CHAPTER 1

UNSTEADY FLOW OF COUPLE STRESS FLUID IN CONTACT WITH A NEWTONIAN FLUID BETWEEN PERMEABLE BEDS
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1.1 INTRODUCTION

Viscous flow through and past porous media has important applications in various branches of engineering and medicine. The viscous fluids that move in biological systems are named as biofluids. Some of the biofluids are blood, saliva, tears, chyme, bile and gastric juice. As the behavior of these fluids is complex (sometimes Newtonian, sometimes non-Newtonian), several mathematical models are proposed to describe the biofluid flow in a biological system. One of the well known biofluids viz., blood is attracting the attention of researchers now a days. The study of blood flow in a stenotic artery requires the knowledge of the basic principles of biomechanics. The constitution of blood and its behavior suggest two fluid mathematical modeling in various situations. The study of such biofluid flows is useful in the design of various artificial physiological systems.

Stokes theory of couple stress fluid is the simplest generalization of the classical theory of fluids which allows the polar effects such as the presence of a non symmetric stress tensor, couple stress and body couples. The polar fluid theory is different from the other fluids theories in the angular momentum effects such as couple-stress and symmetric stress tensor. Both polar and dipolar fluids reduce to the theory of fluids with couple stress introduced by Stokes (1966). Based on the couple stress theory of Stokes, Valnis and Sun (1969) and Wang-Long Li (2003) have studied some theoretical models for blood flow through narrow tubes.

laminar, fully developed flow and heat transfer characteristics of two immiscible viscous and couple stress fluids in a vertical channel bounded by rigid walls. The problem of steady, laminar, fully developed flow and heat transfer in a horizontal channel consisting of a couple stress fluid sandwiched between two clear viscous fluids was discussed by Umavathi et al. (2005). Vajravelu, Sreenadh and Ramesh Babu (2006) investigated the peristaltic pumping of a Herschel – Bulkley fluid in contact with a Newtonian fluid. Ali and Hayat (2007) studied the peristaltic flow of a couple stress fluids in an asymmetric channel. Umavathi et al. (2007) also investigated laminar, fully developed flow and heat transfer of couple stress and viscous fluids in a vertical channel.

Even though a number of two fluid models are available to describe the behavior of blood, most of the works reported so far deal with the study of two fluid flows between rigid walls. In view of the structure of the physiological systems such as artery, it is necessary to study biofluid flows through permeable ducts. Vajravelu et al. (2009) studied the peristaltic transport of a Casson fluid in contact with a Newtonian fluid in a circular tube with permeable wall.

Rathod and Shakera Tanveer (2009) observed pulsatile flow of couple stress fluid through a porous medium with periodic body acceleration and magnetic field. It is found that velocity distribution increases with an increase of both body acceleration and permeability of the porous medium, while it decreases as the magnetic parameter increases.

Keeping in view the practical applications in the field of biomechanics and in the fluid models of mixture of Newtonian and non-Newtonian immiscible fluids, an attempt is made to analyze unsteady fully developed flow of couple stress and Newtonian fluids in a horizontal channel bounded by permeable beds. The flow region between the permeable beds is divided into two regions. Region 1 is occupied by a non-Newtonian couple stress fluid and Region 2 is filled by a Newtonian fluid. The solution for the unsteady problem is obtained using the method of Wang (1971). The results are discussed for various physical parameters of interest.
1.2 MATHEMATICAL FORMULATION OF THE PROBLEM

Consider the flow of two immiscible Newtonian and couple stress fluids between two permeable beds of different permeabilities $k_1$ and $k_2$ (Fig. 1.1). Let $h$ be the height of each of the fluid layers and let $\mu_i$ and $\mu_2$ be their corresponding viscosities and also $\rho_1$ and $\rho_2$ be their corresponding densities of the fluids. The flow region between the permeable beds is divided into two regions. The flow region between the nominal surface of the lower permeable bed $y = -h$ and the interface $y = 0$ is termed as Region 1 where as the flow region between the interface $y = 0$ and the nominal surface of the upper permeable bed $y = h$ is designated as Region 2. The flow in Region 1 is governed by couple stress model and the flow in Region 2 is governed by Navier-Stokes equations. The flow in the lower and upper permeable beds is governed by the generalized Darcy law. Let x-axis be taken along the interface and y-axis is chosen perpendicular to it.

The following assumptions are made in the analysis of the problem:

(a) The porous beds are homogeneous, isotropic and isothermal. The lower and upper beds are maintained at constant temperature $T_i$ and $T_u$ respectively.

(b) The flow in the x-direction is driven by an exponentially time dependent pressure gradient.

(c) The flow is unsteady and fully developed so that all physical characteristics except pressure are functions of $y$ and $t$ only.

