UNSTEADY FLOW OF THREE IMMISCIBLE JEFFREY FLUIDS IN A CHANNEL
CHAPTER 5
UNSTEADY FLOW OF THREE IMMISCIBLE
JEFFREY FLUIDS IN A CHANNEL

5.1 INTRODUCTION

Historically, the most commonly-studied cases of two-phase flow are in large scale power systems. Coal and gas-fired power stations used very large boilers to produce steam for use in turbines. In such cases, pressurised water is passed through the heated pipes and it changes to steam as it moves through the pipe. The design of boilers requires a detailed understanding of two-phase flow heat-transfer and pressure drop behavior, which is significantly different from the single-phase case. Even more critically, nuclear reactors use water to remove heat from the reactor core using two-phase flow. A great deal of study has been performed on the nature of two-phase flow in such cases, so that engineers can design against possible failures in pipe work, loss of pressure and so on. The above two-phase flow cases are for a single fluid occurring by itself as two different phases, such as steam and water. The term two-phase flow is also applied to mixtures of different fluids having different phases, such as air and water, or oil and natural gas. Sometimes even three-phase flow is considered, such as in oil, water and gas pipelines where there might be significant fraction of solids.

The flow and heat transfer aspects of immiscible fluids is of special importance in the petroleum extraction and transport. For example, the reservoir rock of an oil field always contains several immiscible fluids in it pores. Part of the pore volume is occupied by water and the rest may be occupied either by oil (or) gas (or) both. Crude oils often contain dissolved gases which may be released into the reservoir rock when the pressure decreases. These examples show the importance of knowledge of the laws governing immiscible multi-phase flows for proper understanding of the processes involved and interface conditions between the phases. In general, multi-phase flows are driven by gravitational and viscous forces.
Since the introduction of high speed computing equipment, one of the goals of the reservoir engineering research has been to develop more accurate methods of describing fluid movement through underground reservoirs. Various mathematical methods have been developed or used by reservoir engineers to predict reservoir performance.

Interest in problems of machines of systems with more than one phase has been developed rapidly in recent years. Situations which occur frequently are concerned with the motion of liquid or gas which contains a distribution of solid particles. Such situations occur, for example, in the movement of dust laden air, in problems of fluidization, in the use of dust in gas-cooling systems to enhance heat transfer processes and in the processes by which raindrops are formed by the coalescence of small droplets which might be considered as solid particles for the purpose of examining their movement prior to coalescence.

Problems involving immiscible multi-phase flow and heat transfer and multi-component mass transfer also arise in a number of scientific and engineering disciplines. Important applications include petroleum industry, geophysics and plasma physics. In modeling such problems, the presence of a second immiscible fluid phase adds a number of complexities to the nature of interacting transport phenomenon and interface conditions between the phases. In general, multi-phase flows are driven by gravitational and viscous forces.

Many researchers considered the fluid to behave like a Newtonian fluid for physiological peristalsis including the flow of blood in arterioles. But such a model cannot suitable for blood flow unless the non-Newtonian nature of the fluid in included in it. Moreover, the Jeffrey model is relatively simpler linear model using time derivatives instead of convective derivatives, for example the oldroyd-B model does, it represents rheology different from the Newtonian. In spite of its relative simplicity, the Jeffrey model can indicate the changes of the rheology on the peristaltic flow even under the assumption of long wave length, low Reynolds number and small (or) large amplitude ratio.
The investigations of blood flow through arteries are of considerable importance in many cardiovascular diseases, particularly arteriosclerosis. The Jeffrey fluid model may be possible to control blood pressure and its flow behaviour by using an appropriate magnetic field. Rand et al. (1964), Chien (1965), Bugliarello and Sevilla (1970) and Cokelet (1972) have shown experimentally that the fluids such as blood have like non-Newtonian fluids. Lew et al. (1971) have reported that chyme is a non-Newtonian material having plastic like properties.

There has been some theoretical and experimental work on stratified laminar flow of two immiscible fluids in a horizontal pipe investigated by Packham and Shail (1971). Two-phase MHD flow and heat transfer in an inclined channel was investigated by Malashetty and Umavathi (1997). Malashetty et al. (2001) and Umavathi et al. (2004) studied convective flow and heat transfer of immiscible Newtonian fluids in a of horizontal and vertical channels.

Malashetty et al. (2005) studied flow and heat transfer in an inclined channel containing a porous layer sandwiched between two fluid layers. Umavathi et al. (2005) studied the problem of unsteady oscillatory flow and heat transfer of two viscous immiscible fluids through a horizontal channel with isothermal permeable walls. It is observed that the velocity and temperature profiles decrease as the periodic frequency increases. Vajravelu et al. (2005) studied the peristaltic transport of a Herschel–Bulkley fluid in contact with a Newtonian fluid in a channel under peristalsis in a channel.

Rathod and Mahadev (2011) studied the effect of thickness of porous material on the peristaltic pumping of a Jeffrey fluid when the tube wall is provided with non-erodible porous lining. Krishna Kumari et al. (2011) studied the peristaltic pumping of a Jeffrey fluid in a porous tube. Sreenadh et al. (2011) studied the mathematical model is developed to study the steady flow of Casson fluid through an inclined tube of non-uniform cross section with multiple stenoses. Unsteady flow of a Jeffrey fluid in an elastic tube with stenosis has been investigated by Sreenadh et al. (2012).

Motivated by these works unsteady flow and heat transfer of three immiscible Jeffrey fluids in a horizontal channel is investigated. The velocity, the temperature field the interface velocity and the volume flux are obtained in all the three regions. When \( \lambda_1, \lambda_2, \) and \( \lambda_3, \) tend to zero, the results of Shaik Meera (2009) are recovered as a special case for the flow of three Newtonian immiscible fluids in a horizontal channel. The effects of various physical parameters on the velocity and temperature fields are discussed through graphs.

5.2 MATHEMATICAL FORMULATIONS

Consider two-dimensional unsteady flow of three immiscible Jeffrey fluids between horizontal parallel plates, extending in \( x \) and \( z \) directions (Fig. 5.1). The regions \(-h \leq y \leq 0 \) (Region 3), \( 0 \leq y \leq h \) (Region 2) and \( h \leq y \leq 2h \) (Region 1) are filled with three immiscible fluids having densities \( \rho_1, \rho_2, \rho_3 \), dynamic viscosities \( \mu_1, \mu_2, \mu_3 \), and thermal conductivities \( K_1, K_2, K_3 \), respectively, with specific heat at constant pressure \( C_p \). The flow region is divided into three regions. The flow in the all the three Regions are governed by Jeffrey fluid model. The flow in all regions of the channel is assumed to be fully developed and is driven by a common pressure gradient \( -\frac{\partial P}{\partial x} \) and the existence of heat transfer does not affect the pressure gradient. The boundary walls of channel held at different constant temperature, the upper wall is held at a temperature \( T_{w1} \) and the lower wall is held at a temperature \( T_{w2} \) with \( T_{w1} > T_{w2} \). The
continuity of velocity, shear stress, temperature and heat at the two interfaces are assumed. The transport properties of the fluids in all regions are assumed to be constant.

![Physical model](image)

Fig. 5.1 Physical model

In view of these assumptions, the governing equations reduce to

\[ \rho_i \frac{\partial u_i}{\partial t} = \frac{\mu_i}{1 + \lambda_i} \frac{\partial^2 u_i}{\partial y^2} - \frac{\partial p}{\partial x}, \quad i = 1, 2, 3. \tag{5.2.1} \]

\[ \rho_i \ C_p \frac{\partial T_i}{\partial t} = K_i \ \frac{\partial^4 T_i}{\partial y^4} + \frac{\mu_i}{1 + \lambda_i} \left( \frac{\partial u_i}{\partial y} \right)^2, \quad i = 1, 2, 3. \tag{5.2.2} \]

where \( u_1, u_2, u_3 \) are the \( x \)-components of Jeffrey fluid velocity, \( T_1, T_2, T_3 \) are the temperatures, \( K_1, K_2, K_3 \) are the thermal conductivities, \( \rho_1, \rho_2, \rho_3 \) are the densities, \( \mu_1, \mu_2, \mu_3 \) are dynamic viscosities and \( \lambda_1, \lambda_2, \lambda_3 \) denote Jeffrey parameters of the fluids in Region 1, Region 2, Region 3 respectively. \( C_p \) is the specific heat at constant pressure. The boundary conditions on velocity are the no-slip boundary conditions and the boundary conditions on the temperature are isothermal conditions.
The boundary conditions on the velocity are

\[ u_x(2h) = 0 \]

\[ u_x(h) = u_x(h) \]

\[ u_x(0) = u_x(0) \]

\[ u_x(-h) = 0 \]

\[ \frac{\mu_1}{1 + \lambda_1} \frac{\partial u_x}{\partial y} = \frac{\mu_2}{1 + \lambda_2} \frac{\partial u_x}{\partial y}, \quad \text{at} \quad y = h \]  \hspace{1cm} (5.2.3)

\[ \frac{\mu_2}{1 + \lambda_2} \frac{\partial u_x}{\partial y} = \frac{\mu_3}{1 + \lambda_3} \frac{\partial u_x}{\partial y}, \quad \text{at} \quad y = 0 \]

\[ T_x(2h) = T_w \]

\[ T_x(h) = T_x(h) \]

\[ T_x(0) = T_x(0) \]

\[ T_x(-h) = 0 \]

\[ K_1 \frac{\partial T_x}{\partial y} = K_2 \frac{\partial T_x}{\partial y}, \quad \text{at} \quad y = h \]  \hspace{1cm} (5.2.4)

\[ K_2 \frac{\partial T_x}{\partial y} = K_3 \frac{\partial T_x}{\partial y}, \quad \text{at} \quad y = 0 \]
5.3 NON-DIMENSIONALIZATION OF THE FLOW QUANTITIES

It is convenient to introduce the following non-dimensional quantities

\[
\begin{align*}
\bar{u}_1 &= \frac{u_1}{\bar{u}_1}, \quad y^* = \frac{y}{h}, \quad t^* = \frac{t u_1}{h^2}, \quad p = \frac{h^2}{\mu u_1} \left( \frac{\partial p}{\partial x} \right), \quad \theta^* = \frac{T_w - T_*}{T_w - T_1}, \\
Ec &= \frac{u_1}{Cp(T_w - T_1)} \quad \text{(Eckrt number)}, \quad Pr = \frac{\mu Cp}{K_i} \quad \text{(Prandtl number)},
\end{align*}
\]

(5.3.1)

where \( \bar{u}_1 \) is the average velocity in Region 1.

Equations (5.2.1) - (5.2.4) using (5.3.1) take the following form and neglecting the asterisks.

