has a fixed point. This fixed point is then a solution of (6.5). Substituting $u(t)$ in the above equation we get $(\Psi x)(b) = x_1$ which means that the control $u$ steers system (6.5) from the given initial condition $x_0$ to $x_1$ in time $b$ in $Z_T$. Thus the system (6.5) is locally controllable. The remaining part of the proof is similar to that of Theorem 6.1 and hence it is omitted.

### 6.3. NEUTRAL DELAY DIFFERENTIAL EQUATIONS

#### 6.3.1. Preliminaries

Consider the nonlinear neutral time varying multiple delay differential equation of the form

$$\frac{d}{dt}\left[x(t) + F(t, x(t), x(b_1(t)), \ldots, x(b_m(t)))\right] = Ax(t) + G(t, x(t), x(a_1(t)), \ldots, x(a_n(t))), t \in J = [0, a], \quad (6.6)$$

$$x(0) = x_0,$$

where $A$ is the infinitesimal generator of a compact analytic semigroup of bounded linear operators $T(t)$ in a Banach space $X$, $F : [0, a] \times X_{m+1} \to X$, $G : [0, a] \times X_{n+1} \to X$ are continuous functions. The delays $a_i(t), b_j(t)$ are continuous scalar valued functions defined on $J$ such that $a_i(t) \leq t, b_j(t) \leq t$.

Let $A : D(A) \to X$ be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator $T(t)$ defined on a Banach space $X$ with norm $\| \cdot \|$. Let $0 \in \rho(A)$. Then define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D((-A)^\alpha)$ which is dense in $X$. Further $D((-A)^\alpha)$ is a Banach space under the norm

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \text{ for } x \in D((-A)^\alpha).$$

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and is denoted by $X_\alpha$. The imbedding $X_\alpha \hookrightarrow X_\beta$, for $0 < \beta < \alpha \leq 1$, is compact whenever the resolvent operator of $A$ is compact. For semigroup $\{ T(t) \}$ the following properties will be used.

(a) there is a $M_1 > 1$ such that $\| T(t) \| \leq M_1$, for all $0 \leq t \leq a$.

(b) for any $\alpha > 0$, there exists a positive constant $M_2(\alpha) > 0$ such that

$$\| (-A)^\alpha T(t) \| \leq M_2 t^{-\alpha}, 0 < t \leq a.$$  

(6.7)

**Definition 6.3.** A function $x(\cdot)$ is called a mild solution of the system (6.6) if $x(0) = x_0$, the restriction of $x(\cdot)$ to the interval $J$, is continuous and, for each $0 \leq t \leq a$, the function $AT(t-s)F(s,x(s),x(b_1(s)),\ldots,x(b_m(s))), s \in [0,t)$, is integrable and the following integral equation

$$x(t) = T(t)[x_0 + F(0,x_0,x(b_1(0)),\ldots,x(b_m(0)))]$$

$$-F(t,x(t),x(b_1(t)),\ldots,x(b_m(t)))$$

$$- \int_0^t AT(t-s)F(s,x(s),x(b_1(s)),\ldots,x(b_m(s)))ds$$

$$+ \int_0^t T(t-s)G(s,x(s),x(a_1(s)),\ldots,x(a_n(s)))ds$$

(6.8)

is satisfied.

Here we establish the main results by using the Schaefer fixed point theorem [87].

Assume that the following conditions hold:

(C1) For each $t \in J$, the function $G(t,\cdot) : X^{n+1} \to X$ is continuous and, for each $(x_0,x_1,\ldots,x_n) \in X^{n+1}$, the function $G(\cdot,x_0,x_1,\ldots,x_n) : [0,a] \to X$ is strongly measurable.

(C2) For each positive integer $k$, there exists $\alpha_k \in L^1[0,a]$ such that

$$\sup_{\| x_0 \|,\ldots,\| x_n \| \leq k} \| G(t,x_0,x_1,\ldots,x_n) \| \leq \alpha_k(t) \text{ for } t \in J.$$

(C3) The function $F : [0,a] \times X^{m+1} \to X$ is completely continuous and for any bounded set $Q$ in $C([-\tau,a),X)$, the set

$$\{ t \to F(t,x(t),x(a_1(t)),\ldots,x(a_m(t))) : x \in Q \}$$

is compact.
is equicontinuous.