(d) The velocity fields and the pressure distribution vary exponentially with time.

(e) The densities of the two fluids are equal to $\rho_0$ (i.e. $\rho_0 = \rho_1 = \rho_2$).
In view of these assumptions, the governing equations reduce to

Region 1

\[ \frac{\partial u_1}{\partial x} = 0 \quad (1.2.1) \]

\[ \rho \frac{\partial u_1}{\partial t} = -\frac{\partial p}{\partial x} + \mu_1 \frac{\partial^2 u_1}{\partial y^2} - \eta \frac{\partial^4 u_1}{\partial y^4} \quad (1.2.2) \]

\[ \rho \gamma \frac{\partial \delta T}{\partial t} = K_{f1} \frac{\partial^2 T}{\partial y^2} + \mu_1 \left( \frac{\partial u_1}{\partial y} \right)^2 \quad (1.2.3) \]
Region 2

\[ \frac{\partial u_2}{\partial x} = 0 \quad (1.2.4) \]

\[ \rho_0 \frac{\partial u_2}{\partial t} = -\frac{\partial p}{\partial x} + \mu_2 \frac{\partial^2 u_2}{\partial y^2} \quad (1.2.5) \]

\[ \rho_0 C_p \left[ \frac{\partial T_2}{\partial t} \right] = K_{T2} \frac{\partial^2 T_2}{\partial y^2} + \mu_2 \left[ \frac{\partial u_2}{\partial y} \right] \quad (1.2.6) \]

Lower permeable bed

\[ \frac{1}{\varepsilon_1} \frac{\partial Q_1}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} - \frac{\mu_1}{\rho_0 K_{T1}} Q_1 \quad (1.2.7) \]

Upper permeable bed

\[ \frac{1}{\varepsilon_2} \frac{\partial Q_2}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} - \frac{\mu_2}{\rho_0 K_{T2}} Q_2 \quad (1.2.8) \]

where \( u_1 \) and \( u_2 \) are velocity components in Regions 1 and 2, in lower and upper beds, respectively. Further \( \varepsilon_1, \varepsilon_2 \) are porosity parameters, \( t \) is the time, \( p \) is the pressure, \( \eta \) is the material constant, \( K_{T1} \) and \( K_{T2} \) are thermal conductivities, \( C_p \) is the specific heat at constant pressure in the Regions 1 and 2.
The boundary conditions are

\[ u_1 = u_{h_1}, \quad \frac{\partial u_1}{\partial y} = \frac{\alpha}{\sqrt{k_1}} (u_{h_1} - Q_1) \quad \text{at} \quad y = -h \]

\[ u_1 = u_{h_2}, \quad \mu_1 \frac{\partial u_1}{\partial y} - \eta \frac{\partial^3 u_1}{\partial y^3} = \mu_2 \frac{\partial u_2}{\partial y} \quad \text{at} \quad y = 0 \quad (1.2.9) \]

\[ u_2 = u_{h_2}, \quad \frac{\partial u_2}{\partial y} = -\frac{\alpha}{\sqrt{k_2}} (u_{h_2} - Q_2) \quad \text{at} \quad y = h \]

\[ \frac{\partial^2 u_1}{\partial y^2} = 0 \quad \text{at} \quad y = -h \]

\[ \frac{\partial^2 u_1}{\partial y^2} = 0 \quad \text{at} \quad y = 0 \]

\[ T_1 = T_{m}, \quad \text{at} \quad y = -h \]

\[ T_1 = T_2 \quad \text{at} \quad y = 0 \]

\[ \frac{\partial T_1}{\partial y} = \frac{\partial T_2}{\partial y} \quad \text{at} \quad y = 0 \quad (1.2.10) \]

\[ T_2 = T_{*} \quad \text{at} \quad y = h \]

Here \( u_{h_1} \) and \( u_{h_2} \) are the slip velocities at the permeable beds. \( T_{m} \) and \( T_{*} \) are the constant temperatures of the lower and the upper beds and \( \alpha \) is the slip parameter.
1.3 NON-DIMENSIONALIZATION OF THE FLOW QUANTITIES

It is convenient to introduce the following non-dimensional quantities.

\[ u_i^* = \frac{u_i}{u_{av}}, \quad y^* = \frac{y}{h}, \quad x^* = \frac{x}{h}, \quad t^* = \frac{t}{\sqrt{h^2/\nu}}, \quad v = \frac{\mu}{\rho_0} \]

\[ P = \frac{h^2}{\mu u_{av}} \left( \frac{\partial p}{\partial x} \right), \quad Pr = \frac{\rho_0 \nu c_p}{K_{T_1}} \quad \text{(Prandtl number)} \]

\[ \theta_i = \frac{T_i - T_{e_u}}{u_1^2 \frac{\mu_1}{k_1}} \]

\[ \Gamma = \frac{\mu h^2}{\eta} \quad \text{(Couple stress parameter)}, \quad \beta = \frac{\mu_1}{\mu_i}, \quad \kappa = \frac{K_{12}}{K_{T_1}}, \quad \sigma = \frac{h}{\sqrt{k_i}} \]

\[ Q'_i = Q_i, \quad (i=1,2) \]

