CHAPTER IV
IV. CONTROLLABILITY OF NEUTRAL FUNCTIONAL EVOLUTION INTEGRODIFFERENTIAL SYSTEMS WITH INFINITE DELAY

4.1. INTRODUCTION

The problem of controllability of abstract functional differential and integrodifferential systems has been studied by several authors by means of fixed point principles [11,79]. Most of these works are in connection with finite delays. Since many control systems arising from realistic models heavily depend on infinite delay [22,52,57,93], there is an increasing interest to study the controllability of partial functional differential and integrodifferential systems with infinite delay. Han et al [49] investigated the controllability problem of integrodifferential systems by considering the initial condition in some abstract phase space. Wang and Wang [91] studied the neutral functional differential systems with infinite delay by using the Schaefer fixed point theorem whereas Fu [36] discussed the same problem by means of Sadovskii's fixed point theorem. Recently Balachandran and Anandhi [8] obtained the controllability of neutral functional integrodifferential infinite delay systems in Banach spaces with the help of the Nussbaum fixed point theorem. In this chapter we establish a set of sufficient conditions for the controllability of neutral functional evolution integrodifferential systems in abstract spaces. The results are established by using the Nussbaum fixed point theorem and generalize the results of [8,36,91].

4.2. PRELIMINARIES

Consider the neutral integrodifferential system of the form

\[
\frac{d}{dt}[x(t) - h(t, x_t)] = A(t)x(t) + f(t, x_t) + Bu(t) + \int_{-\infty}^{t} g(t, s, x_s)ds, \quad t \geq 0, \quad x_0 = \phi \in \Omega,
\]

(4.1)

where \( \Omega \) is an open subset of a phase space \( \mathcal{B} \), the state variable \( x(\cdot) \) takes values in the Banach space \( X \) with norm \( \| \cdot \| \), \( x_t \) represents the function \( x_t : (-\infty, 0] \to X \)
defined by \( x_t(\theta) = x(t + \theta), -\infty < \theta < 0 \) which belongs to \( B \), the control function \( u(\cdot) \) is given in \( L^2(J, U) \), a Banach space of admissible control functions with \( U \) as a Banach space and \( J = [0, a] \) and \( B \) is a bounded linear operator from \( U \) into \( X \). The nonlinear operators \( h : J \times \Omega \to X, f : J \times \Omega \to X \) and \( g : J \times J \times \Omega \to X \) are continuous. Here \( A(t) : D(A(t)) \to X \) is the infinitesimal generator of an analytic semigroup.

Throughout this Chapter, \( X \) will be a Banach space with norm \( \| \cdot \| \). For the family \( \{A(t) : 0 \leq t \leq a\} \) of linear operators, we assume the following hypotheses:

(B1) The domain \( D(A) \) of \( \{A(t) : 0 \leq t \leq a\} \) is dense in the Banach space \( X \) and independent of \( t \). \( A(t) \) is a closed linear operator.

(B2) For each \( t \in [0, a] \), the resolvent \( R(\lambda, A(t)) = (\lambda I - A(t))^{-1} \) of \( A(t) \) exists for all \( \lambda \) with \( Re \lambda \leq 0 \) and \( \|R(\lambda, A(t))\| \leq C(|\lambda| + 1)^{-1} \).

(B3) For any \( t, s, r \in [0, a] \), there exist \( 0 < \delta < 1 \) and \( K > 0 \) so that

\[
\| (A(t) - A(\tau)) A^{-1}(s) \| \leq K |t - \tau|^\delta.
\]

Statements (B1)-(B3) imply that there exists a family of bounded linear operators \( \{\Phi(t, s) : 0 \leq s \leq t \leq T\} \) with \( \|\Phi(t, s)\| \leq C|t - s|^\delta-1 \) such that the operator valued function \( W(t, \tau) \) can be defined \( 0 \leq \tau \leq t \leq T \) by

\[
W(t, \tau) = e^{-\tau A(t)} + \int_\tau^t e^{-(t-s)A(s)} \Phi(s, \tau) ds.
\]

Here the family of linear operators \( \{e^{-\tau A(t)} : \tau \geq 0\} \) represents the analytic semigroup generated by \( -A(t) \). Assumption (B2) guarantees that \( A(t) \) generates an analytic semigroup. The family of linear operators \( \{W(t, \tau) : 0 \leq \tau \leq t \leq T\} \) is strongly jointly continuous in \( \tau \) and \( t \) and maps \( X \) into \( D(A) \) if \( t > \tau \).