(C4) There exist $\beta \in (0,1)$ and a constant $c_1 \geq 0$ such that
\[ \|(-A)^\beta F(t, u(t))\| \leq M_3, \quad t \in J. \]

(C5) There exists an integrable function $m : [0, a] \to [0, \infty)$ such that
\[ \|G(t, x(t), x(a_1(t)), \ldots, x(a_n(t)))\| \leq (n + 1)m(t)\Omega(\|x(t)\|) \]
where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(C6)
\[ \int_0^a \tilde{m}(s)ds < \int_c^\infty \frac{ds}{s + \Omega(s)}, \]
where
\[ c = M_1[\|x_0\| + M_3M_4] + M_3M_4 + \frac{M_3M_2a^\beta}{\beta}, \]
\[ M_4 = \|(A)^{-\beta}\| \text{ and } \tilde{m}(t) = M_1m(t)(n + 1)^2. \]

Now let us take
\[ (t, x(t), x(b_1(t)), \ldots, x(b_m(t))) = (t, u(t)) \]
\[ (t, x(t), x(a_1(t)), \ldots, x(a_n(t))) = (t, v(t)) \]

6.3.2. MAIN RESULT

Theorem 6.3. If the assumptions (C1)-(C6) are satisfied then the problem (6.6) has a mild solution on $J$.

Proof. Consider the Banach space $Z = C(J, X)$ with norm
\[ \|x\| = \sup\{|x(t)| : t \in J\}. \]

To prove the existence of mild solution of (6.6), we have to apply Schaefer's theorem for the following operator equation
\[ x(t) = \lambda \Psi x(t), \quad 0 < \lambda < 1, \quad (6.9) \]
where \( \Psi : Z \to Z \) is defined as

\[
(\Psi x)(t) = T(t)[x_0 + F(0,u(0))] - F(t,u(t)) - \int_0^t A(T(t-s)F(s,u(s))ds \\
+ \int_0^t T(t-s)G(s,v(s))ds.
\]

Then we have

\[
\|x(t)\| \leq M_1[\|x_0\| + M_3M_4] + M_3M_4 + M_1 \int_0^t M_3(t-s)^{\beta-1}ds \\
+ M_1 \int_0^t (n+1)m(s)\Omega(\|v(s)\|)ds \\
\leq M_1[\|x_0\| + M_3M_4] + M_3M_4 + \frac{M_3M_2a^\beta}{\beta} + M_1 \int_0^t (n+1)m(s)\Omega(\|v(s)\|)ds.
\]

Denoting the right hand side of above inequality as \( \mu(t) \),

\[
\|x(t)\| \leq \mu(t) \text{ and } \mu(0) = c = M_1[\|x_0\| + M_3M_4] + M_3M_4 + \frac{M_3M_2a^\beta}{\beta}
\]

\[
\mu'(t) = M_1(n+1)m(t)\Omega(\|v(t)\|) \\
\leq M_1(n+1)^2m(t)\Omega(\mu(t)) \\
\leq \hat{m}(t)[\Omega(\mu(t))].
\]

This implies

\[
\int_{\mu(0)}^{\mu(t)} \frac{ds}{\Omega(s)} \leq \int_0^a \hat{m}(s)ds < \int_0^\infty \frac{ds}{\Omega(s)}, \quad 0 \leq t \leq a. \quad (6.10)
\]

Inequality (6.10) implies that there is a constant \( K \) such that \( \mu(t) \leq K, t \in [0,a] \), and hence we have \( \|x\| = \sup\{|x(t)| : t \in J\} \leq K \) where \( K \) depends only on \( a \) and on the functions \( \hat{m} \) and \( \Omega \).

We shall now prove that the operator \( \Psi : Z \to Z \) is a completely continuous operator. Let \( B_k = \{x \in Z : \|x\|_1 \leq k\} \) for some \( k \geq 1 \). We first show that \( \Psi \) maps \( B_k \) into an equicontinuous family.