In view of the above non-dimensional quantities, the basic equations (1.2.1) to (1.2.8) and the boundary conditions (1.2.9), (1.2.10) can be expressed in non-dimensional form, dropping asterisks, as

Region 1

\[ \frac{\partial u_i}{\partial x} = 0 \quad (1.3.2) \]

\[ \frac{\partial u_i}{\partial t} = -P + \frac{\partial^2 u_i}{\partial y^2} \frac{1}{\Gamma} \frac{\partial^4 u_i}{\partial y^4} \quad (1.3.3) \]

\[ \frac{\partial^2 \theta_i}{\partial y^2} - Pr \frac{\partial \theta_i}{\partial t} \left[ \frac{\partial u_i}{\partial y} \right]^2 = 0 \quad (1.3.4) \]
Region 2

\[ \frac{\partial u_2}{\partial x} = 0 \quad (1.3.5) \]

\[ \frac{\partial u_2}{\partial t} = -P + \beta \frac{\partial^2 u_2}{\partial y^2} \quad (1.3.6) \]

\[ K \frac{\partial^2 \theta}{\partial y^2} - Pr \frac{\partial \theta_2}{\partial t} + \beta \left( \frac{\partial u_1}{\partial y} \right)^2 = 0 \quad (1.3.7) \]

\[ \frac{1}{\varepsilon_1} \frac{\partial Q_1}{\partial t} = -P - \sigma_1^2 Q_1 \quad (1.3.8) \]

\[ \frac{1}{\varepsilon_2} \frac{\partial Q_2}{\partial t} = -\frac{1}{\beta} P - \sigma_2^2 Q_2 \quad (1.3.9) \]

The boundary conditions are

\[ u_1 = u_{b_1}, \quad \frac{\partial u_1}{\partial y} = \alpha \sigma_1 \left( u_{b_1} - Q_1 \right) \quad \text{at } y = -h \]

\[ u_1 = u_2, \quad \frac{\partial u_1}{\partial y} - \frac{1}{\Gamma} \frac{\partial^3 u_1}{\partial y^3} = \beta \frac{\partial u_2}{\partial y} \quad \text{at } y = 0 \]

\[ u_2 = u_{b_2}, \quad \frac{\partial u_2}{\partial y} = -\alpha \sigma_2 \left( u_{b_2} - Q_2 \right) \quad \text{at } y = h \]

\[ \frac{\partial^3 u_1}{\partial y^3} = 0 \quad \text{at } y = -h \]

\[ \frac{\partial^3 u_1}{\partial y^2} = 0 \quad \text{at } y = 0 \]
1.4 SOLUTION OF THE PROBLEM

In view of assumption (d), it follows that

\[ P = P_0 e^{ct}, \]
\[ u_1(y, t) = s_1(y) e^{ct}, \]
\[ u_2(y, t) = s_2(y) e^{ct}, \]
\[ u_{\beta_1} = s_{\beta_1} e^{ct}, \]
\[ u_{\beta_2} = s_{\beta_2} e^{ct}, \]
\[ Q_1 = s_{Q_1} e^{ct}, \]
\[ Q_2 = s_{Q_2} e^{ct}, \]
\[ \theta_1(y, t) = \theta_{10}(y) e^{2ct}, \]
\[ \theta_2(y, t) = \theta_{20}(y) e^{ct}, \]

Using (1.4.1), the governing equations (1.3.2) to (1.3.9) for non-porous and porous Regions can be written as
Region 1

\[
\frac{d^4 s_1}{dy^4} - \Gamma \frac{d^2 s_1}{dy^2} + c \Gamma s_1 + \Gamma P_0 = 0
\]  
(1.4.2)

\[
\frac{d^2 \theta_{10}}{dy^2} - 2 \Pr \alpha \theta_{10} = -\left[ \frac{ds_1}{dy} \right]^2
\]  
(1.4.3)

\[
s_{\theta_1} = -\frac{P_0}{c + \varepsilon_1 \sigma_1^2}
\]  
(1.4.4)