**Region 1**

\[
\frac{\partial u_1}{\partial t} = \frac{1}{1 + \lambda_1} \left[ \frac{\partial^2 u_1}{\partial y^2} \right] + P
\]

(5.3.2)

\[
P_{Pr} \frac{\partial \theta_1}{\partial t} = \frac{\partial^2 \theta_1}{\partial y^2} + Ec \quad \frac{\mu_1}{K_i (1 + \lambda_1)} \left[ \frac{\partial u_1}{\partial y} \right]^2
\]

(5.3.3)

**Region 2**

\[
\frac{\partial u_2}{\partial t} = \frac{\eta_1 \alpha_1}{1 + \lambda_2} \left[ \frac{\partial^2 u_2}{\partial y^2} \right] + \eta_1 P
\]

(5.3.4)

\[
\frac{\partial \theta_2}{\partial t} = \frac{1}{Pr} \beta \eta_1 \frac{\partial^2 \theta_2}{\partial y^2} + \alpha_1 \frac{Ec}{(1 + \lambda_2)} \left[ \frac{\partial u_2}{\partial y} \right]^2
\]

(5.3.5)
Region 3

\[ \frac{\partial u_i}{\partial t} = \frac{\eta_2 \alpha_i}{1 + \lambda_2} \left[ \frac{\partial^2 u_i}{\partial y^2} \right] + \eta_2 P \quad (5.3.6) \]

\[ \frac{\partial \theta_i}{\partial t} = \frac{1}{Pr} \beta_i \eta_i \frac{\partial^2 \theta_i}{\partial y^2} + \frac{\alpha_i \eta_i E \beta_i}{1 + \lambda_2} \left[ \frac{\partial u_i}{\partial y} \right] \quad (5.3.7) \]

where

\[ \alpha_1 = \frac{\mu_1}{\mu_i}, \quad \alpha_2 = \frac{\mu_2}{\mu_i}, \quad \beta_1 = \frac{K_1}{K_i}, \quad \beta_2 = \frac{K_2}{K_i}, \quad \eta_1 = \frac{\rho_1}{\rho_i}, \quad \eta_2 = \frac{\rho_2}{\rho_i}, \] are the ratios of the velocities, thermal conductivities and densities respectively. The non-dimensional forms of boundary and interface conditions are given by

\[ u_i(2) = 0 \]
\[ u_i(1) = u_i(1) \]
\[ u_i(0) = u_i(0) \]
\[ u_i(-1) = 0 \quad (5.3.8) \]

\[ \frac{\mu_1}{1 + \lambda_1} \frac{\partial u_1}{\partial y} = \frac{\mu_2}{1 + \lambda_2} \frac{\partial u_2}{\partial y}, \quad at \ y = 1 \]

\[ \frac{\mu_2}{1 + \lambda_2} \frac{\partial u_2}{\partial y} = \frac{\mu_1}{1 + \lambda_1} \frac{\partial u_1}{\partial y}, \quad at \ y = 0 \]

\[ \theta_i(2) = 1 \]
\[ \theta_i(1) = \theta_i(1) \]
\[ \theta_i(0) = \theta_i(0) \]
\[ \theta_i(-1) = 0 \]

\[ \frac{\partial \theta_i}{\partial y} = \beta_i \frac{\partial \theta_i}{\partial y}, \quad at \ y = 1 \]

\[ \frac{\partial \theta_i}{\partial y} = \beta_i \frac{\partial \theta_i}{\partial y}, \quad at \ y = 0 \quad (5.3.9) \]
5.4 SOLUTION OF THE PROBLEM

The governing momentum equation, along with the energy equations from (5.3.2)-(5.3.7) are solved subject to the boundary and interface conditions (5.3.8) and (5.3.9) for the velocity and temperature distributions in the three regions. These equations are highly nonlinear partial differential equations that cannot be solved in closed form. However, it can be reduced to ordinary differential equations by assuming.

\[ u_i(y, t) = u_{i0}(y) + e^{i\omega t} u_{i1}(y) \]

\[ \theta_i(y, t) = \theta_{i0}(y) + e^{i\omega t} \theta_{i1}(y) \quad \text{where } i=1, 2, 3 \]  \hspace{1cm} (5.4.1)

\[ P_i(y, t) = P_{i0}(y) + e^{i\omega t} P_{i1}(y) \]

By substituting equation (5.4.1) in equations (5.3.2) to (5.3.9), equating the coefficients of harmonic and non-harmonic terms and neglecting the higher order terms of \( O(\omega t)^3 \) we obtain the following pairs of equations for \( (u_i, \theta_i) \) (non-periodic coefficients) of \( O(\omega t)^0 \) and \( (u_{i1}, \theta_{i1}) \) (periodic coefficients) of the order of \( O(\omega t)^1 \) as given below:

Region 1

\[
\frac{d^2 u_{i0}}{dy^2} + (1 + \lambda_i) P_{i0} = 0 \hspace{1cm} (5.4.2)
\]

\[
\frac{d^2 \theta_{i0}}{dy^2} + \frac{Ec \cdot Cp \cdot \mu_i}{1 + \lambda_i} \left( \frac{du_{i0}}{dy} \right)^2 = 0 \hspace{1cm} (5.4.3)
\]

\[
\frac{d^2 u_{i1}}{dy^2} - i\omega(1 + \lambda_i) u_{i1} + P_{i1}(1 + \lambda_i) = 0 \hspace{1cm} (5.4.4)
\]

\[
\frac{d^2 \theta_{i1}}{dy^2} - i\omega \Pr \theta_{i1} + 2\frac{E_c \cdot C_p \cdot \mu_i}{1 + \lambda_i} \frac{du_{i0}}{dy} \frac{du_{i1}}{dy} = 0 \hspace{1cm} (5.4.5)
\]
Region 2

\[ \frac{d^2 u_{20}}{dy^2} + \left( \frac{1 + \lambda_s}{\alpha_1} \right) P_{20} = 0 \quad (5.4.6) \]

\[ \frac{d^2 \theta_{20}}{dy^2} + \frac{Pr \, Ec \, \alpha_1 \left( \frac{du_{20}}{dy} \right)^2}{\beta_1 (1 + \lambda_s)} = 0 \quad (5.4.7) \]

\[ \frac{d^2 u_{21}}{dy^2} - i \omega \frac{(1 + \lambda_s)}{\alpha_1 \eta_1} u_{21} + \frac{(1 + \lambda_s)}{\alpha_1} P_{21} = 0 \quad (5.4.8) \]

\[ \frac{d^2 \theta_{21}}{dy^2} - i \omega \frac{Pr}{\eta_1 \beta_1} \theta_{21} + 2 \frac{\alpha_1 Ec Pr \left( \frac{du_{20}}{dy} \frac{du_{21}}{dy} \right)}{\beta_1 (1 + \lambda_s)} = 0 \quad (5.4.9) \]

Region 3

\[ \frac{d^2 u_{30}}{dy^2} + \frac{(1 + \lambda_s)}{\alpha_2} P_{30} = 0 \quad (5.4.10) \]

\[ \frac{d^2 \theta_{30}}{dy^2} + \frac{Pr \, Ec \, \alpha_1 \left( \frac{du_{30}}{dy} \right)^2}{\beta_1 (1 + \lambda_s)} = 0 \quad (5.4.11) \]

\[ \frac{d^2 u_{31}}{dy^2} - i \omega \frac{(1 + \lambda_s)}{\alpha_2 \eta_2} u_{31} + \frac{(1 + \lambda_s)}{\alpha_2} P_{31} = 0 \quad (5.4.12) \]

\[ \frac{d^2 \theta_{31}}{dy^2} - i \omega \frac{Pr}{\eta_2 \beta_2} \theta_{31} + 2 \frac{\alpha_2 Ec Pr \left( \frac{du_{30}}{dy} \frac{du_{31}}{dy} \right)}{\beta_2 (1 + \lambda_s)} \]
The corresponding boundary and interface conditions given by (5.3.8) and (5.3.9) become

\[ u_{10}(2) = 0 \]

\[ u_{10}(1) = u_{20}(1) \]  

(5.4.14)

\[ u_{20}(0) = u_{30}(0) \]

\[ u_{30}(-1) = 0 \]

\[ \frac{\mu_1}{1 + \lambda_1} \frac{du_{10}}{dy} = \frac{\mu_2}{1 + \lambda_2} \frac{du_{20}}{dy} \quad \text{at } y = 1 \]

\[ \frac{\mu_3}{1 + \lambda_2} \frac{du_{20}}{dy} = \frac{u_3}{1 + \lambda_3} \frac{du_{30}}{dy} \quad \text{at } y = 0 \]

\[ \theta_{10}(2) = 0 \]

\[ \theta_{10}(1) = \theta_{20}(1) \]  

(5.4.15)

\[ \theta_{20}(0) = \theta_{30}(0) \]

\[ \theta_{30}(-1) = 0 \]

\[ \frac{d\theta_{10}}{dy} = \beta_1 \frac{d\theta_{20}}{dy} \quad \text{at } y = 1 \]

\[ \frac{d\theta_{20}}{dy} = \frac{\beta_1}{\beta_2} \frac{d\theta_{30}}{dy} \quad \text{at } y = 0 \]
Solving (5.4.2) to (5.4.13) subject to the conditions (5.4.14) to (5.4.17) we obtain the velocity and the temperature fields as follows.