Conditions (B1) - (B3) imply that, for each \( t \in [0, T] \), the integral

\[
(-A)^{-\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-s A(t)} ds
\]

exists for each \( \alpha \in (0, 1] \). The operator defined by (4.2) is a bounded linear operator and \( (-A)^{-\alpha}(t)(-A)^{-\beta}(t) = (-A)^{-(\alpha+\beta)}(t) \). The operator \( (-A)^\alpha(t) = \)
$((\mathcal{A}^{-\alpha(t)})^{-1}$ is a closed linear operator with $D((\mathcal{A}^{-\alpha(t)})$ dense in $X$ and $D((\mathcal{A}^{\alpha(t)}) \subset D((\mathcal{A}^{\beta(t)})$ if $\alpha \geq \beta$. The operator $(\mathcal{A}^{-\alpha(t)})$ generates a strongly continuous semigroup of bounded operators and $(\mathcal{A}^{-\alpha(t)}) = [(\mathcal{A}^{-1(t)})]^{\alpha}$ for positive integer $\alpha$.

Further, for any $\alpha > 0$, we set $(-\mathcal{A})^{a(t)} = [(\mathcal{A}^{-\alpha(t)})^{-1}$. This definition is reasonable since from the equality $(-\mathcal{A})^{a(t)}v = 0$, it follows that $v = 0$ and thus there exists an inverse $[(\mathcal{A}^{-\alpha(t)})]^{-1}$. The last is clear for integer $\alpha$. If $\alpha$ is fractional and $[\alpha]$ is its integral part, then

$$(-\mathcal{A})^{[\alpha]^{-1}(t)v = (-\mathcal{A})^{[\alpha]^{-1}+\alpha(t)}(-\mathcal{A})^{-\alpha(t)}v = 0$$

and consequently $v = 0$, since $[\alpha] + 1$ is an integer.

For $\alpha > 0$, the operators $(-\mathcal{A})^{\alpha(t)}$ are no longer bounded. They have domains of definition $D([-\mathcal{A}]^{\alpha(t)}])$ dense in $X$ where $D([-\mathcal{A}]^{\alpha(t)}) \subset D([-\mathcal{A}]^{\beta(t)})$ for $\alpha \geq \beta$. Further $D([-\mathcal{A}]^{\alpha(t)})$ is a Banach space with the norm $\|x\|_{\alpha} = \|(-\mathcal{A})^{\alpha(t)}x\|$ and it is denoted by $X_{\alpha}(t)$. For any $\alpha, \beta$ and for $v \in D((-\mathcal{A})^{\gamma(t)})$, where $\gamma = \max\{\alpha, \beta, \alpha + \beta\}$, the identity $(-\mathcal{A})^{\alpha(t)}(-\mathcal{A})^{\beta(t)}v = (-\mathcal{A})^{\beta(t)}(-\mathcal{A})^{\alpha(t)}v = (-\mathcal{A})^{\alpha+\beta(t)}v$ holds. Finally, for any $\alpha < \beta < \gamma$, an inequality of moments

$$\|(-\mathcal{A})^{\beta\gamma(t)}v\| \leq C(\alpha, \beta, \gamma)\|(-\mathcal{A})^{\gamma(t)}v\|^{\frac{\beta-\alpha}{\gamma-\alpha}}(-\mathcal{A})^{\alpha}(t)v\|^{\frac{\gamma-\beta}{\gamma-\alpha}}, \quad (v \in D((-\mathcal{A})^{\gamma(t)})$$

holds. Finally (B1) implies that $((-\mathcal{A})^{-\beta(t)}$ is compact for all $\beta > 0$ and the inclusion $X_{\alpha}(t) \subset X_{\beta}(t)$ is compact for $\alpha > \beta > 0$. The results given above for semigroups of linear operators, evolution systems and fractional powers of operators can be found in Friedman [35], Pazy [78] and Sobolevskii [88].

The family $\{\mathcal{A}(t) : 0 \leq t \leq a\}$ generates a unique linear evolution system $\{U(t, s) : 0 \leq s \leq t \leq a\}$ satisfying the following properties:

(a) $U(t, s) \in L(X)$, the space of bounded linear transformations on $X$ whenever $0 \leq s \leq t \leq a$ and for each $x \in X$, the mapping $(t, s) \rightarrow U(t, s)x$ is continuous.

(b) $U(t, s)U(s, \tau) = U(t, \tau)$ for $0 \leq \tau, s \leq t \leq a$.

(c) $U(t, t) = I$. 

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For the evolution system \( \{U(t, s) : 0 \leq s \leq t \leq a\} \), the following properties are well known.

(i) for any \( t, r \in [0, a] \) and \( \epsilon > 0 \),
\[
\|(-A)^\alpha(t)U(t, \tau)(-A)^{-\beta}(\tau)\| \leq C|t - \tau|^{\beta - \alpha}
\]
for \( 0 \leq \beta \leq \alpha \leq 1 + \epsilon \).