Let \( x \in B_k \) and \( t_1, t_2 \in [0,a] \). Then, if \( 0 < t_1 < t_2 < a \),

\[
\|(\Psi x)(t_1) - (\Psi x)(t_2)\|
\]
\[
\leq \|(T(t_1) - T(t_2))[x_0 + F(0, u(0))]| + \|F(t_1, u(t_1)) - F(t_2, u(t_2))\| \\
+ \| \int_0^{t_1} A[T(t_1 - s) - T(t_2 - s)]F(s, u(s))ds\| + \| \int_{t_1}^{t_2} AT(t_2 - s)F(s, u(s))ds\| \\
+ \| \int_0^{t_1} [T(t_1 - s) - T(t_2 - s)]G(s, v(s))ds\| + \| \int_{t_1}^{t_2} T(t_2 - s)G(s, v(s))ds\| \\
\leq \|(T(t_1) - T(t_2))[x_0 + F(0, u(0))]| + \|F(t_1, u(t_1)) - F(t_2, u(t_2))\| \\
+ \int_0^{t_1} \|A[T(t_1 - s) - T(t_2 - s)]\| M_3 M_4 ds + \int_{t_1}^{t_2} \|AT(t_2 - s)\| M_3 M_4 ds \\
+ \int_0^{t_1} \|T(t_1 - s) - T(t_2 - s)\| \alpha_k(s) ds + \int_{t_1}^{t_2} \|T(t_2 - s)\| \alpha_k(s) ds.
\]
The right hand side is independent of \(x \in B_k\) and tends to zero as \(t_1 \to t_2\) since \(F\) is completely continuous and the compactness of \(T(t)\) for \(t > 0\) implies continuity in the uniform operator topology. Thus \(\Psi\) maps \(B_k\) into an equicontinuous family of functions.

It is easy to see that \(\Psi B_k\) is uniformly bounded. Next we show \(\overline{\Psi B_k}\) is compact. Since we have shown \(\Psi B_k\) is equicontinuous collection, by the Arzela-Ascoli theorem, it suffices to show that \(\Psi\) maps \(B_k\) into a precompact set in \(X\).

Let \(0 < t < a\) be fixed and let \(\epsilon\) be a real number satisfying \(0 < \epsilon < t\). For \(x \in B_k\), we define
\[
(\Psi_x(t)) = T(t)[x_0 + F(0, u(0))] - F(t, u(t)) - \int_{t-\epsilon}^{t} AT(t - s)F(s, u(s))ds \\
+ \int_0^{t-\epsilon} T(t - s)G(s, v(s))ds \\
= T(t)[x_0 + F(0, u(0))] - F(t, u(t)) - T(\epsilon) \int_0^{t-\epsilon} AT(t - s - \epsilon)F(s, u(s))ds \\
+ T(\epsilon) \int_0^{t-\epsilon} T(t - s - \epsilon)G(s, v(s))ds.
\]
Since \(T(t)\) is a compact operator, the set \(Y_t = \{(\Psi_x)(t) : x \in B_k\}\) is precompact in \(X\) for every \(\epsilon, 0 < \epsilon < t\). Moreover, for every \(x \in B_k\) we have
\[
\|(\Psi x(t)) - (\Psi_x(t))\| \leq \int_t^{t-\epsilon} \|AT(t - s)F(s, u(s))\| ds + \int_{t-\epsilon}^{t} \|T(t - s)G(s, v(s))\| ds \\
\leq \int_t^{t-\epsilon} \|AT(t - s)F(s, u(s))\| ds + \int_{t-\epsilon}^{t} \|T(t - s)\| \alpha_k(s) ds.
\]
Therefore there are precompact sets arbitrarily close to the set \(\{(\Psi x)(t) : x \in B_k\}\).

Hence, the set \(\{(\Psi x)(t) : x \in B_k\}\) is precompact in \(X\).
It remains to show that $\Psi : Z \to Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \to x$ in $Z$. Then there is an integer $q$ such that $\|x_n(t)\| \leq q$ for all $n$ and $t \in J$; so $x_n \in B_r$ and $x \in B_r$. By (C2)

$$G(t, v_n(t)) \to G(t, v(t))$$

for each $t \in J$ and since

$$\|G(t, v_n(t)) - G(t, v(t))\| \leq 2\alpha_q(t),$$

we have, by dominated convergence theorem,

$$\|\Psi x_n - \Psi x\| = \sup_{t \in J} \|[F(t, u_n(t)) - F(t, u(t))]$$

$$+ \int_0^t A(t - s)[F(s, u_n(s)) - F(s, u(s))]ds$$

$$+ \int_0^t T(t - s)[G(s, u_n(s)) - G(s, u(s))]ds\|$$

$$\leq \|F(t, u_n(t)) - F(t, u(t))\|$$

$$+ \int_0^t \|A(t - s)[F(s, u_n(s)) - F(s, u(s))]\|ds$$

$$+ \int_0^t \|T(t - s)[G(s, u_n(s)) - G(s, u(s))]\|ds$$

$$\to 0, \text{ as } n \to \infty.$$ 

Thus $\Psi$ is continuous. This completes the proof that $\Psi$ is completely continuous.