\[
\begin{align*}
    u_{11}(2) &= 0 \\
    u_{11}(1) &= u_{31}(1) \hspace{2cm} (5.4.16) \\
    u_{21}(0) &= u_{31}(0) \\
    u_{31}(-1) &= 0 \\
    \frac{\mu_1}{1 + \lambda_1} \frac{du_{11}}{dy} &= \frac{\mu_2}{1 + \lambda_2} \frac{du_{21}}{dy} \hspace{0.5cm} \text{at } y = 1 \\
    \frac{\mu_2}{1 + \lambda_2} \frac{du_{21}}{dy} &= \frac{u_1}{1 + \lambda_1} \frac{du_{31}}{dy} \hspace{0.5cm} \text{at } y = 0 \\
    \theta_{11}(2) &= 0 \\
    \theta_{11}(1) &= \theta_{31}(1) \\
    \theta_{21}(0) &= \theta_{31}(0) \\
    \theta_{31}(-1) &= 0 \\
    \frac{d\theta_{11}}{dy} &= \beta_1 \frac{d\theta_{31}}{dy} \hspace{0.5cm} \text{at } y = 1 \hspace{2cm} (5.4.17) \\
    \frac{d\theta_{21}}{dy} &= \beta_3 \frac{d\theta_{31}}{dy} \hspace{0.5cm} \text{at } y = 0
\end{align*}
\]
Region 1

\[ u_{10} = C_4 y + C_1 - \frac{P_{10}}{\omega} (1 + \lambda_1) \frac{y^4}{4} \]  
(5.4.18)

\[ u_{11} = e^{\sigma y} \left[ A_1 \cos m_2 y + A_2 \sin m_2 y \right] - i \frac{P_{11}}{\omega} \]  
(5.4.19)

\[ \theta_{10} = C_{11} y + C_{12} + e_2 y^3 + e_3 y^3 + e_4 y^4 \]  
(5.4.20)

\[ \theta_{11} = e^{\sigma y} \left[ A_{11} \cos(12 y) + A_{12} \sin m_2 y \right] + g_{14} ye^{\sigma y} \cos(m_2 y) + g_{30} ye^{\sigma y} \sin(m_2 y) \]  
(5.4.21)

\[ + g_{32} ye^{\sigma y} \sin(m_2 y) + \frac{P_{21}}{\omega} \]  
(5.4.22)

\[ u_{20} = C_3 y + C_4 - \frac{P_{20}}{\omega} (1 + \lambda_2) \frac{y^4}{4} \]  
(5.4.23)

\[ u_{21} = e^{\sigma y} \left[ A_3 \cos m_2 y + A_4 \sin m_2 y \right] - i \frac{P_{21}}{\omega} \]  
(5.4.24)

\[ \theta_{20} = C_{13} y + C_{14} + e_2 y^3 + e_3 y^3 + e_4 y^4 \]  
(5.4.25)

\[ \theta_{21} = e^{\sigma y} \left[ A_{13} \cos(4 y) + A_{14} \sin m_2 y \right] + g_{30} ye^{\sigma y} \cos(m_2 y) \]  
\[ + g_{32} ye^{\sigma y} \sin(m_2 y) + g_{40} ye^{\sigma y} \sin(m_2 y) + g_{41} ye^{\sigma y} \cos(m_2 y) \]  
(5.4.25)

\[ + i \left( f_{30} ye^{\sigma y} \sin(m_2 y) + f_{32} ye^{\sigma y} \cos m_2 y + f_{40} ye^{\sigma y} \cos m_2 y + f_{41} ye^{\sigma y} \sin m_2 y \right) \]
Region 3

\[ u_{30} = C_3 y + C_6 - \frac{P_{30}}{\alpha_2} \frac{(1 + \lambda_4) y^2}{2} \]  

\[ u_{31} = e^{m\nu} \left[ A_5 \cos m_b y + A_6 \sin m_b y \right] - i \frac{\eta_3 P_{31}}{\omega} \]  

\[ \eta_{30} = C_{13} y + C_{16} + e_{13} y^2 + e_{14} y^3 + e_{15} y^4 \]  

(5.4.28)

\[ \eta_{31} = e^{m\nu} \left[ A_{13} \cos m_b y + A_{16} \sin m_b y \right] + g_{36} e^{m\nu} \cos(m_b y) \]  

\[ + g_{38} e^{m\nu} \sin(m_b y) + g_{44} e^{m\nu} \cos(m_b y) + g_{46} e^{m\nu} \cos(m_b y) \]  

\[ + i \left[ f_{36} e^{m\nu} \sin(m_b y) + f_{38} e^{m\nu} \cos(m_b y) + f_{44} e^{m\nu} \cos(m_b y) + f_{46} e^{m\nu} \sin(m_b y) \right] \]  

(5.4.29)
Velocity and temperature distributions can be expressed in the following form:

**Region 1**

\[
\begin{align*}
    u_t &= C_1 y + C_2 - P(1 + \lambda_1) \frac{\nu}{2} \sin \omega t \left[ e^{-\nu t} \left( g_{11} \cos m_1 y + g_{12} \sin m_1 y \right) - g_1 \right] \\
    &\quad + i \cos \omega t \left[ e^{-\nu t} \left( g_{11} \cos m_1 y + g_{12} \sin m_1 y \right) - g_1 \right] \\
    \theta_t &= C_{11} y + C_{12} + \epsilon_1 y^2 + \epsilon_2 y^3 + e_j y^4 \theta_{11} \\
    &\quad + \cos \omega t \left[ e^{-\nu t} \left( X_{11} \cos m_1 y + X_{12} \sin m_1 y \right) + g_{31} y e^{-\nu t} \cos(m_1 y) \\
    &\quad + g_{36} y e^{-\nu t} \sin(m_1 y) + g_{46} e^{-\nu t} \sin(m_1 y) + g_{46} e^{-\nu t} \cos(m_1 y) \right] \\
    &\quad - \sin \omega t \left[ e^{-\nu t} \left( R_{11} \cos m_1 y + R_{12} \sin m_1 y \right) + f_{14} y e^{-\nu t} \sin(m_1 y) \\
    &\quad + f_{36} y e^{-\nu t} \cos(m_1 y) + f_{46} e^{-\nu t} \cos(m_1 y) + f_{46} e^{-\nu t} \cos(m_1 y) \right] \\
    &\quad + i \left[ e^{-\nu t} \left( X_{11} \cos m_1 y + X_{12} \sin m_1 y \right) + g_{31} y e^{-\nu t} \cos(m_1 y) \\
    &\quad + g_{36} y e^{-\nu t} \sin(m_1 y) + g_{46} e^{-\nu t} \sin(m_1 y) + g_{46} e^{-\nu t} \cos(m_1 y) \right] \\
    &\quad + \cos \omega t \left[ e^{-\nu t} \left( R_{11} \cos m_1 y + R_{12} \sin m_1 y \right) + f_{14} y e^{-\nu t} \sin(m_1 y) \\
    &\quad + f_{36} y e^{-\nu t} \cos(m_1 y) + f_{46} e^{-\nu t} \cos(m_1 y) + f_{46} e^{-\nu t} \cos(m_1 y) \right] \\
\end{align*}
\]

(5.4.30)

\[
\begin{align*}
    &\left\{ + \sin \omega t \left[ e^{-\nu t} \left( X_{11} \cos m_1 y + X_{12} \sin m_1 y \right) + g_{31} y e^{-\nu t} \cos(m_1 y) \\
    &\quad + g_{36} y e^{-\nu t} \sin(m_1 y) + g_{46} e^{-\nu t} \sin(m_1 y) + g_{46} e^{-\nu t} \cos(m_1 y) \right] \\
    &\quad + i \left[ e^{-\nu t} \left( X_{11} \cos m_1 y + X_{12} \sin m_1 y \right) + g_{31} y e^{-\nu t} \cos(m_1 y) \\
    &\quad + g_{36} y e^{-\nu t} \sin(m_1 y) + g_{46} e^{-\nu t} \sin(m_1 y) + g_{46} e^{-\nu t} \cos(m_1 y) \right] \\
    &\quad + \cos \omega t \left[ e^{-\nu t} \left( R_{11} \cos m_1 y + R_{12} \sin m_1 y \right) + f_{14} y e^{-\nu t} \sin(m_1 y) \\
    &\quad + f_{36} y e^{-\nu t} \cos(m_1 y) + f_{46} e^{-\nu t} \cos(m_1 y) + f_{46} e^{-\nu t} \cos(m_1 y) \right] \\
\end{align*}
\]

(5.4.31)
\[
\begin{aligned}
\theta_2 &= C_{13} y + C_{14} + e_6 y^3 + e_7 y^4 + e_8 y^4,
\end{aligned}
\]

\[
\begin{aligned}
&\left\{ e^{n\omega} \left[ A_{13} \cos (A_{14} \sin (m_2 y) + g_{36} y e^{n\omega} \cos (m_2 y) + g_{32} y e^{n\omega} \sin (m_2 y) + g_{41} e^{n\omega} \cos (m_2 y) \right] \\
&+ \cos \omega t \left[ f_{36} e^{n\omega} \sin (m_2 y) + f_{32} e^{n\omega} \cos (m_2 y) + f_{41} e^{n\omega} \sin (m_2 y) + f_{41} e^{n\omega} \cos (m_2 y) \right] \\
&- \sin \omega t \left[ f_{36} e^{n\omega} \sin (m_2 y) + f_{32} e^{n\omega} \cos (m_2 y) + f_{41} e^{n\omega} \sin (m_2 y) + f_{41} e^{n\omega} \cos (m_2 y) \right] \\
&+ i \left\{ e^{n\omega} \left[ A_{13} \cos (A_{14} \sin (m_2 y) + g_{36} y e^{n\omega} \cos (m_2 y) + g_{32} y e^{n\omega} \sin (m_2 y) + g_{41} e^{n\omega} \cos (m_2 y) \right] \\
&+ \sin \omega t \left[ f_{36} e^{n\omega} \sin (m_2 y) + f_{32} e^{n\omega} \cos (m_2 y) + f_{41} e^{n\omega} \sin (m_2 y) + f_{41} e^{n\omega} \cos (m_2 y) \right] \\
&- \sin \omega t \left[ f_{36} e^{n\omega} \sin (m_2 y) + f_{32} e^{n\omega} \cos (m_2 y) + f_{41} e^{n\omega} \sin (m_2 y) + f_{41} e^{n\omega} \cos (m_2 y) \right] \\
&\right\}
\end{aligned}
\]
\[ u_3 = C_2 y + C_6 - \frac{a_3}{a_2} \frac{y^2}{2} - \sin \omega t \left\{ e^{\omega t} \left[ g_{16} \cos m_6 y + g_{36} \sin m_6 y \right] - g_3 \right\} + i \cos \omega t \left\{ e^{\omega t} \left[ g_{19} \cos m_9 y + g_{39} \sin m_9 y \right] - g_3 \right\} \] (5.4.34)

\[ \theta_3 = C_{13} y + C_{16} + e_{10} y^2 + e_{11} y^3 + e_{12} y^4 \]

\[ + \cos \omega t \left\{ e^{m_3 y} \left[ A_{15} \cos m_{15} y + A_{16} \sin m_{16} y \right] + g_{36} y e^{m_6 y} \cos (m_6 y) + g_{33} y e^{m_3 y} \sin (m_3 y) \right\} \]

\[ - \sin \omega t \left\{ f_{36} y e^{m_6 y} \sin (m_6 y) + f_{33} y e^{m_3 y} \cos m_3 y + f_{44} y e^{m_4 y} \cos (m_4 y) + f_{43} y e^{m_3 y} \sin (m_3 y) \right\} \]