Region 2

\[
\frac{d^2 s_2}{dy^2} - \frac{c}{\beta} s_2 - \frac{P_0}{\beta} = 0
\]  
(1.4.5)

\[
\frac{d^2 \theta_{20}}{dy^2} - \frac{2 \Pr \alpha \theta_{20}}{K} + \frac{\beta}{K} \left[ \frac{ds_2}{dy} \right]^2 = 0
\]  
(1.4.6)

\[
s_{\theta_2} = -\frac{P_0}{\beta \left( c + \varepsilon_2 \sigma_2^2 \right)}
\]  
(1.4.7)

The corresponding boundary conditions are

\[
s_1 = s_{\theta_1}, \quad \frac{ds_1}{dy} = \alpha \sigma_1 \left( s_\theta - s_{\theta_1} \right) \quad \text{at} \quad y = -1
\]  
(1.4.8)

\[
s_1 = s_2, \quad \frac{ds_1}{dy} - \frac{1}{\Gamma} \frac{d^2 s_1}{dy^2} = \beta \frac{ds_2}{dy} \quad \text{at} \quad y = 0
\]  
(1.4.9)

\[
s_2 = s_{\theta_1}, \quad \frac{ds_2}{dy} = -\alpha \sigma_2 \left( s_\theta - s_{\theta_1} \right) \quad \text{at} \quad y = 1
\]  
(1.4.10)
Solving (1.4.2) to (1.4.7) subject to the conditions (1.4.8) to (1.4.16) we obtain the velocity and temperature fields as follows:

Region 1

\[ s_1 = A_1 \cosh m_1 y + A_2 \sinh m_1 y + A_3 \cosh m_2 y + A_4 \sinh m_2 y - \frac{P_0}{c} \]  

\[ s_{B1} = A_1 b_1 - A_2 b_2 + A_3 b_3 - A_4 b_4 - \frac{P_0}{c} \]  

\[ \theta_{10} = A_{10} \cosh n_1 y + A_{11} \sinh n_1 y - \{d_{20} \cosh 2m_1 y + d_{21} \cosh 2m_2 y \]  

\[ + d_{22} \sinh 2m_1 y + d_{23} \sinh 2m_2 y + d_{24} \cosh m_1 y + d_{25} \cosh m_2 y \]  

\[ + d_{26} \sinh m_1 y + d_{27} \sinh 2m_1 y + d_{28} \} \]
\[ s_2 = A_4 \cosh \sqrt{\frac{c}{\beta}} y + A_5 \sinh \sqrt{\frac{c}{\beta}} y - \frac{P_0}{c} \]  
(1.4.20)

\[ s_{B2} = A_5 b_{31} + A_6 b_{12} - \frac{P_0}{c} \]  
(1.4.21)

\[ \theta_{20} = A_{12} \cosh n_1 y + A_{13} \sinh n_2 y + d_{20} \cosh 2m_3 y + d_{30} \sinh 2m_3 y + d_{31} \]  
(1.4.22)

where

\[ m_1 = \sqrt{\lambda_1}, \quad \lambda_1 = \frac{\Gamma + \sqrt{\Gamma^2 - 4cl}}{2}, \quad m_2 = \sqrt{\lambda_2}, \quad \lambda_2 = \frac{\Gamma + \sqrt{\Gamma^2 - 4cl}}{2}, \]

\[ b_1 = \cosh m_1, \quad b_2 = \sinh m_1, \quad b_3 = \cosh m_2, \quad b_4 = \sinh m_2, \]

\[ b_5 = -(\alpha \sigma, b_1 + m_1 b_2), \quad b_6 = \alpha \sigma, b_2 + m_1 b_1, \quad b_7 = \alpha \sigma, b_3 + m_1 b_4, \]

\[ b_8 = \alpha \sigma, b_4 + m_1 b_5, \quad b_9 = -\alpha \sigma, \left[ \frac{P}{c} + s_{\chi_0} \right], \quad b_{10} = b_1 - b_3, \quad b_{11} = \frac{b_1 m_1^2}{m_1^2}, \]

\[ b_{12} = b_3 m_1^2 - b_1 m_1^2, \]

\[ b_{13} = b_6 m_2^2, \quad b_{14} = b_4 m_2^2, \quad b_{15} = b_4 m_1^2, \]

\[ b_{16} = m_2^2 b_3 b_5 - m_2^2 b_3 b_3, \quad b_{17} = m_2^2 b_4 b_5 - m_2^2 b_4 b_4, \quad b_{18} = m_2^2 b_4 b_5 - m_2^2 b_4 b_4, \]

\[ b_{19} = b_9 b_1 m_2^2, \quad b_{20} = b_1 m_1^2, \quad b_{21} = b_2 m_1^2, \]

\[ b_{22} = b_3 m_1^2, \quad b_{23} = b_4 m_2^2, \quad b_{24} = b_4 (b_3 + b_4), \]

\[ b_{25} = b_6 b_{15} + b_{16} b_1, \quad b_{26} = b_6 b_{15} + b_{16} b_1, \quad b_{27} = b_6 b_{15} + b_1 b_1, \]