(ii) there is a constant \( M_0 > 0 \) such that \( \|(-A)^{-\beta}(s)\| \leq M_0 \).

(iii) there is a constant \( M_1 > 0 \) such that \( \|U(t, s)\| \leq M_1 \) for all \( 0 \leq t \leq a \).

(iv) the family of operators \( \{U(t, s), t > s\} \) is continuous in \( t \) in the uniform operator topology uniformly for \( s \).

We need the following fixed point theorem due to Nussbaum [75].

**The Nussbaum Fixed Point Theorem** : Let \( S \) be a closed, bounded and convex subset of a Banach space \( X \). Let \( \Phi_1, \Phi_2 \) be a continuous mappings from \( S \) into \( X \) such that

(i) \( (\Phi_1 + \Phi_2)S \subset S \).

(ii) \( \|\Phi_1\omega_1 - \Phi_1\omega_2\| \leq k\|\omega_1 - \omega_2\| \) for all \( \omega_1, \omega_2 \in S \) where \( K \) is a constant and \( 0 \leq k \leq 1 \).

(iii) \( \overline{\Phi_2(S)} \) is compact.

Then the operator \( \Phi_1 + \Phi_2 \) has a fixed point in \( S \).

Let \( B_r[x] \subset \Omega \) be a closed ball centered at \( x \) with radius \( r > 0 \). We shall make the following assumptions on the system (4.1):

(H1) \( h : J \times B \to X \) is a continuous function and there exist constants \( L_1, L_2 > 0 \) such that the function \( (-A)^\beta(s)h \) satisfies the Lipschitz condition
\[
\|(-A)^\beta(s)h(s_1, \psi_1) - (-A)^\beta(s)h(s_2, \psi_2)\| \leq L_1[|s_1 - s_2| + \|\psi_1 - \psi_2\|],
\]
for every \( 0 < s_1, s_2 < a \) and \( \psi_1, \psi_2 \in \Omega \).
(H2) \((-A)^{p}(s)h\) and \(f\) are continuous and there exist constants \(M_{2}, M_{3} > 0\) such that
\[
\|(-A)^{p}(s)h(s, \psi)\| \leq M_{2} \quad \text{and} \quad \|f(t, \psi)\| \leq M_{3},
\]
for every \(0 \leq s \leq t \leq a\) and \(\psi \in B_{r}[\phi]\).

(H3) \(g: J \times J \times B \to X\) is continuous and there exists \(N_{1} > 0\) such that
\[
\|g(t, s, \eta)\| \leq N_{1},
\]
for every \(0 \leq s \leq t \leq a\) and \(\eta \in B_{r}[\phi]\).

(H4) For each \(\phi \in B\),
\[
q(t) = \lim_{a \to \infty} \int_{-a}^{0} g(t, s, \phi(s))ds
\]
exists and is continuous. Further, there exists a positive constant \(N_{2} > 0\) such that \(\|q(t)\| \leq N_{2}\).

(H5) There is a compact set \(V \subseteq X\) such that \(U(t, s)f(s, \psi), U(t, s)Bu(s), U(t, s)g(s, \tau, \eta)\) and \(U(t, s)q(s) \in V\) for all \(\psi, \eta \in B_{r}[\phi]\) and \(0 \leq \tau \leq s \leq t \leq a\).

(H6) The linear operator \(W\) from \(L^{2}(J, U)\) into \(X\) defined by
\[
Wu = \int_{0}^{a} U(a, s)Bu(s)ds
\]
induces an invertible operator \(\tilde{W}\) defined on \(L^{2}(J; U)/KerW\) and there exists a positive constant \(K_{1} > 0\) such that \(\|B\tilde{W}^{-1}\| \leq K_{1}\).

Let \(y(\cdot): (-\infty, a) \to X\) be the function defined by
\[
y(t) = \begin{cases} \frac{U(t, 0)\phi(0)}{0 \leq t \leq a}, \\ \phi(t), & \text{if } -\infty < t < 0. \end{cases}
\]
Then \(y_{0} = \phi\) and the map \(t \to y_{t}\) is continuous and for fixed \(\epsilon > 0\), the following conditions hold:

(i) \(\|y_{t} - \phi\|_{B} \leq \epsilon\).
(ii) \(\|(U(t, 0) - I)g(t, y_{t})\| \leq \epsilon\).
For simplicity, let us take

\[
K^* = \max\{K(t) : 0 \leq t \leq a\}, \quad \omega = (1 + aM_1K_1)M_0L_1K^*,
\]

\[
L^* = K^*M_0L_1\left(1 + \frac{a^{\beta-\alpha}}{\beta - \alpha}\right)
\]

and

\[
M^* = a\left[(1 + aM_1K_1)M_0M_1L_1 + M_1K_1\|\phi\|
+ M_1\|\phi(0)\| + aM_1(M_3 + N_2 + aN_1) + M_0M_2C \frac{a^{\beta-\alpha}}{\beta - \alpha}\right]
+ M_1(M_3 + N_2 + aN_1) + (1 + aM_1K_1)(M_0M_1L_1 + 1)\epsilon
+ M_0M_2C \frac{a^{\beta-\alpha}}{\beta - \alpha}.
\]

Let \( k = \frac{r - \epsilon}{K^*} \). Assume further that

(H7)

(i) \( 0 \leq L^* < 1 \).

(ii) \( M^* \leq (1 - \omega)k \).

Then the mild solution of the system (4.1) is given by

\[
x(t) = U(t,0)[\phi(0) - h(0,\phi)] + h(t,x_t) + \int_0^t U(t,s)A(s)h(s,x_s)ds
+ \int_0^t U(t,s)\left[Bu(s) + q(s) + f(s,x_s) + \int_0^s g(s,\tau,x_\tau)d\tau\right]ds.
\]

(4.3)

**Definition 4.1.** The system (4.1) is said to be **controllable** on the interval \( J \) if for every initial function \( \phi \in \Omega \) and \( x_1 \in X \), there exists a control \( u \in L^2(J,U) \) such that the solution \( x(\cdot) \) of (4.1) satisfies \( x(a) = x_1 \).

**4.3. MAIN RESULTS**

**Theorem 4.1.** If the hypotheses (H1)-(H7) are satisfied, then the system (4.1) is controllable on \( J \).
Proof. Using the hypotheses \((H6)\) for an arbitrary function \(x(\cdot)\), define the control
\[
u(t) = \tilde{W}^{-1} \left[ x_1 - U(a, 0)(\phi(0) - h(0, \phi)) - h(a, x_a) - \int_0^a U(a, s)A(s)h(s, x_a)ds \right.
- \int_0^a U(a, s)f(s, x_a)ds - \int_0^a U(a, s)q(s)ds
- \left. \int_0^a U(a, s) \int_0^s g(s, \tau, x_\tau) d\tau ds \right](t).
\]
Define the set
\[
B_k = \{ z \in C(J; X) : z(0) = 0, \|z(t)\| \leq k, \ 0 \leq t \leq a \}.
\]
Clearly \(B_k\) is a nonempty, bounded, convex and closed set in \(C(J; X)\). For each \(z \in C(J; X)\) with \(z(0) = 0\), we denote by \(\bar{z}\) the function defined by
\[
\bar{z}(t) = \begin{cases} 
    z(t), & 0 \leq t \leq a, \\
    0, & -\infty < t < 0.
\end{cases}
\]
If \(x(t)\) satisfies \((4.3)\), then we can decompose it as \(x(t) = z(t) + y(t), 0 \leq t \leq a\), which implies that \(x_t = \bar{z}_t + y_t\), for every \(0 \leq t \leq a\) and that the function \(z(t)\) satisfies
\[
z(t) = -U(t, 0)h(0, \phi) + h(t, \bar{z}_t + y_t) + \int_0^t U(t, s)A(s)h(s, \bar{z}_s + y_s)ds
+ \int_0^t U(t, s)B\tilde{W}^{-1} \left[ x_1 - U(a, 0)(\phi(0) - h(0, \phi)) - h(a, \bar{z}_a + y_a) 
- \int_0^a U(a, s)A(s)h(s, \bar{z}_s + y_s)ds - \int_0^a U(a, s)f(s, \bar{z}_s + y_s)ds
- \int_0^a U(a, s)q(s)ds - \int_0^a U(a, s) \int_0^s g(s, \tau, \bar{z}_\tau + y_\tau) d\tau ds \right]d\eta
+ \int_0^t U(t, s) \left[ f(s, \bar{z}_s + y_s) + q(s) + \int_0^s g(s, \tau, \bar{z}_\tau + y_\tau) d\tau \right]ds.
\]
Define the operator \(\Psi = \Psi_1 + \Psi_2\) on \(B_k\) by
\[
\Psi_1 z(t) = -U(t, 0)h(0, \phi) + h(t, \bar{z}_t + y_t) + \int_0^t U(t, s)A(s)h(s, \bar{z}_s + y_s)ds
\]
and
\[
\Psi_2 z(t) = \int_0^t \hat{U}(t, s)B\tilde{W}^{-1} \left[ x_1 - U(a, 0)(\phi(0) - h(0, \phi)) - h(a, \bar{z}_a + y_a) 
- \int_0^a U(a, s)A(s)h(s, \bar{z}_s + y_s)ds - \int_0^a U(a, s)f(s, \bar{z}_s + y_s)ds
- \int_0^a U(a, s)q(s)ds - \int_0^a U(a, s) \int_0^s g(s, \tau, \bar{z}_\tau + y_\tau) d\tau ds \right]d\eta
+ \int_0^t U(t, s) \left[ f(s, \bar{z}_s + y_s) + q(s) + \int_0^s g(s, \tau, \bar{z}_\tau + y_\tau) d\tau \right]ds.
\]
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Now we shall show that the operator $\Psi$ has a fixed point. In order to apply the Nussbaum fixed point theorem for the operator $\Psi$, we prove the following assertions:

(i) $\Psi_1$ and $\Psi_2$ are well defined.

(ii) $\Psi_1$ satisfies the Lipschitz condition.

(iii) $\Psi_2$ is relatively compact.

(iv) $(\Psi_1 + \Psi_2)B_k \subseteq B_k$.

We can easily see that if $z(\cdot) \in B_k$, then $\bar{z}_t + y_t \in B_r[\phi]$ for all $0 \leq t \leq a$. From (A1) and (H7) we see that

$$|\bar{z}_t + y_t - \phi|_B \leq |\bar{z}_t|_B + |y_t - \phi|_B \leq k^*k + \epsilon \leq r - \epsilon + \epsilon = r.$$

Now, for $0 \leq t \leq a$,

$$||\Psi_1 z(t)|| \leq || - U(t,0)h(0,\phi) + h(t,\bar{z}_t + y_t) + \int_0^t U(t,s)A(s)h(s,\bar{z}_s + y_s)ds||$$

$$\leq ||U(t,0)(-A)^{-\beta}(s)[(-A)^{\beta}(s)h(t,\bar{z}_t + y_t) - (-A)^{\beta}(s)h(0,\phi)]||$$

$$+ ||(I - U(t,0))h(t,y_t)|| + ||(-A)^{-\beta}(s)|| ||(-A)^{\beta}(s)h(t,\bar{z}_t + y_t) - (-A)^{\beta}(s)h(t,y_t)||$$

$$+ \int_0^t ||(-A)^{-\alpha}(t)|| ||(-A)^{\alpha}(t)U(t,s)(-A)^{2-\beta}(s)||$$

$$\times ||(-A)^{\beta}(s)h(s,\bar{z}_s + y_s)||ds$$

$$\leq ||(-A)^{-\beta}(s)|| ||M_1L_1(t + \epsilon)|| + \epsilon + ||(-A)^{-\beta}(s)|| ||L_1K^*k$$

$$+ \int_0^t ||(-A)^{-\alpha}(t)|| |C|t - s|^{\beta - \alpha - 1}M_2ds$$

$$\leq M_0||M_1L_1(a + \epsilon)|| + \epsilon + M_0L_1K^*k + M_0M_2C\frac{a^{\beta - \alpha}}{\beta - \alpha}$$

and

$$||\Psi_2 z(t)|| \leq ||\int_0^t U(t,\eta)B\tilde{W}^{-1} [x_1 - U(a,0)[\phi(0) - h(0,\phi) - h(a,\bar{z}_a + y_a)$$

$\quad - \int_0^a U(a,s)A(s)h(s,\bar{z}_s + y_s)ds - \int_0^a U(a,s)f(s,\bar{z}_s + y_s)ds]ds$$

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Thus we have

\[ \| \Psi z(t) \| \leq \| \Psi_1 z(t) \| + \| \Psi_2 z(t) \| \]

\[ \leq M_0 M_1 L_1 (a + \epsilon) + \epsilon + M_0 L_1 K^* k + M_0 M_2 C \frac{a^{\beta - \alpha}}{\beta - \alpha} + aM_1 (M_3 + N_2 + aN_1) + aM_1 (M_3 + N_2 + aN_1). \]
\[ \begin{align*}
&+CM_0 M_2 \frac{\alpha^2 - \alpha}{\beta - \alpha} + aM_1 (M_3 + N_2 + aN_1) + aM_1 (M_3 + N_2 + aN_1) \\
\leq & (1 + aM_1 K_1) M_0 L_1 K^* k + a \left[ (1 + aM_1 K_1) M_0 M_1 L_1 + M_1 K_1 \right] \left[ \| x_1 \| \\
&+M_1 \| \phi(0) \| + aM_1 (M_3 + N_2 + aN_1) + M_0 M_2 C \frac{\alpha^2 - \alpha}{\beta - \alpha} \right] \\
&+M_1 (M_3 + N_2 + aN_1) \\
&+ (1 + aM_1 K_1) (M_0 M_1 L_1 + 1) \varepsilon + M_0 M_2 C \frac{\alpha^2 - \alpha}{\beta - \alpha} \\
\leq & \omega k + (1 - \omega) k = k.
\end{align*} \]