\[ \left\{ \begin{align*}
& \cos \omega t \left\{ f_{36} y e^{m_6 y} \sin (m_6 y) + f_{33} y e^{m_3 y} \cos m_3 y + f_{44} y e^{m_4 y} \cos (m_4 y) + f_{43} y e^{m_3 y} \sin (m_3 y) \right\} \\
& + i \left\{ e^{m_3 y} \left[ A_{15} \cos m_{15} y + A_{16} \sin m_{16} y \right] + g_{36} y e^{m_6 y} \cos (m_6 y) + g_{33} y e^{m_3 y} \sin (m_3 y) \right\} \\
& + g_{44} y e^{m_4 y} \sin (m_4 y) + g_{43} y e^{m_3 y} \cos m_3 y \end{align*} \right\} \] (5.4.35)
\[
m_1 = w(1 + \lambda_1), \quad m_2 = \sqrt{\frac{m_1}{2}}, \quad m_3 = \frac{w(1 + \lambda_2)}{\alpha_1 \eta_1}, \quad m_4 = \sqrt{\frac{m_3}{2}}, \quad m_5 = \frac{w(1 + \lambda_3)}{\alpha_2 \eta_2}, \quad m_6 = \sqrt{\frac{m_5}{2}}; \]

\[
a_i = -P(1 + \lambda_i); \quad a_2 = -P(1 + \lambda_2); \quad a_3 = P(1 + \lambda_3); \quad \alpha_1 = \frac{\mu_1}{\mu_i}; \quad \alpha_2 = \frac{\mu_2}{\mu_i}; \quad \alpha_3 = \frac{\alpha_1}{\alpha_1};
\]

\[
L_1 = \frac{1 + \lambda_1}{1 + \lambda_2}; \quad L_2 = \frac{1 + \lambda_2}{1 + \lambda_3};
\]

\[
a = \frac{a_2}{2a_1} \frac{a_3}{2}; \quad a_4 = a_t L_1; \quad a_5 = L_1 a_2 - a_4; \quad a_7 = \frac{a_1}{2a_2}; \quad a_8 = a_t L_2;
\]

\[
a_9 = -2a_1 - a_4; \quad a_{10} = 1 + a_3; \quad a_{11} = a_5 - a_5; \quad a_{12} = a_1 a_{10}; \quad a_{13} = \frac{a_1 + a_{11}}{1 + a_{12}};
\]

\[
b_i = \cos 2m_i; \quad b_2 = \sin 2m_i; \quad b_3 = \cos m_i; \quad b_4 = \sin m_i;
\]

\[
b_5 = e^{2m_5}; \quad b_{10} = e^{2m_5}; \quad b_{11} = e^{m_3} b_5; \quad b_{12} = e^{m_4} b_5; \quad b_1 = e^{m_3} b_5;
\]

\[
b_{13} = e^{-m_3} b_5; \quad b_{15} = e^{-m_3} b_5; \quad b_{16} = e^{-m_5} b_5;
\]

\[
m_1 = \frac{\mu_1 m_1}{1 + \lambda_1}; \quad m_2 = \frac{\mu_2 m_2}{1 + \lambda_2}; \quad m_3 = \frac{\mu_3 m_3}{1 + \lambda_3}; \quad g_1 = \frac{p}{w}; \quad g_2 = \frac{p \eta}{w}; \quad g_3 = \frac{p \eta_2}{w};
\]

\[
g_4 = g_t - g_5; \quad g_5 = g_3 - g_5;
\]

\[
b_1 = m_3 [b_{11} - b_{13}]; \quad b_1 = m_3 [b_{11} + b_{13}]; \quad b_{17} = m_3 [b_{13} - b_{14}];
\]

\[
b_{20} = m_3 [b_{13} + b_{14}]; \quad b_{21} = \frac{b_{15}}{b_{16}};
\]

\[
g_6 = \frac{g_6 b_1 + g_1}{b_{16}}; \quad g_7 = -[g_5 + g_6]; \quad g_8 = g_3 + g_5; \quad m_5 = \frac{m_1}{m_5};
\]

\[
b_{22} = b_{22} b_{14} + b_{15} m_{10}; \quad b_{24} = b_{24} b_{14} + b_{15} m_{10}; \quad b_{25} = b_{25} b_{14} - b_{15} m_{10}; \quad b_{26} = b_{26} b_{15} - b_{17} m_{14};
\]

\[
b_{27} = b_{12} b_{20} - b_{14} m_{18}; \quad b_{28} = b_{28} b_{14} + b_{17} m_{18}; \quad b_{29} = b_{29} b_{15} + b_{17} m_{15}; \quad b_{30} = b_{30} b_{15} - b_{17} m_{15};
\]

\[
b_{31} = b_{31} b_{15} - b_{19} m_{10}; \quad b_{33} = b_{33} b_{14} - b_{25} m_{35}; \quad b_{34} = b_{34} b_{15} + b_{25} m_{35};
\]

\[
b_{35} = b_{35} b_{15} - b_{19} m_{10}; \quad b_{36} = b_{36} b_{15} - b_{19} m_{10};
\]

\[
g_9 = g_9 b_1 + g_1 m_{10}; \quad g_{10} = g_1 b_{20}; \quad g_{11} = g_1 b_{20}; \quad g_{12} = g_3 b_{14} - g_3 b_{15};
\]

\[
g_{13} = g_9 b_1 - g_{11} b_{15}; \quad g_{14} = g_{12} b_{14} - g_{11} b_{15}; \quad g_{15} = \frac{g_{14}}{b_{16}};
\]

\[
g_{16} = \frac{g_3 - b_2 g_{13}}{b_{16}}; \quad g_{17} = g_1 b_1 - g_5;
\]

\[
g_{18} = \frac{g_3 b_{13} - g_{11} b_{15}}{b_{16}}; \quad g_{19} = g_3 - g_5;
\]

\[
g_{20} = \frac{g_3 b_{13} - g_{11} b_{15}}{b_{16}}; \quad g_{21} = \frac{g_3 - b_2 g_{13}}{m_{10}}; \quad g_{22} = \frac{g_3 b_{13} - g_{11} b_{15}}{m_{10}}; \quad g_{23} = \frac{g_3 - b_2 g_{13}}{m_{10}};
\]