\[ b_{28} = b_5 b_{22} + b_{20} b_1, \quad b_{29} = b_6 b_{22} - b_{21} b_1, \quad b_{30} = b_4 b_{22} - b_{23} b_1, \]

\[ b_{31} = b_9 b_{22}, \quad b_{32} = b_9 (b_{22} - b_{20}), \quad b_{33} = b_9 b_{22} + b_{21} b_{13}, \]

\[ 17 \]
\[ b_{34} = b_{16} b_{22} + b_{23} b_{15} \quad b_{35} = b_{17} b_{22} \quad b_{36} = b_{24} b_{20} - b_{28} b_{23} \]
\[ b_{37} = b_{26} b_{29} - b_{30} b_{25} \quad b_{38} = b_{27} b_{29} - b_{31} b_{25} \quad b_{39} = b_{24} b_{21} - b_{28} b_{23} \]
\[ b_{40} = b_{26} b_{33} - b_{34} b_{25} \quad b_{41} = b_{23} b_{33} - b_{35} b_{23} \quad b_{42} = b_{30} b_{40} - b_{39} b_{31} \]
\[ b_{43} = b_{38} b_{40} - b_{41} b_{37} \]
\[ b_{44} = \frac{b_{41}}{b_{42}} \]
\[ A_4 = b_{44} \]
\[ A_5 = \frac{m_1^2}{m_2^2} A_1 \]
\[ A_6 = \frac{1}{b_{16}} \left[ b_{17} - A_5 b_{15} \right] \]
\[ A_7 = \frac{1}{b_{10}} \left[ A_{10} b_{12} - A_4 b_{11} \right] \]
\[ n_1 = 2 \Pr c_1, \quad d_1 = \frac{A_1^2 m_1^2}{2}, \quad d_2 = \frac{A_1^2 m_1^2}{2}, \]
\[ d_3 = \frac{A_1^2 m_2^2}{2}, \quad d_4 = \frac{A_1^2 m_2^2}{2}, \quad d_5 = A_1 A_2 m_2^2, \]
\[ d_6 = A_3 A_4 m_2^2, \quad d_7 = m_1 m_2 \left( A_1 A_3 + A_2 A_4 \right), \quad d_8 = m_1 m_2 \left( A_1 A_4 - A_2 A_3 \right) \]
\[ d_9 = m_1 m_2 \left( A_1 A_4 + A_2 A_3 \right), \quad d_{10} = m_1 m_2 \left( A_1 A_4 - A_2 A_3 \right), \quad d_{11} = d_1 + d_2 \]
\[ d_{12} = d_1 + d_4, \quad d_{13} = -d_1 + d_2 - d_3 + d_4, \quad m_4 = m_1 + m_2 \]
\[ m_5 = m_1 - m_2 \]
\[ d_{20} = \frac{d_{11}}{4m_1^2 - n_1}, \quad d_{21} = \frac{d_{12}}{4m_2^2 - n_1} \]
\[ d_{22} = \frac{d_5}{4m_1^2 - n_1}, \quad d_{23} = \frac{d_6}{4m_2^2 - n_1} \]
\[ d_{24} = \frac{d_7}{m_1^2 - n_1}, \quad d_{25} = \frac{d_8}{m_2^2 - n_1} \]
\[ d_{26} = \frac{d_9}{m_1^2 - n_1}, \quad d_{27} = \frac{d_{10}}{m_2^2 - n_1} \]
\[ d_{28} = \frac{-d_{12}}{n_1}, \quad n_2 = \frac{2\rho c}{K}, \quad n_3 = -\frac{\beta}{K} \]
\[ d_{14} = \frac{A_2 m_1^2}{2} \quad d_{15} = \frac{A_5 m_2^2}{2} \quad d_{16} = A_4 \frac{d_{17}}{2} \]

\[ d_{17} = d_{14} + d_{15} \quad d_{18} = d_{17} - d_{14} \quad d_{29} = \frac{n_1 d_{17}}{4m_1^2 - n_2} \]

\[ d_{30} = \frac{n_2 d_{17}}{4m_2^2 - n_2} \quad d_{31} = -\frac{n_1 d_{17}}{n_2} \]

\[ d_{32} = -d_{29} \cosh 2m_1 - d_{21} \cosh 2m_2 + d_{22} \sinh 2m_1 + d_{23} \sinh 2m_2 - d_{24} \cosh m_4 - d_{25} \cosh m_5 + d_{26} \sinh m_4 + d_{27} \sinh m_5 - d_{28} \]