Hence \( \Psi(B_k) \subseteq B_k \). Next we shall show that the operator \( \Psi_1 \) satisfies the Lipschitz condition. Let \( v, w \in B_k \) and, for each \( 0 \leq t \leq a \), we have

\[ \begin{align*}
\| \Psi_1 v(t) - \Psi_2 w(t) \| & \leq \| h(t, \bar{v}_t + y_t) - h(t, \bar{w}_t + y_t) \| \\
&+ \int_0^t \| \frac{(-A)^{-\alpha}(t)}{\|(-A)^{\alpha}(t)\|} \| (A)^{\alpha}(t) U(t, s)(-A)^{1-\beta}(s) \| \\
&\times \| (-A)^{\beta}(s)[h(s, \bar{v}_s + y_s) - h(s, \bar{w}_s + y_s)] \| ds \\
&\leq \| (A)^{-\alpha}(t) \| \| (A)^{\alpha}(t) \| (A)^{\beta}(s)[h(t, \bar{v}_t + y_t) - h(t, \bar{w}_t + y_t)] \\
&+ \| (A)^{-\alpha}(t) \| \int_0^t C \| t - s \|^\alpha L_1 |\bar{v}_s - \bar{w}_s| ds \\
&\leq M_0 L_1 |\bar{v}_t - \bar{w}_t| + CM_0 L_1 |\bar{v}_s - \bar{w}_s| B \frac{\alpha^2 - \alpha}{\beta - \alpha} \\
&\leq \left[ K^* M_0 L_1 + K^* M_0 L_1 C \frac{\alpha^2 - \alpha}{\beta - \alpha} \right] \sup_{0 \leq s \leq a} \| v(s) - w(s) \| \\
&\leq L^* \sup_{0 \leq s \leq a} \| v(s) - w(s) \|. \quad (4.4)
\end{align*} \]

Thus

\[ \| \Psi_1 v(t) - \Psi_2 w(t) \| \leq L^* \| v - w \| \]

and so \( \Psi_1 \) satisfies Lipschitz condition with \( L^* < 1 \).

Finally, we prove that \( \Psi_2 \) is relatively compact in \( B_k \). To prove this, first we shall show that \( \Psi_2 \) maps \( B_k \) into a precompact subset of \( B_k \). We now show that, for every fixed \( t \in J \), the set \( V(t) = \{ \Psi_2 z(t) : z \in B_k \} \) is precompact in \( X \). Let \( 0 < t \leq a \) be fixed, \( 0 < \epsilon < t \); for \( z \in B_k \), we define

\[ \begin{align*}
\Psi_2 z(t) &= \int_0^{t-\epsilon} U(t, \eta) B \tilde{W}^{-1} [x_1 - U(a, 0)[\phi(0) - h(0, \phi)] - h(a, z_\alpha + y_\alpha) \\
&\quad - \int_0^a U(a, s) A(s) h(s, z_\alpha + y_\alpha) ds - \int_0^a U(a, s) f(s, z_\alpha + y_\alpha) ds
\end{align*} \]
\[-\int_0^t U(a, s)q(s)ds - \int_0^t U(a, s) \int_0^s g(s, \tau, \bar{z}_\tau + y_\tau)d\tau ds \right] d\eta \]
\[+ \int_{t-\epsilon}^t U(t, s) \left[ f(s, \bar{z}_s + y_s) + q(s) + \int_0^s g(s, \tau, \bar{z}_\tau + y_\tau)d\tau \right] ds. \]

Now by the assumption \((H5)\), the set \(V_\epsilon(t) = \{\Psi_2z(t) : z \in B_k\}\) is relatively compact in \(X\) for every \(\epsilon, 0 < \epsilon < t\). Moreover, for \(z \in B_k\), we have