\[130\]
\[ e_1 = \frac{E_c C_p \mu_4}{1 + \lambda_4}; \quad e_2 = \frac{E_c C_1^2}{2}; \quad e_3 = \frac{-e_c C_1 a_1}{3}; \quad e_4 = \frac{e_c a_1^3}{12}; \]
\[ e_5 = \frac{-P_e E_c \alpha_1}{\beta_1(1 + \lambda_2)}; \quad e_6 = \frac{E_c C_3^2}{2}; \quad e_7 = \frac{e_c C_1 a_1}{3 \alpha_1}; \quad e_8 = \frac{e_c a_1^2}{12 \alpha_1^2}; \]
\[ e_9 = \frac{-P_e E_c \alpha_2}{\beta_1(1 + \lambda_3)}; \quad e_{10} = \frac{E_c C_3^2}{2}; \quad e_{11} = \frac{e_c C_1 a_1}{3 \alpha_2}; \quad e_{12} = \frac{e_c a_1^2}{12 \alpha_2^2}; \]
\[ d_1 = 1 - 4e_2 - 8e_3 - 16e_4; \quad d_2 = e_6 + e_8 - e_7 - e_9; \quad d_3 = e_{11} - e_{10} - e_{12}; \]
\[ d_4 = \beta_1(2e_6 + 3e_7 + 4e_8) - 2e_2 - 3e_3 - 4e_4; \quad \beta_3 = \frac{\beta_1}{\beta_2}; \quad d_5 = \frac{-1}{\beta_2}; \]
\[ d_6 = d_3 - \frac{d_4}{\beta_2}; \quad d_7 = d_1 - d_2; \quad d_8 = 1 + d_3; \quad d_9 = d_1 + d_6; \quad d_{10} = d_1 d_2 - 2d_5; \]
\[ d_{11} = 1 + d_4; \quad d_{12} = d_1 + d_5; \quad d_{13} = d_4 d_5; \quad d_{14} = d_1 d_3 - d_1 d_4; \quad d_{15} = d_1 d_2; \]
\[ d_{16} = d_3 d_4; \quad d_{17} = d_3 d_5; \quad d_{18} = d_4 d_5; \quad d_{19} = d_1 d_4 - d_1 d_5; \quad d_{20} = d_2 d_4 - d_2 d_5; \quad d_{21} = 2 - d_2 d_3; \quad d_{22} = 2 - d_2 d_4; \]
\[ d_{23} = 2 - d_2 d_5; \quad d_{24} = 2 d_2 + d_3 d_4; \quad d_{25} = 4 d_2 - d_3 d_4; \quad d_{26} = 4 d_2 - d_3 d_5; \quad d_{27} = 4 d_2 - d_3 d_5; \]
\[ d_{28} = 2 - d_2 d_3; \quad d_{29} = 2 - d_2 d_4; \quad d_{30} = 2 - d_2 d_5; \quad d_{31} = d_1 d_2 d_3 - d_1 d_2 d_5; \]
\[ d_{32} = d_1 d_2 + d_3 d_4; \quad d_{33} = d_1 d_2 + d_3 d_4; \]
\[ m_1 = w P; \quad m_{12} = \sqrt{\frac{m_1}{2}}; \]
\[ D_1 = 2e_c c_1 m_2 (A_2 + A_4); \quad D_2 = 2e_c c_1 m_2 (A_2 - A_4); \quad D_3 = 2e_c c_1 m_1 (A_2 - A_4); \]
\[ D_4 = 2e_c c_2 m_2 (A_2 - A_4); \quad D_5 = 2e_c c_2 m_2 (A_2 - A_4); \quad D_6 = 2e_c c_2 m_2 (A_2 - A_4); \]
\[ D_7 = 2e_c c_3 m_4 (A_4 - A_2); \quad D_8 = 2e_c c_3 m_4 (A_4 - A_2); \quad D_9 = 2e_c c_3 m_4 (A_4 - A_2); \]
\[ D_{10} = 2e_c c_3 m_6 (A_6 - A_2); \quad D_{11} = 2e_c c_5 m_6 (A_6 - A_2); \quad D_{12} = 2e_c c_5 m_6 (A_6 - A_2); \]
\[ m_{11}^2 - 4m_4^2 = m_{11}; \quad f_{22} = \frac{2D_1 m_1^3}{m_{11}}; \quad f_{23} = \frac{2D_1 m_1^2}{m_{11}}; \quad f_{24} = \frac{-2D_1 m_1^2}{m_{11}}; \]
\[ f_{25} = \frac{8D_1 m_1^2 + 2m_2 m_3 D_1}{(m_{11})^2}; \quad f_{26} = \frac{2D_1 m_1^3}{m_{11}}; \quad f_{27} = \frac{D_1 (8m_1^3 + 2m_2 m_3^2)}{(m_{11})^2}; \]
\[ g_{22} = \frac{m_{11} D_1}{m_{11}}; \quad g_{23} = \frac{-m_{11} D_1}{m_{11}}; \quad g_{24} = \frac{-m_{11} D_1}{m_{11}}; \quad g_{25} = \frac{-8m_1 D_1 m_1^3}{(m_{11})^2}; \]
\[ g_{26} = \frac{-m_{11} D_1}{m_{11}}; \quad g_{27} = \frac{8m_1 D_1 m_1^3}{(m_{11})^2}; \quad m_{13} = \frac{w P}{\eta \beta_1}; \quad m_{14} = \frac{m_{11}}{2}; \quad a_{15} = \frac{a_2}{\alpha_1}; \]
\[ f_{28} = \frac{2D_1m_4^2}{mm_2} \]; \[ f_{29} = \frac{-2D_1m_4^3}{mm_2} \]; \[ f_{30} = \frac{-2D_1m_4^2}{mm_2} \]; \[ f_{31} = \frac{D_1(8m_4^2 + 2m_4m_1^1)}{(mm_2)^2} \];
\[ f_{32} = \frac{2D_2m_6^2}{mm_2} \]; \[ f_{33} = \frac{D_2(8m_6^2 + 2m_4m_1^1)}{(mm_2)^2} \]; \[ g_{28} = \frac{-D_1m_4^2}{mm_2} \]; \[ g_{29} = \frac{-D_1m_4^3}{mm_2} \]; \[ g_{30} = \frac{-8D_1m_4^3}{(mm_2)^2} \]; \[ g_{31} = \frac{-8D_1m_4^1}{mm_2} \]; \[ g_{32} = \frac{-8D_1m_4^3}{(mm_2)^2} \]; \[ m_{15} = \frac{w_P}{\eta_1^2} \]; \[ m_{16} = \sqrt{\frac{m_{15}}{2}} \]; \[ a_{15} = \frac{a_1}{a_2} \]; \[ m_{15} - 4m_4^1 = mm_4 \];
\[ f_{34} = \frac{-2D_1m_6^2}{mm_3} \]; \[ f_{35} = \frac{-2D_1m_6^3}{mm_3} \]; \[ f_{36} = \frac{-2D_1m_6^2}{mm_3} \]; \[ f_{37} = \frac{D_1(8m_6^2 + 2m_4m_1^1)}{(mm_3)^2} \];
\[ f_{38} = \frac{2D_2m_8^2}{mm_3} \]; \[ f_{39} = \frac{D_2(8m_8^2 + 2m_4m_1^1)}{(mm_3)^2} \]; \[ g_{34} = \frac{-m_{15}D_1}{mm_3} \]; \[ g_{35} = \frac{-m_{15}D_1}{mm_3} \]; \[ g_{36} = \frac{-m_{15}D_1m_4^1}{(mm_3)^2} \]; \[ g_{37} = \frac{-8m_{15}D_1m_4^1}{(mm_3)^2} \]; \[ g_{38} = \frac{m_{15}D_1}{mm_3} \]; \[ g_{39} = \frac{8m_{15}D_1m_4^1}{(mm_3)^2} \];
\[ f_{40} = f_{23} + f_{25} \]; \[ f_{41} = f_{22} + f_{27} \]; \[ f_{42} = f_{26} + f_{31} \]; \[ f_{43} = f_{24} + f_{31} \]; \[ f_{44} = f_{31} + f_{37} \]; \[ f_{45} = f_{39} + f_{34} \]; \[ g_{40} = g_{23} + g_{25} \]; \[ g_{41} = g_{22} + g_{27} \]; \[ g_{42} = g_{26} + g_{31} \]; \[ g_{43} = g_{24} + g_{31} \]; \[ g_{44} = g_{33} + g_{37} \]; \[ g_{45} = g_{39} + g_{34} \];
\[ m_{20} = e^{2m_1} \cos 2m_1 \]; \[ m_{31} = e^{2m_1} \sin 2m_1 \]; \[ m_{22} = e^{m_2} \sin m_2 \];
\[ m_{23} = e^{m_2} \cos m_2 \]; \[ m_{24} = e^{m_2} \cos m_2 \]; \[ m_{25} = e^{m_2} \sin m_2 \];
\[ X_1 = e^{2m_1} \{ \cos 2m_1[2g_{14} + g_{41}] + \sin 2m_1[2g_{26} + g_{40}] \} \]
\[ R_1 = e^{2m_1} \{ \cos 2m_1[2f_{16} + f_{40}] + \sin 2m_1[2f_{24} + f_{41}] \} \]
\[ R_2 = [f_{41} + f_{24}] e^{m_2} \sin m_2 + [f_{40} + f_{36}] e^{m_2} \cos m_2 \]
\[ R_3 = [f_{39} + f_{43}] e^{m_2} \sin m_2 + [f_{40} + f_{36}] e^{m_2} \cos m_2 \]
\[ X_2 = [g_{41} + g_{36}] e^{m_2} \cos m_2 + [g_{40} + g_{36}] e^{m_2} \sin m_2 \]
\[ X_3 = [g_{41} + g_{36}] e^{m_2} \cos m_2 + [g_{40} + g_{36}] e^{m_2} \sin m_2 \]
\[ X_4 = X_3 - X_2 \]; \[ R_5 = f_{44} - f_{43} \]; \[ X_5 = g_{45} - g_{43} \]; \[ m_{26} = e^{m_2} \cos m_4 \]
\[ m_{27} = e^{-m_2} \sin m_4 \]; \[ R_6 = [f_{44} - f_{43}] e^{-m_2} \sin m_6 + [f_{46} - f_{43}] e^{-m_2} \sin m_6 \]
\[ X_6 = [g_{45} - g_{46}] e^{-m_2} \sin m_6 \]
\[ q_1 = \cos m_{12} - \sin m_{12} \]; \[ q_2 = \cos m_{12} + \sin m_{12} \]; \[ q_3 = q_1 m_{12} e^{m_2} \]
\[ q_4 = m_2 q_2 e^{m_1} \]
\[ q_6 = q_5 f_2 e^{m_1} \]
\[ q_8 = q_7 f_3 e^{m_1} \]
\[ q_{11} = \cos m_2 + \sin m_2 \]
\[ r_2 = q_3 g_2 e^{m_1} \]
\[ q_{13} = \cos m_4 - \sin m_4 \]
\[ q_{16} = q_{14} q_{16} \beta e^{m_1} \]
\[ q_{18} = q_{17} f_{30} e^{m_1} \]
\[ q_{20} = q_{19} f_{32} e^{m_1} \]
\[ q_{23} = \cos m_4 + \sin m_4 \]
\[ r_6 = q_{17} g_{32} e^{m_1} \]
\[ R_7 = \beta_7 [q_{18} + q_{20} + q_{22} + q_{24}] - q_6 - q_8 + q_{10} - q_{12} \]
\[ X_7 = \beta_7 [r_3 + r_6 + r_7 + r_9] - r_1 - r_2 - r_3 - r_5 \]
\[ R_8 = \beta_8 [f_{38} + m_6 f_{44} + m_8 f_{45}] - f_{32} - m_4 f_{43} - f_{41} \]
\[ X_8 = \beta_8 [g_{36} + m_8 g_{44} + m_6 g_{45}] - g_{30} - m_6 g_{43} - m_8 g_{43} \]
\[ q_{25} = \frac{m_2 m_{14} - m_2 m_{16} - m_6 m_{20}}{m_{27}} \]
\[ X_9 = \frac{X_{27} m_{27} - m_{16} X_4 + m_{16} X_6}{m_{27}} \]
\[ X_{10} = X_0 - X_1 \]
\[ X_{11} = m_{23} X_{10} + X_4 m_{14} \]
\[ X_{12} = q_{10} X_4 - m_{25} X_7 \]
\[ q_{24} = q_{23} q_{16} - m_{25} q_{14} \]
\[ X_{13} = q_{10} X_4 - m_{25} X_7 \]
\[ q_{34} = q_{25} q_{16} - m_{25} q_{14} \]
\[ q_{35} = q_{26} q_{31} - q_{28} R_{28} \]
\[ X_{14} = q_{31} X_{11} - q_{28} X_{12} \]
\[ q_{37} = q_{29} q_{34} - q_{35} q_{32} \]
\[ R_{15} = q_{34}R_{12} - q_{31}R_{13} \quad X_{15} = q_{34}X_{12} \quad q_{30} = q_{35}q_{38} - q_{36}q_{37} \]
\[ R_{16} = q_{38}R_{14} - q_{31}R_{15} \quad X_{16} = q_{38}X_{14} - q_{36}X_{15} \quad R_{17} = \frac{R_{16}}{q_{30}} \]
\[ X_{17} = \frac{X_{16}}{q_{39}} \quad R_{18} = -\frac{R_1 - m_{20}R_{17}}{m_{21}} \quad X_{18} = -\frac{X_1 - m_{20}X_{17}}{m_{21}} \]
\[ R_{19} = \frac{R_{11} - q_{20}R_{17} - q_{22}R_{18}}{q_{24}} \quad X_{19} = \frac{X_{11} - q_{20}X_{17} - q_{22}X_{18}}{q_{24}} \quad X_{20} = X_{19} - X_5 \quad R_{21} = \frac{R_{20}m_{26} + R_{26}}{m_{27}} \quad X_{21} = \frac{X_{20}m_{26} + X_{26}}{m_{27}} \]
\[ R_{22} = \frac{m_{21}R_{17} + m_{23}R_{18} - m_{24}R_{19} - R_1}{m_{25}} \]
\[ X_{22} = \frac{m_{22}X_{17} + m_{23}X_{18} - m_{24}X_{19} - X_4}{m_{25}} \]

5.5 RATE OF HEAT TRANSFER (NUSELT NUMBER - \( N_u \))

The rate of heat transfer (Nusselt number) through the channel wall to the fluid is given by

\[ N_u = \left[ \frac{d\theta}{dy} \right] \quad \text{at} \quad y = -1, 2 \quad (5.5.1) \]

Based on the analytical solutions reported above the rate of heat transfer at the bottom wall is given by

\[ N_{u1} = \left[ \frac{d\theta}{dy} \right]_{y = -1} \]