Finally the set $\zeta(\Psi) = \{x \in Z : x = \lambda \Psi x, \lambda \in (0, 1)\}$ is bounded, as we proved in the first step. Consequently, by Schaefer's theorem, the operator $\Psi$ has a fixed point in $Z$. This means that any fixed point of $\Psi$ is a mild solution of (6.6) on $J$ satisfying $(\Psi x)(t) = x(t)$.

6.3.3. Application

As an application of Theorem 6.3, we shall consider the equation (6.6) with a control parameter as

$$\frac{d}{dt} [x(t) + F(t, x(t), x(b_1(t)), \cdots, x(b_n(t)))]$$

$$= Ax(t) + Bw(t) + G(t, x(t), x(a_1(t)), \cdots, x(a_n(t))),$$

$$t \in J = [0, a],$$

$$x(0) = x_0,$$

(6.11)

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where $B$ is a bounded linear operator from $U$, a Banach space, to $X$ and $w \in L^2(J, U)$.

In this case the mild solution of (6.11) is given by

$$
x(t) = T(t)[x_0 + F(0, u(0))] - F(t, u(t)) - \int_0^t AT(t-s)F(s, u(s))ds
+ \int_0^t T(t-s)[Bw(s)ds + G(s, v(s))]ds.
$$

**Definition 6.4.** The system (6.11) is said to be *locally controllable* on the interval $J$ if for any subset $Y \subset X$ and for every $x_0, x_1 \in Y$, there exists a control $w \in L^2(J, U)$ such that the solution $x(\cdot)$ of (6.11) satisfies $x(a) = x_1$. Let

$$X_r = \{x \in X : \|x\| \leq r\} \text{ for some } r > 0 \text{ and } Z_r = C^1(J, X_r).$$

To establish the controllability result for the system (6.11), we need the following additional hypotheses.

(C7) The linear operator $W : L^2(J, U) \rightarrow X$ defined by

$$Ww = \int_0^a T(a-s)Bw(s)ds$$

induces an invertible operator $\tilde{W}$ defined on $L^2(J, U)/\ker W$ and there exists a positive constant $M_5 > 0$ such that $\|B\tilde{W}^{-1}\| \leq M_5$.

(C8)

$$\int_0^a \tilde{m}(s)ds < \int_0^\infty \frac{ds}{\Omega(s)},$$

where

$$c = M_1[\|x_0\| + M_3c_1] + M_3M_4 + \frac{M_3M_2a^\beta}{\beta} + M_1Na$$

and

$$N = M_5[\|x_1\| + M_1(\|x_0\| + M_3M_4) + M_3M_4 + \frac{M_3M_2a^\beta}{\beta}
+ M_1 \int_0^a m(s)(n+1)\Omega(r)ds].$$
Theorem 6.4. If the hypotheses (C1)-(C8) are satisfied, then the system (6.11) is controllable.

Proof. Using the hypotheses (C7), for an arbitrary function \( x(t) \), define the control

\[
w(t) = W^{-1} \left\{ x_1 - T(a)[x_0 + F(0, u(0))] + F(a, u(a)) + \int_0^a AT(a-s)F(s, u(s))ds - \int_0^a T(a-s)G(s, v(s))ds \right\}(t).
\]

We shall show that when using this control the operator \( \Phi : Z_r \to Z_r \) defined by

\[
(\Phi x)(t) = T(t)[x_0 + F(0, u(0))] - F(t, u(t)) - \int_0^t AT(t-s)F(s, u(s))ds + \int_0^t T(t-s)[Bw(s) + G(s, v(s))]ds, \quad t \in J,
\]

has a fixed point. This fixed point is then a solution of (6.11). Substituting \( w(t) \) in the above equation, we get \( (\Phi x)(a) = x_1 \) which means that the control \( w \) steers the system (6.11) from the given initial condition \( x_0 \) to \( x_1 \) in time \( a \). Thus the system (6.11) is controllable. The remaining part of the proof is similar to that of Theorem 6.3 and hence it is omitted.

Remark: The results established in this chapter generalise the results of [14,40].

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