\[\|\Psi_2z(t) - \Psi_2z(t)\| \leq \int_{t-\epsilon}^t \|U(t, \eta)\| \|B^W\| \|x_1\| + \|U(a, 0)\phi(0)\|
\]
\[-\int_0^a \|U(a, s)\| \|A(s)\| h(a, \bar{z}_a + y_a)ds - \int_0^a \|U(a, s)\| f(s, \bar{z}_s + y_s)ds \]
\[-\int_0^a \|U(a, s)\| q(s)ds - \int_0^a \|U(a, s)\| \int_0^s g(s, \tau, \bar{z}_\tau + y_\tau)d\tau ds \right] d\eta \]
\[+ \int_{t-\epsilon}^t \|U(t, s)\| \left[ f(s, \bar{z}_s + y_s) + q(s) + \int_0^s g(s, \tau, \bar{z}_\tau + y_\tau)d\tau \right] ds \]
\[\leq \epsilon M_1 K_1 \|x_1\| + M_1 \|\phi(0)\| + M_0 M_1 L_1(a + |\phi - y_a|) + \epsilon
\]
\[+ M_0 K^* L k + M_0 \int_0^a C M_2 |t - s|^\beta - 1 - \alpha ds + a M_1 (M_3 + N_2 + a N_1) \]
\[+ \epsilon M_1 (M_3 + N_2 + a N_1). \]

Since there are precompact sets arbitrarily close to the set \(V(t)\), the set \(V(t)\) is also precompact in \(X\). We now show that the image of \(B_k\)

\[\Psi_2(B_k) = \{\Psi_2z : z \in B_k\}\]

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is an equicontinuous family of functions. Let $0 < t_1 < t_2$. Then

$$
\|\Psi_2z(t_1) - \Psi_2z(t_2)\|
\leq \int_0^{t_1} \left| U(t_1, \eta) - U(t_2, \eta) \right| B\tilde{W}^{-1} \left[ x_1 - U(a, 0)(\phi(0) - h(0, \phi)) 
- h(a, \bar{z}_a + y_a) - \int_0^a U(a, s)A(s)h(s, \bar{z}_s + y_s)ds - \int_0^a U(a, s)q(s)ds 
- \int_0^a U(a, s)f(s, \bar{z}_s + y_s)ds - \int_0^a \int_0^s U(a, s)g(s, \tau, \bar{z}_\tau + y_\tau)d\tau ds \right] d\eta 
\leq \int_0^{t_1} \left\| U(t_1, \epsilon) - U(t_2, \epsilon) \right\| \left\| U(\epsilon, \eta) \right\| K_1 \left[ \|x_1\| + M_1\|\phi(0)\| + M_0M_1L_1(a + \epsilon) + \epsilon 
+ M_0L_1K^*k + CM_0M_2\frac{\alpha^{\beta-\alpha}}{\beta - \alpha} + aM_1(M_3 + N_2 + aN_1) \right] d\eta 
+ (t_1 - t_2)M_1K_1 \left[ \|x_1\| + M_1\|\phi(0)\| + M_0M_1L_1(a + \epsilon) + \epsilon + M_0L_1K^*k 
+ CM_0M_2\frac{\alpha^{\beta-\alpha}}{\beta - \alpha} + aM_1(M_3 + N_2 + aN_1) \right] 
+ \int_0^{t_1} \left\| U(t_1, \epsilon) - U(t_2, \epsilon) \right\| \left\| U(\epsilon, s) \right\| \left[ M_3 + N_2 + aN_1 \right] ds 
+ (t_1 - t_2)M_1(M_3 + N_2 + aN_1).
$$

Since $h(s, \bar{z}_s + y_s)$ is continuous and $U(\epsilon, s)f(s, \bar{z}_s + y_s), U(\epsilon, s)g(s, \tau, \bar{z}_\tau + y_\tau)$ and $U(\epsilon, s)q(s)$ are included in the compact set $V$ for all $0 \leq s \leq a$ and all $z \in B_k$, the functions $U(\cdot)z$ for $z \in V$ are equicontinuous. Hence $\Psi_2(B_k)$ is an equicontinuous family of functions. Also $\Psi_2(B_k)$ is bounded in $C(J, X)$ and so, by the Arzela-Ascoli theorem, $\Psi_2(B_k)$ is precompact. Hence it follows from the Nussbaum fixed point theorem [75], there exists a fixed point $z \in B_k$ such that $\Psi z(t) = z(t)$. Since we have $x(t) = z(t) + y(t)$, it follows that $x(t)$ is a mild solution of (4.1) on $[0, a]$ satisfying $x(a) = x_1$. Thus the system (4.1) is controllable on $J$. 

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4.4. EXAMPLE

Consider the following partial integro-differential equation of the form

\[
\frac{\partial}{\partial t}[z(t, x) + \int_{-\infty}^{t} \int_{0}^{\pi} a(s - t, \eta, x)z(s, \eta)d\eta ds] = a(t, x) \frac{\partial^2}{\partial x^2} z(t, x) + \mu(t, x) + k_0(x)z(t, x) \\
+ \int_{-\infty}^{t} k(s - t)z(s, x)ds,
\]

(4.5)

\[
z(t, 0) = z(t, \pi) = 0, \quad t \in J = [0, a], \\
z(t, x) = \phi(t, x), \quad t \leq 0, \quad 0 \leq x \leq \pi,
\]

where \(z_t - a(t, x)z_{xx}\) is a uniformly parabolic differential operator with \(a(t, x)\) continuous on \(0 \leq x \leq \pi, 0 \leq t \leq a\) and is uniformly Holder continuous in \(t\).