134
At the top wall, it is given by

\[
N_{u_1} = C_{11} - 2e_{10} + 3e_{11} - 4e_{12} + \cos \alpha \theta \\
\begin{align*}
&= e^{-i\psi} \cos \theta \left[ m_{j_1} R_{j_1} + R_{j_1} m_{j_1} \right] + e^{-i\psi} \sin \theta \left[ -m_{j_1} R_{j_1} + R_{j_1} m_{j_1} \right] \\
&\quad + e^{i\psi} \cos \theta \left[ R_{j_1} + m_{j_1} R_{j_1} \right] + e^{i\psi} \sin \theta \left[ -R_{j_1} + m_{j_1} R_{j_1} \right] \\
&\quad - \sin \theta \left[ f_{j_1} - m_{j_1} f_{j_1} + m_{j_1} f_{j_1} + f_{j_1} \right] \\
&\quad + e^{-i\psi} \cos \theta \left[ f_{j_1} - m_{j_1} f_{j_1} - m_{j_1} f_{j_1} - f_{j_1} \right] \\
&\quad + e^{i\psi} \sin \theta \left[ f_{j_1} - m_{j_1} f_{j_1} - m_{j_1} f_{j_1} - f_{j_1} \right] \\
&\quad (5.5.2)
\end{align*}
\]

\[
N_{u_2} = \left[ \frac{d\theta}{dy} \right]_{y=1} \\
\begin{align*}
&= e^{-i\psi} \cos 2m_{11} \left[ m_{i_1} x_{i_1} + x_{i_1} m_{i_1} \right] + e^{i\psi} \sin 2m_{11} \left[ m_{i_1} x_{i_1} - x_{i_1} m_{i_1} \right] \\
&\quad + e^{i\psi} \sin 2m_{11} \left[ -2m_{i_1} x_{i_1} + x_{i_1} m_{i_1} \right] \\
&\quad + e^{-i\psi} \cos 2m_{11} \left[ 2f_{i_1} + f_{i_1} + m_{i_1} f_{i_1} + m_{i_1} f_{i_1} \right] \\
&\quad + e^{i\psi} \sin 2m_{11} \left[ f_{i_1} + 2m_{i_1} f_{i_1} - m_{i_1} f_{i_1} + m_{i_1} f_{i_1} \right] \\
&\quad (5.5.4)
\end{align*}
\]
5.6 MASS FLUX

The dimensionless mass flow rate per unit with the channel is

\[ Q = Q_0 e^{i\omega t} \]  \hspace{1cm} (5.6.1)

where

\[ Q_0 = F_1 + F_2 + F_3 \]

and here

\[ F_1 = \int_{y_1}^{y_2} u_1(y) dy \]

\[ = \frac{3C_a}{2} + C_a - \frac{7a_1}{2} - \sin \omega t \left[ \frac{e^{i\omega t}}{2m_1} \cos 2m_1 \left( g_{1s} - g_{1c} \right) + \frac{e^{i\omega t}}{2m_1} \sin 2m_1 \left( g_{1s} + g_{1c} \right) + \frac{e^{i\omega t}}{2m_1} \cos m_1 \left( g_{1s} - g_{1c} \right) \right. \]

\[ + \frac{e^{i\omega t}}{2m_2} \sin m_1 \left( g_{1s} + g_{1c} \right) - 3g_1 \]

\hspace{1cm} (5.6.2)

\[ F_2 = \int_{y_2}^{y_3} u_2(y) dy \]

\[ = \frac{C_3}{2} + C_3 - \frac{a_1}{6\alpha_1} \sin \omega t \left[ e^{i\omega t} \cos m_1 \left( g_{1u} - g_{1b} \right) + e^{i\omega t} \sin m_1 \left( g_{1u} + g_{1b} \right) - 2m_1 g_3 + 2m_1 g_3 - g_1 \right] \]

\hspace{1cm} (5.6.3)

\[ F_3 = \int_{y_3}^{y_4} u_3(y) dy \]

\[ = -\frac{C_5}{2} + C_5 - \frac{a_1}{6\alpha_1} \sin \omega t \left[ e^{-i\omega t} \cos m_1 \left( g_{1b} - g_{1u} \right) - e^{-i\omega t} \sin m_1 \left( g_{1b} + g_{1u} \right) - 2m_1 (g_{1c} - g_{1b}) + g_1 \right] \]

\hspace{1cm} (5.6.4)
5.7 INTERFACE VELOCITY

Taking $y = 0$ in the equations (5.4.32) or (5.4.34) we get the interface velocity which separates Region 2 and 3 as

$$U_{01} = u_2 \big|_{y=0} \quad \text{or} \quad u_1 \big|_{y=0}$$

$$= C_4 - \sin \omega t \left[ g_{14} - g_1 \right] + i \cos \omega t \left[ g_{14} - g_2 \right]$$

(or)

$$= C_6 - \sin \omega t \left[ g_{16} - g_3 \right] + i \cos \omega t \left[ g_{16} - g_4 \right]$$  \hspace{1cm} (5.7.1)

Taking $y = 1$ in the equations (5.4.30) or (5.4.32) we get the interface velocity separates Region 1 and 2 as

$$U_{02} = u_1 \big|_{y=1} \quad \text{or} \quad u_2 \big|_{y=1}$$

$$= C_3 + C_4 + \frac{a}{2a_1} - \sin \omega t \left[ e^{\alpha t} \left( g_{1a} \cos m_1 + g_{2a} \sin m_1 \right) - g_1 \right] + i \left[ \cos \omega t \left[ e^{\alpha t} \left( g_{1a} \cos m_1 + g_{2a} \sin m_1 \right) - g_1 \right] \right]$$

(or)

$$= C_3 + C_4 + \frac{a}{2} - \sin \omega t \left[ e^{\alpha t} \left( g_{1a} \cos m_1 + g_{2a} \sin m_1 \right) - g_1 \right] + i \left[ \cos \omega t \left[ e^{\alpha t} \left( g_{1a} \cos m_1 + g_{2a} \sin m_1 \right) - g_1 \right] \right]$$  \hspace{1cm} (5.7.3)
In this chapter, representative flow and heat transfer for unsteady flow of three immiscible Jeffrey fluids through a horizontal channel is investigated. The solutions for velocity and temperature fields are evaluated numerically for different values of physical parameters such as viscosity ratio, pressure, conductivity ratio, frequency parameter and periodic frequency parameter which are depicted in Figures 5.2 to 5.22. The Jeffrey fluids in all the three regions are different.

For simply we assume that $p_{10} = p_{11} = p_{20} = p_{21} = p_{30} = p_{31} = p$.

Flow solutions are depicted graphically with three immiscible fluids having pressure gradient $P$, viscosity ratio $\alpha$, $\alpha_1 \left(= \frac{\mu_1}{\mu_1}\right)$, Jeffrey parameters $\lambda_1, \lambda_2, \lambda_3$, ratio of densities $\eta \left(= \frac{\rho_1}{\rho_1}\right)$, $\eta_1 \left(= \frac{\rho_1}{\rho_1}\right)$, Eckert number $E_c$, Prandtl number $Pr$ and ratio of thermal conductivities $\beta_1 \left(= \frac{k_1}{k_1}\right)$, $\beta_2 \left(= \frac{k_1}{k_1}\right)$ on velocity and temperature in three Regions of the channel.

The variation of velocity with $y$ is calculated, from equations (5.4.30), (5.4.32) and (5.4.34), for different values of pressure gradient $P$ and is shown in Fig. 5.2, for fixed $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$ and $t = 0.01$. We observe that the velocity increases with the increase in the pressure gradient $P$. For a given $P$, the velocity increases in Region 3 and it attains the maximum value at $y = 0.5$, in Region 2 and decreases in Region 1.

The variation of velocity with $y$ is calculated, for different values of ratio of viscosities parameter $\alpha$, and is shown in Fig. 5.3, for fixed $P = 1$, $\alpha_1 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$ and $t = 0.01$. We observe that the velocity increases
with the decrease in the ratio of viscosities $\alpha_i$. For a given $\alpha_i$, the velocity increases in Region 3 and it attains the maximum value at $y = 0.5$, in Region 2 and decreases in Region 1.

The variation of velocity with $y$ is calculated, for different values of ratio of viscosities parameter $\alpha_2$ and is shown in Fig. 5.4, for fixed $P = 1$, $\alpha_1 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$ and $t = 0.01$. We observe that the velocity increases with the decrease in the ratio of viscosities $\alpha_2$. For a given $\alpha_2$, the velocity increases in Region 3 and it attains the maximum value at $y = 0.5$, in Region 2 and decreases in Region 1.

The variation of velocity with $y$ is calculated, for different values of Jeffrey parameter $\lambda_1$ and is shown in Fig. 5.5, for fixed $P = 1$, $\alpha_1 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$ and $t = 0.01$. We observe that the velocity increases with the increase in the Jeffrey parameter $\lambda_1$. For a given $\lambda_1$, the velocity increases in Region 3 and it attains the maximum value at $y = 0.5$ in Region 2 and decreases in Region 1.

The variation of velocity with $y$ is calculated, for different values of Jeffrey parameter $\lambda_2$ and is shown in Fig. 5.6, for fixed $P = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_3 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$ and $t = 0.01$. We observe that the velocity increases with the increase in the Jeffrey parameter $\lambda_2$. For a given $\lambda_2$, the velocity increases in Region 3 and it attains the maximum value at $y = 0.5$, in Region 2 and decreases in Region 1.

The variation of velocity with $y$ is calculated, for different values of Jeffrey parameter $\lambda_3$ and is shown in Fig. 5.7, for fixed $P = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$ and $t = 0.01$. We observe that the velocity increases with the
increase in the Jeffrey parameter $\lambda_3$. For a given $\lambda_3$, the velocity increases in Region 3 and it attains the maximum value at $y = 0.5$ in Region 2 and decreases in Region 1.

The variation of velocity with $y$ is calculated, for different values of ratio of densities $\eta_1$ and is shown in Fig. 5.8, for fixed $P = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 0.5$, $\omega = 0.01$ and $t = 0.01$. We observe that the velocity increases with the increase in the ratio of densities $\eta_1$. For a given $\eta_1$, the velocity increases in Region 3 and it attains the maximum value at $y = 0.5$, in Region 2 and decreases in Region 1.

The variation of velocity with $y$ is calculated, for different values of ratio of densities $\eta_2$ and is shown in Fig. 5.9, for fixed $P = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 1$, $\omega = 0.01$ and $t = 0.01$. We observe that the velocity increases with the increase in the ratio of densities $\eta_2$. For a given $\eta_2$, the velocity increases in Region 3 and it attains the maximum values at $y = 0.5$ in Region 2 and decreases in Region 1.