\[ d_{33} = d_{29} + d_{31} + d_{20} + d_{21} + d_{24} + d_{25} + d_{28} \quad d_{34} = d_{29} \cosh 2m_5 + d_{29} \sinh 2m_5 + d_{31} - 1 \]

\[ d_{35} = -2m_1 d_{22} - 2m_2 d_{33} - m_4 d_{26} - m_5 d_{17} - 2m_3 d_{30} \]

\[ d_{36} = \cosh n_1 \quad d_{37} = \sinh n_1 \quad d_{28} = \cosh n_2 \]

\[ d_{39} = \sinh n_2 \quad d_{40} = d_{35} d_{26} + d_{32} \quad d_{44} = \frac{n_2 d_{39}}{n_2} \]

\[ d_{42} = \frac{d_{35} d_{26} + n_2 d_{32}}{n_2} \quad d_{43} = -\left(d_{40} d_{41} + d_{42} d_{37}\right) \quad d_{44} = d_{35} d_{41} + d_{32} d_{37} \]

\[ A_{12} = \frac{d_{35}}{d_{44}} \quad A_{10} = A_{12} + d_{33} \quad A_{11} = \frac{1}{d_{37}} \left[A_{10} d_{36} + d_{32}\right] \quad A_{13} = \frac{1}{n_2} \left[A_{11} n_1 + d_{33}\right] \]

1.5 RATE OF HEAT TRANSFER

The rate of heat transfer (Nusselt number) through the channel wall to the fluid is (no dimensional form) given by.

\[ Nu = \left[ \frac{d \theta}{dy} \right]_{y=1,-1} \quad (1.5.1) \]
Based on the analytical solutions reported above the rate of heat transfer at the bottom wall is given by

\[ Nu_1 = \left[ \frac{d\theta_1}{dy} \right]_{y=-1} \]  

\[ = e^{2\alpha} \left\{ -n_1 A_{10} \sinh n_1 + n_1 \cosh n_1 \cosh 2m_1 \cosh 2m_2 - 2m_3 d_{13} \sinh 2m_1 \right\} \] 

\[ = e^{2\alpha} \left\{ -m_3 d_{13} \cosh m_3 - m_3 d_{13} \cosh m_3 \right\} \]  

At the top wall, it is given by

\[ Nu_2 = \left[ \frac{d\theta_2}{dy} \right]_{y=1} \]  

\[ = e^{2\alpha} \left\{ n_2 A_{12} \sinh n_2 + n_2 A_{13} \cosh n_2 + 2m_3 d_{23} \sinh 2m_3 + 2m_3 d_{23} \cosh 2m_3 \right\} \] 

\[ = e^{2\alpha} \left\{ -m_3 d_{13} \cosh m_3 - m_3 d_{13} \cosh m_3 \right\} \]  

1.6 MASS FLUX

The dimensionless mass flow rate per unit width of the channel is

\[ Q = Q_0 e^{\epsilon t} \]  

Where

\[ Q_0 = F_1 + F_2 \]
and here

\[ F_1 = \int_{-1}^{0} s_1(y) dy \]

\[ = \frac{A_2}{m_1} (1 - \cosh m_1) + \frac{A_4}{m_2} (1 - \cosh m_2) + \frac{A_1}{m_1} (\sinh m_1) + \frac{A_4}{m_2} (\sinh m_2) - \frac{P_0}{c} \]  \hspace{1cm} (1.6.3)

And

\[ F_2 = \int_{0}^{1} s_2(y) dy \]

\[ = \frac{A_3}{m_3} (\sinh m_3) + \frac{A_6}{m_3} (\cosh m_3 - 1) - \frac{P_0}{c} \]  \hspace{1cm} (1.6.4)

1.7 INTERFACE VELOCITY

Taking \( y = 0 \) in the equation (1.4.17) or equation (1.4.20) we get the interface velocity as

\[ u_0 = s_0 e^{ci} \]  \hspace{1cm} (1.7.1)

where

\[ s_0 = A_1 + A_3 - \frac{P_0}{c} \]  \hspace{1cm} (1.7.2)

or

\[ s_0 = A_1 - \frac{P_0}{c} \]  \hspace{1cm} (1.7.3)
1.8 RESULTS AND DISCUSSION

In this chapter, unsteady flow and heat transfer of couple stress and Newtonian fluids through a horizontal channel bounded by permeable beds is investigated and the results are discussed for various physical parameters.

Flow solutions are depicted graphically for couple stress parameter $\Gamma$, viscosity ratio $\beta$, pressure $P_0$, conductivity ratio $K$, slip parameter $\alpha$, permeability parameters $\sigma_1, \sigma_2$ and Prandtl number $Pr$, on velocity and temperature in both regions of the channel.