Let us take \(X = U = L^2(0, \pi), x(t) = z(t, \cdot)\) and \(u(t) = \mu(t, \cdot)\) where \(\mu : J \times [0, \pi] \rightarrow [0, \pi]\) is continuous and \(A(t)\) be defined by

\[
A(t)z = -a(t, x)z''
\]

with the domain

\[
D(A) = \{z(\cdot) \in X : z, z' \text{ absolutely continuous}, \ z'' \in X, z(0) = z(\pi) = 0\}.
\]

Then \(A(t)\) generates an evolution system \(U(t, s)\) satisfying assumptions (B1)-(B3).

Let \(B\) denote the space \(C_r \times L^2(g, X)\) with \(r = 0\), as defined in [54]. It is clear that \(B\) is isomorphic and isometric to the space \(X \times L^2_\mu((-\infty, 0] \times [0, \pi])\) where \(\mu\) is the measure \(\mu(\theta, \tau) = g(\theta)d\theta d\tau\). Next we assume that the following conditions hold:

(a) The function \(a(\cdot)\) is measurable and

\[
\int_{0}^{\pi} \int_{-\infty}^{0} \int_{0}^{\pi} (a^2(\theta, \eta, \xi)/g(\theta))d\eta d\theta d\xi < \infty.
\]

(b) The function \(\frac{\partial}{\partial \psi} a(\theta, \eta, \psi)\) is measurable; \(a(\theta, \eta, \pi) = 0; \ a(\theta, \eta, 0) = 0\) and

\[
N_1^* = \int_{0}^{\pi} \int_{-\infty}^{0} \int_{0}^{\pi} \frac{1}{g(\theta)} \left(\frac{\partial}{\partial \psi} a(\theta, \eta, \psi)\right)^2 d\eta d\theta d\psi < 1.
\]

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(c) The function $k_0 \in L^\infty([0, \pi])$.

(d) The function $k(\cdot)$ is measurable and
\[ \int_{-\infty}^{0} \frac{k^2(\theta)}{g(\theta)} d\theta \leq \infty. \]

(e) The function $\phi$ defined by $\phi(\theta)(\xi) = \phi(\theta, \xi)$ belongs to $\mathcal{B}$.

We define $h : [0, \infty) \times \mathcal{B} \to X$, $f : [0, \infty) \times \mathcal{B} \to X$ and $g : [0, \infty) \times [0, \infty) \times \mathcal{B} \to X$ by $h(t, \phi) = P_1(\phi)$, $f(t, \phi) = P_2(\phi)$ and $g(t, s, \phi) = P_3(\phi)$ where
\begin{align*}
P_1(\phi) &= \int_{-\infty}^{t} \int_{0}^{\pi} a(s - t, \eta, x) \phi(s, \eta) d\eta ds, \\
P_2(\phi) &= k_0(x) \phi(t, x), \\
P_3(\phi) &= \int_{-\infty}^{0} k(s - t) \phi(s, x) ds.
\end{align*}

From (a) and (c), it is clear that $P_1, P_2$ and $P_3$ are bounded linear operators on $\mathcal{B}$. Further $P_1(\phi) \in D((-A)^{1/2}(t_0))$ and $\|(-A)^{1/2}(t_0) P_1\| \leq N^*_1 < 1$.

For some $t_0 \in [0, a]$, the operator $A^{1/2}(t_0)$ can be defined on
\[ D(A^{1/2}(t_0)) = \{ z(\cdot) \in X : z \text{ absolutely continuous, } z' \in X, z(0) = z(\pi) = 0 \}. \]

by
\[ A^{1/2}(t_0) z = -(a(t_0, x))^{1/2} z'' \]
for $z \in D(A^{1/2}(t_0))$ (see [80]).

Assume that there exists an invertible operator $W^{-1}$ defined on $L^2(J, U)/\ker W$ by
\[ W u = \int_{0}^{b} U(a, s) B u(s) ds \]
and satisfies the condition (H6).

With this choice of $A(t)$, $h$, $f$, $g$ and $B = I$, the identity operator, we see that the equation (4.5) is an abstract formulation of (4.1). Also all the conditions stated in the Theorem 4.1 are satisfied and hence the system (4.5) is controllable on $J$. 