From the equations (5.4.31), (5.4.33) and (5.4.35), we have calculated the temperature as a function of $y$ for different values of pressure gradient $P$ and is shown in Fig. 5.10, for fixed $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$, $Pr = 0.01$, $Ec = 0.001$, $K_1 = 1$, $K_2 = 1$, $K_3 = 0.5$ and $t = 0.01$. We observe that the temperature increases with the decrease in the pressure gradient $P$. For a given $P$, the temperature increases with decreasing pressure gradient $P$ from $y = -1$ to $y = 2$. The temperature attains the maximum value at $y = 2$, in Region 1 and the temperature increases with increasing $y$ ($-1 \leq y \leq 2$) from Region 3 to Region 1.
The variation temperature as a function of $y$, for fixed $P = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$, $Pr = 0.01$, $Ec = 0.001$, $K_1 = 1$, $K_2 = 1$, $K_3 = 0.5$ and $t = 0.01$ and for different values of Jeffrey parameter $\lambda_1$, is shown in Fig. 5.11. We observe that the temperature increases with the increase in the Jeffrey parameter $\lambda_1$. For a given $\lambda_1$, the temperature increases with increasing of in the Jeffrey parameter $\lambda_1$ from $y = -1$ to $y = 2$. The temperature attains the maximum value at $y = 2$, in Region 1 and the temperature increases with increasing $y (-1 \leq y \leq 2)$ from Region 3 to Region 1.

The variation temperature as a function of $y$, for fixed $P = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$, $Pr = 0.01$, $Ec = 0.001$, $K_1 = 1$, $K_2 = 1$, $K_3 = 0$ and $t = 0.01$ and for different values of Jeffrey parameter $\lambda_2$ is shown in Fig. 5.12. We observe that the temperature increases with the increase in the Jeffrey parameter $\lambda_2$. For a given $\lambda_2$, the temperature increases with increasing of in the Jeffrey parameter $\lambda_2$ from $y = -1$ to $y = 2$. The temperature attains the maximum value at $y = 2$, in Region 1 and the temperature increases with increasing $y (-1 \leq y \leq 2)$ from Region 3 to Region 1.

The variation temperature as a function of $y$, for fixed $P = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$, $Pr = 0.01$, $Ec = 0.001$, $K_1 = 1$, $K_2 = 1$, $K_3 = 0.5$ and $t = 0.01$ for different values of Jeffrey parameter $\lambda_3$ is shown in Fig. 5.13. We observe that the temperature increases with the increase in the Jeffrey parameter $\lambda_3$. For a given $\lambda_3$, the temperature increases with the increase in the Jeffrey parameter $\lambda_3$ from $y = -1$ to $y = 2$. The temperature attains the maximum value at $y = 2$ in Region 1 and the temperature increases with increasing $y (-1 \leq y \leq 2)$ from Region 3 to Region 1.
The variation temperature as a function of $y$, for fixed $P = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$, $Pr = 0.01$, $Ec = 0.001$, $K_1 = 1$, $K_2 = 1$, $K_3 = 0.5$ and $t = 0.01$ and for different values of ratio of viscosities $\alpha_1$ is shown in Fig. 5.14. We observe that the temperature increases with the decrease in the ratio of viscosities $\alpha_1$. For a given $\alpha_1$, the temperature increases with decreasing ratio of viscosities $\alpha_1$, from $y = -1$ to $y = 2$. The temperature attains the maximum value at $y = 2$, in Region 1 and the temperature increases with increasing $y$ ($-1 \leq y \leq 2$) from Region 3 to Region 1.

The variation temperature as a function of $y$, for fixed $P = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$, $Pr = 0.01$, $Ec = 0.001$, $K_1 = 1$, $K_2 = 1$, $K_3 = 0.5$ and $t = 0.01$ and for different values of ratio of viscosities $\alpha_2$ is shown in Fig. 5.15. We observe that the temperature increases with the decrease in the ratio of viscosities $\alpha_2$. For a given $\alpha_2$, the temperature increases with decreasing in the ratio of viscosities $\alpha_2$, from $y = -1$ to $y = 2$. The temperature attains the maximum value at $y = 2$, in Region 1 and the temperature increases with increasing $y$ ($-1 \leq y \leq 2$) from Region 3 to Region 1.

The variation temperature as a function of $y$, for fixed $P = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$, $Pr = 0.01$, $Ec = 0.001$, $K_1 = 1$, $K_2 = 1$, $K_3 = 0.5$ and $t = 0.01$ and for different values of ratio of densities $\eta_1$ is shown in Fig. 5.16. We observe that the temperature increases with the increase in the ratio of densities $\eta_1$. For a given $\eta_1$, the temperature increases with the increasing of in the ratio of densities $\eta_1$, from $y = -1$ to $y = 2$. The temperature attains the maximum value at $y = 2$ in Region 1 and the temperature increases with increasing $y$ ($-1 \leq y \leq 2$) from Region 3 to Region 1.
The variation temperature as a function of $y$, for fixed $P = 1, \alpha_1 = 1, \alpha_2 = 1, \lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 1, \eta_1 = 1, \omega = 0.01, Pr = 0.01, Ec = 0.001, K_1 = 1, K_2 = 1, K_3 = 0.5$ and $t = 0.01$ and for different values of ratio of density $\eta_2$ is shown in Fig. 5.17. We observe that the temperature increases with the increase in the ratio of densities $\eta_2$. For a given $\eta_2$, the temperature increases with the increasing of in the ratio of densities $\eta_2$, from $y = -1$ to $y = 2$. The temperature attains the maximum value at $y = 2$, in Region 1 the temperature increases with the increasing $y (-1 \leq y \leq 2)$ from Region 3 to Region 1.

The variation temperature as a function of $y$, for fixed $P = 1, \alpha_1 = 1, \alpha_2 = 1, \lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 1, \eta_1 = 1, \eta_2 = 0.5, \omega = 0.01, Pr = 0.01, K_1 = 1, K_2 = 1, K_3 = 0.5$ and $t = 0.01$ and for different values of Eckert number $Ec$, is shown in Fig. 5.18. We observe that the temperature increases with the decrease in the Eckert number $Ec$. For a given Eckert number $Ec$, the temperature increases with decreasing of in the Eckert number $Ec$, from $y = -1$ to $y = 2$. The temperature attains the maximum value at $y = 2$, in Region 1 and the temperature increases with increasing $y (-1 \leq y \leq 2)$ from Region 3 to Region 1.

From given (5.4.31), (5.4.33) and (5.4.35), the variation temperature is calculated as a function of $y$, for fixed $P = 1, \alpha_1 = 1, \alpha_2 = 1, \lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 1, \eta_1 = 1, \eta_2 = 0.5, \omega = 0.01, Ec = 0.001, K_1 = 1, K_2 = 1, K_3 = 0.5$ and $t = 0.01$ and for different values of Prandtl number $Pr$ is shown in Fig. 5.19. We observe that the temperature increases with the increase in the Prandtl number $Pr$. For a given Prandtl number $Pr$, the temperature increases with the increasing of in the Prandtl number $Pr$, from $y = -1$ to $y = 2$. The temperature attains the maximum value at the interface $y = 1$ and the temperature increases with increasing $y (-1 \leq y \leq 2)$ from Region 3 to Region 1.
The variation temperature is calculated as a function of $y$, for fixed $P = 1$, $\alpha_i = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$, $Pr = 0.01$, $Ec = 0.001$, $K_2 = 1$, $K_3 = 0.5$ and $t = 0.01$ and for different values of thermal conductivity $K_1$ is shown in Fig. 5.20. We observe that the temperature increases with the increase in the thermal conductivity $K_1$. For a given thermal conductivity $K_1$, the temperature increases with increasing of in the thermal conductivity $K_1$, from $y = -1$ to $y = 2$. The temperature attains the maximum value at $y = 2$, in Region 1 and the temperature increases with increasing $y$ ($-1 \leq y \leq 2$) from Region 3 to Region 1.

The variation temperature as a function of $y$, for fixed $P = 1$, $\alpha_i = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$, $Pr = 0.01$, $Ec = 0.001$, $K_1 = 1$, $K_2 = 0.5$ and $t = 0.01$ and for different values of thermal conductivity $K_2$ is shown in Fig. 5.21. We observe that the temperature increases with the increase in the thermal conductivity $K_2$. For a given thermal conductivity $K_2$, the temperature increases with the increasing of in the thermal conductivity $K_2$, from $y = -1$ to $y = 2$. The temperature attains the maximum value at $y = 2$, in Region 1 and the temperature increases with increasing $y$ ($-1 \leq y \leq 2$) from Region 3 to Region 1.

The variation temperature as a function of $y$, for fixed $P = 1$, $\alpha_i = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 1$, $\eta_1 = 1$, $\eta_2 = 0.5$, $\omega = 0.01$, $Pr = 0.01$, $Ec = 0.001$, $K_1 = 1$, $K_2 = 1$ and $t = 0.01$ and for different values of thermal conductivity $K_3$ is shown in Fig. 5.22. We observe that the temperature increases with the decrease in the thermal conductivity $K_3$. For a given thermal conductivity $K_3$, the temperature increases with the decreasing of in the thermal conductivity $K_3$, from $y = -1$ to $y = 2$. The temperature attains the maximum
value at \( y = 2 \), in Region 1 and the temperature increases with increasing \( y \) \((-1 \leq y \leq 2)\)
from Region 3 to Region 1.

<table>
<thead>
<tr>
<th>( \lambda_1 )</th>
<th>( \alpha_1 = 0.5 )</th>
<th>( \alpha_1 = 1 )</th>
<th>( \alpha_1 = 1.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.8509</td>
<td>1.8994</td>
<td>2.5995</td>
</tr>
<tr>
<td>0.6</td>
<td>1.8309</td>
<td>1.8605</td>
<td>2.6437</td>
</tr>
<tr>
<td>0.7</td>
<td>1.9128</td>
<td>1.9181</td>
<td>2.671</td>
</tr>
<tr>
<td>0.8</td>
<td>1.9488</td>
<td>1.9582</td>
<td>2.8449</td>
</tr>
<tr>
<td>0.9</td>
<td>1.9749</td>
<td>1.9791</td>
<td>2.928</td>
</tr>
<tr>
<td>1</td>
<td>1.9802</td>
<td>2.0131</td>
<td>2.961</td>
</tr>
<tr>
<td>1.1</td>
<td>2.0123</td>
<td>2.046</td>
<td>3.0066</td>
</tr>
<tr>
<td>1.2</td>
<td>2.0427</td>
<td>2.0665</td>
<td>3.0734</td>
</tr>
<tr>
<td>1.3</td>
<td>2.0468</td>
<td>2.0629</td>
<td>3.1384</td>
</tr>
<tr>
<td>1.4</td>
<td>2.0956</td>
<td>2.12</td>
<td>3.2124</td>
</tr>
<tr>
<td>1.5</td>
<td>2.0972</td>
<td>2.144</td>
<td>3.2963</td>
</tr>
</tbody>
</table>

Table 5.1: Interface velocity \((u_{01})\) between (Region 2 and Region 3 at \( y = 0 \)) for different
values of \( \lambda_1 \), with effect of ratio of viscosities \( \alpha_1 \), with fixed \( p = 1 \),
\[ \lambda_2 = 1, \lambda_3 = 1, \eta_1 = 1, \eta_2 = 1 \text{ and } \alpha_2 = 0.5. \]

The effects of Jeffrey parameter \( \lambda_1 \), with effect of ratio of viscosities \( \alpha_1 \), on the
interface velocity \( u_{01} \) (between Region 2 and Region 3), is shown in Table 5.1. It is
found that the interface velocity increases with the increment of Jeffrey parameter \( \lambda_1 \),
with effect of ratio of viscosities \( \alpha_1 \). For a given Jeffrey parameter \( \lambda_1 \), the interface
velocity increases with increasing of ratio of viscosities \( \alpha_1 \).
Table 5.2: Interface velocity \((u_{o2})\) between (Region 1 and Region 2 at \(y=1\)) for different values of \(\lambda_1\), with effect of ratio of viscosities \(\alpha_1\left(=\frac{\mu_2}{\mu_1}\right)\), with fixed \(P=1\).