The variation of velocity with $y$ is calculated, from equations (1.4.17), (1.4.18), (1.4.20) and (1.4.21), for different values of pressure gradient $P_0$ and is shown in Fig. 1.2, for fixed $\Gamma = 3$, $\beta = 0.5$, $\alpha = 0.5$, $\sigma_1 = \sigma_2 = 1$, $c = 1$ and $t = 0.01$. We observe that the velocity increases with the decrease in the pressure gradient $P_0$. For given $P_0$, the velocity attains the maximum values at $y = 0$ i.e at the interface of couple stress and Newtonian fluids.

The variation of velocity with $y$ is calculated, for different values of couple stress parameter $\Gamma$ and is shown in Fig. 1.3, for fixed $P_0 = -0.1$, $\beta = 0.5$, $\alpha = 0.5$, $\sigma_1 = \sigma_2 = 1$, $c = 1$ and $t = 0.01$. We observe that the velocity increases with the decrease in the couple stress parameter $\Gamma$. For given $\Gamma$, the velocity attains the maximum value at $y = 0$, at the interface of couple stress and Newtonian fluids.

The variation of velocity with $y$ is calculated, for different values of ratio of viscosities $\beta$ and is shown in Fig. 1.4, for fixed $P_0 = -0.1$, $\Gamma = 3$, $\alpha = 0.5$, $\sigma_1 = \sigma_2 = 1$, $c = 1$ and $t = 0.01$. It is seen that the velocity increases with the decreasing of ratio of viscosities $\beta$. For given ratio of viscosities $\beta$, the velocity attains the maximum value
at \( y = 0 \). Further the slip velocity at the upper bed is found to be greater than the slip velocity at the lower bed.

The variation of velocity with \( y \) is calculated, for different values of slip parameter \( \alpha \) and is shown in Fig. 1.5, for fixed \( P_0 = -0.1, \Gamma = 3, \beta = 0.5, \sigma_1 = \sigma_2 = 1, c = 1 \) and \( t = 0.01 \). We observe that the velocity increases with the increases of slip parameter \( \alpha \) in Region 1. For given slip parameter \( \alpha \), the velocity attains the maximum value at \( y = 0 \).

The variation of velocity with \( y \) is calculated, for different values of permeability parameter \( \sigma_1 \) and is shown in Fig. 1.6, for fixed \( P_0 = -0.1, \Gamma = 3, \alpha = 0.5, \beta = 0.5, \sigma_2 = 1, c = 1 \) and \( t = 0.01 \). We observe that the velocity increases with the increase in the permeability parameter \( \sigma_1 \). For given permeability parameter \( \sigma_1 \), the velocity attains the maximum value at \( y = 0 \).

The variation of velocity with \( y \) is calculated, for different values of permeability parameter \( \sigma_2 \) and is shown in Fig. 1.7, for fixed \( P_0 = -0.1, \Gamma = 3, \alpha = 0.5, \beta = 0.5, \sigma_1 = 1, c = 1 \) and \( t = 0.01 \). We observe that the velocity increases with the decrease in the permeability parameter \( \sigma_2 \). For given permeability parameter \( \sigma_2 \), the velocity attains the maximum value at \( y = 0 \).

From the equations (1.4.19) and (1.4.22), we have calculated the temperature as a function of \( y \), for fixed \( \Gamma = 3, \beta = 0.5, \alpha = 0.5, \sigma_1 = \sigma_2 = 1, Pr = 0.7, K = 1, c = 1 \) and \( t = 0.01 \) and for different values of pressure gradient \( P_0 \) and is shown in Fig. 1.8. We observe that the temperature increases with the decrease in the pressure gradient \( P_0 \). For given \( P_0 \), the temperature increases with the increment in \( y \) in Region 1 and decreases in Region 2.
The variation temperature is evaluated as a function of $y$, for fixed $P_0 = -0.1$, $\beta = 0.5$, $\alpha = 0.5$, $\sigma_1 = \sigma_2 = 1$, $Pr = 0.7$, $K = 1$, $c = 1$ and $t = 0.01$ and for different values of couple stress parameter $\Gamma$ and is shown in Fig. 1.9. It is observed that the temperature increases with the increase in the couple stress parameter $\Gamma$. For given $\Gamma$, the temperature increases with the increment in $y$ in Region 1 and also increases in Region 2. It is observed that for a given $\Gamma$, the temperature attains the maximum value at $y = 1$, i.e. at the upper permeable bed.

The variation temperature is computed as a function of $y$, for fixed $P_0 = -0.1$, $\Gamma = 3$, $\alpha = 0.5$, $\sigma_1 = \sigma_2 = 1$, $Pr = 0.7$, $K = 1$, $c = 1$ and $t = 0.01$ and for different values of ratio of viscosities $\beta$ and is shown in Fig. 1.10. It is observed that the temperature increases with the decrease in the ratio of viscosities $\beta$. For given $\beta$, the temperature increases with the increment in $y$ in Region 1 and is also increases in Region 2. It is observed that for a given $\beta = 0.1$, the temperature increases for $-1 \leq y \leq 0.2$ (Region 1) and decreases for $0.2 \leq y \leq 1$ (Region 2).

The variation temperature is calculated as a function of $y$, for fixed $P_0 = -0.1$, $\Gamma = 3$, $\alpha = 0.5$, $\beta = 0.5$, $\sigma_1 = \sigma_2 = 1$, $Pr = 0.7$, $c = 1$ and $t = 0.01$, and for different values of ratio of thermal conductivities $K$ and is shown in Fig. 1.11. It is observe that the temperature increases with the decrease in the ratio of conductivities $K$.

The variation temperature with $y$ is calculated for fixed $P_0 = -0.1$, $\Gamma = 3$, $\alpha = 0.5$, $\beta = 0.5$, $\sigma_1 = \sigma_2 = 1$, $K = 1$, $c = 1$ and $t = 0.01$ and for different values of Prandtl number $Pr$ and is shown in Fig. 1.12. It is observe that the temperature increases with the increase in Prandtl number. For given $Pr$, the temperature increases with the increment in $y$ in Region 1 and attains the maximum value in Region 2.
The variation temperature with \( y \) is calculated for fixed \( P_0 = -0.1, \Gamma = 3, \beta = 0.5, \sigma_1 = \sigma_2 = 1, \text{ Pr } = 0.7, K = 1, c = 1 \) and \( \tau = 0.01 \), and for different values of slip parameter \( \alpha \) and \( \gamma \). It is observed that the temperature increases with the increase in the slip parameter \( \alpha \). For given \( \alpha \), the temperature increases with the increment in \( y \) in Region 1 and attains the maximum value in Region 2.

<table>
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<th>( \Gamma )</th>
<th>( \alpha = 0.5 )</th>
<th>( \alpha = 1 )</th>
<th>( \alpha = 1.5 )</th>
<th>( \beta = 0.5 )</th>
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Table 1.1: Interface velocity for different values of \( \Gamma, \alpha \) and \( \beta \) with fixed \( P = -0.1, \sigma_1 = \sigma_2 = 1, K = 1, \text{ Pr } = 0.7 \).

The effects of the slip parameter \( \alpha \) and the ratio of viscosities \( \beta \) on the interface velocity \( S_0 \) is shown in Table 1.1. It is found that the interface velocity decreases with the increment in the couple stress parameter \( \Gamma \), with effect of slip parameter \( \alpha \). For a given couple stress parameter \( \Gamma \), the interface velocity increases with increasing slip parameter \( \alpha \). And also here observed that the interface velocity decreases with the increment in the couple stress parameter \( \Gamma \), for fixed \( \beta \). For a given couple stress parameter \( \Gamma \), the interface velocity decreases with increasing the ratio of viscosities \( \beta \).
Table 1.2: Nusselt numbers for different values of $\Gamma$.

<table>
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<td>4</td>
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The effect of couple stress parameter $\Gamma$ on the Nusselt numbers $N_u_1$ and $N_u_2$ are shown in Table 1.2. It is found that the Nusselt numbers ($N_u_1$ and $N_u_2$) increase with the increment in the couple stress parameter $\Gamma$. 
Fig. 1.2 Velocity profiles for different values of pressure gradient $P$
Fig. 1.3 Velocity profiles for different values of couple stress parameter $\Gamma$. 
Fig. 1.4 Velocity profiles for different values of ratio of viscosities $\beta$. 

Region 1
Couple stress fluid

Region 2
Newtonian fluid
Fig. 1.5 Velocity profiles for different values of slip parameter $\alpha$. 
Fig. 1.6 Velocity profiles for different values of $\sigma_1$
Fig. 1.7 Velocity profiles for different values of $\sigma_2$.
Fig. 1.8 Temperature profiles for different values of pressure gradient $P$.
Fig. 1.9 Temperature profiles for different values of couple stress parameter $\Gamma$. 

Region 1
Couple stress fluid

Region 2
Newtonian fluid

$\Gamma = 5$

$\Gamma = 3$

$\Gamma = 0.7$
Fig. 1.10 Temperatures profiles for different values of ratio of viscosities $\beta$
Fig. 1.11 Temperature profiles for different values of ratio of thermal conductivities $K$. 

Region 1
Couple stress fluid

Region 2
Newtonian fluid

$K = 1$

$K = 0.5$

$K = 1.5$
Fig. 1.12  Temperatures profile for different values of Prandtl number \( \text{Pr} \).
Fig. 1.13 Temperatures profile for different values of slip parameter $\alpha$.