\(
\begin{array}{cccc}
\lambda_1 & \alpha_1 = 0.5 & \alpha_1 = 1 & \alpha_1 = 1.5 \\
0.5 & 1.59 & 1.6264 & 2.0388 \\
0.6 & 1.6742 & 1.7043 & 2.1438 \\
0.7 & 1.7562 & 1.7795 & 2.2459 \\
0.8 & 1.8362 & 1.8521 & 2.3451 \\
0.9 & 1.9141 & 1.9222 & 2.4416 \\
1 & 1.99 & 1.99 & 2.5355 \\
1.1 & 2.0641 & 2.0556 & 2.6268 \\
1.2 & 2.1363 & 2.149 & 2.7157 \\
1.3 & 2.2069 & 2.2105 & 2.8022 \\
1.4 & 2.2757 & 2.284 & 2.8866 \\
1.5 & 2.3429 & 2.3577 & 2.9687 \\
\end{array}
\)

The effects of Jeffrey parameter \(\lambda_1\), with effect of ratio of viscosities \(\alpha_1\), on the interface velocity \(u_{o2}\) (between Region 1 and Region 2 at \(y=1\)), is shown in Table 5.2. It is observe that the interface velocity increases with the increment of Jeffrey parameter with effect of ratio of viscosities \(\alpha_1\). For a given Jeffrey parameter \(\lambda_1\), the interface velocity increases with increasing of ratio of viscosities \(\alpha_1\).
Table 5.3: Interface velocity \((u_{01})\) between (Region 2 and Region 3 at \(y=0\)) for different values of \(\lambda_1\), with effect of ratio of viscosities \(\alpha_2 = \left(\frac{\mu_2}{\mu_1}\right)\), with fixed \(P = 1\).

\[
\begin{array}{c|ccc}
\lambda_1 & \alpha_2 = 0.5 & \alpha_2 = 1 & \alpha_2 = 1.5 \\
\hline
0.5 & 2.7009 & 1.7994 & 1.3791 \\
0.6 & 2.7657 & 1.8605 & 1.3893 \\
0.7 & 2.812 & 1.8981 & 1.4175 \\
0.8 & 2.8965 & 1.9282 & 1.4297 \\
0.9 & 3.0259 & 1.9791 & 1.4601 \\
1 & 3.0578 & 2.0131 & 1.4615 \\
1.1 & 3.154 & 2.046 & 1.4696 \\
1.2 & 3.1692 & 2.0665 & 1.5019 \\
1.3 & 3.2203 & 2.0906 & 1.5187 \\
1.4 & 3.2467 & 2.12 & 1.5287 \\
1.5 & 3.3131 & 2.144 & 1.5451 \\
\end{array}
\]

The effects of Jeffrey parameter \(\lambda_1\), with effect of ratio of viscosities \(\alpha_2\), on the interface velocity \(u_{01}\) (between Region 2 and Region 3) is shown in Table 5.3. It is found that the interface velocity increases with the increment of Jeffrey parameter with effect of ratio of viscosities \(\alpha_2\). For a given Jeffrey parameter \(\lambda_1\), the interface velocity decreases with increasing of ratio of viscosities \(\alpha_2\).
Table 5.4: Interface velocity ($u_{02}$) between (Region 1 and Region 2 at $y=1$) for different values of $\lambda_1$, with effect of ratio of viscosities $\alpha_2 \left(= \frac{\mu_1}{\mu} \right)$, with fixed $\lambda = 1$.

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\alpha_2 = 0.5$</th>
<th>$\alpha_2 = 1$</th>
<th>$\alpha_2 = 1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.99</td>
<td>1.6264</td>
<td>1.4534</td>
</tr>
<tr>
<td>0.6</td>
<td>2.0953</td>
<td>1.7043</td>
<td>1.5285</td>
</tr>
<tr>
<td>0.7</td>
<td>2.1978</td>
<td>1.7795</td>
<td>1.6014</td>
</tr>
<tr>
<td>0.8</td>
<td>2.2977</td>
<td>1.8521</td>
<td>1.6722</td>
</tr>
<tr>
<td>0.9</td>
<td>2.3951</td>
<td>1.9222</td>
<td>1.7412</td>
</tr>
<tr>
<td>1</td>
<td>2.49</td>
<td>1.99</td>
<td>1.8082</td>
</tr>
<tr>
<td>1.1</td>
<td>2.5826</td>
<td>2.0556</td>
<td>1.8734</td>
</tr>
<tr>
<td>1.2</td>
<td>2.6729</td>
<td>2.119</td>
<td>1.9369</td>
</tr>
<tr>
<td>1.3</td>
<td>2.7611</td>
<td>2.1805</td>
<td>1.9987</td>
</tr>
<tr>
<td>1.4</td>
<td>2.8471</td>
<td>2.24</td>
<td>2.059</td>
</tr>
<tr>
<td>1.5</td>
<td>2.9312</td>
<td>2.2977</td>
<td>2.1177</td>
</tr>
</tbody>
</table>

The effects of Jeffrey parameter $\lambda_1$, with effect of ratio of viscosities $\alpha_2$, on the interface velocity $u_{02}$ (between Region 1 and Region 2 at $y=1$), is shown in Table 5.4. It is observe that the interface velocity increases with the increment of Jeffrey parameter $\lambda_1$ with effect of ratio of viscosities $\alpha_2$. For a given Jeffrey parameter $\lambda_1$, the interface velocity decreases with increasing of ratio of viscosities $\alpha_1$. 
Table 5.5: Interface velocity \( u_{01} \) between (Region 2 and Region 3 at \( y=0 \)) for different values of \( \lambda_3 \), with effect of ratio of viscosities \( \alpha_1 \left( \frac{\mu_2}{\mu_1} \right) \), with fixed \( P=1 \).

\[
\begin{array}{|c|c|c|c|}
\hline
\lambda_3 & \alpha_1=0.5 & \alpha_1=1 & \alpha_1=1.5 \\
\hline
0.5 & 1.5002 & 1.5981 & 2.5363 \\
0.6 & 1.6902 & 1.7402 & 2.7517 \\
0.7 & 1.7717 & 1.8422 & 2.7573 \\
0.8 & 1.8372 & 1.8827 & 2.7794 \\
0.9 & 1.9224 & 1.9235 & 2.8695 \\
1 & 1.9802 & 2.0131 & 2.9682 \\
1.1 & 2.0223 & 2.0183 & 3.0028 \\
1.2 & 2.2272 & 2.1429 & 3.0642 \\
1.3 & 2.1892 & 2.493 & 3.1226 \\
1.4 & 2.278 & 2.5599 & 3.1781 \\
1.5 & 2.3678 & 2.4232 & 3.1822 \\
\hline
\end{array}
\]

The effects of Jeffrey parameter \( \lambda_3 \), with effect of ratio of viscosities \( \alpha_1 \), on the interface velocity \( u_{01} \) (between Region 2 and Region 3), is shown in Table 5.5. It is found that the interface velocity increases with the increment of Jeffrey parameter with effect of ratio of viscosities \( \alpha_1 \). For a given Jeffrey parameter \( \lambda_3 \), the interface velocity increases with increasing of ratio of viscosities \( \alpha_1 \).
Table 5.6: Interface velocity \( u_{02} \) between (Region 1 and Region 2 at \( y=1 \)) for different values of \( \lambda_3 \), with effect of ratio of viscosities \( \alpha_1 \left( = \frac{\mu_2}{\mu_1} \right) \), with fixed \( P = 1 \),

\[
\lambda_2 = 1, \lambda_1 = 1, \eta_i = 1, \eta_2 = 1 \text{ and } \alpha_2 = 0.5.
\]

The effects of Jeffrey parameter \( \lambda_3 \), with effect of ratio of viscosities \( \alpha_1 \), on the interface velocity \( u_{02} \) (between Region 1 and Region 2 at \( y=1 \)), is shown in Table 5.6. It is found that the interface velocity increases with the increment of Jeffrey parameter \( \lambda_3 \) with effect of ratio of viscosities \( \alpha_1 \). For a given Jeffrey parameter \( \lambda_3 \), the interface velocity increases with increasing of ratio of viscosities \( \alpha_1 \).
Table 5.7: Interface velocity \( u_{01} \) between (Region 2 and Region 3 at \( y=0 \)) for different values of \( \lambda_3 \), with effect of ratio of viscosities \( \alpha_2 \left( = \frac{\eta_2}{\mu_1} \right) \), with fixed \( P = 1 \),

\[ \lambda_2 = 1, \lambda_4 = 1, \eta_1 = 1, \eta_3 = 1 \text{ and } \alpha_1 = 0.5. \]

The effects of Jeffrey parameter \( \lambda_3 \), with effect of ratio of viscosities \( \alpha_2 \), on the interface velocity \( u_{01} \) (between Region 2 and Region 3), is shown in Table 5.7. It is found that the interface velocity increases with the increment of Jeffrey parameter \( \lambda_3 \) with effect of ratio of viscosities \( \alpha_2 \). For a given Jeffrey parameter \( \lambda_3 \), the interface velocity decreases with increasing of ratio of viscosities \( \alpha_2 \).
Table 5.8. Interface velocity ($u_{0z}$) between (Region 1 and Region 2 at $y=1$) for different values of $\lambda_3$, with effect of ratio of viscosities $\alpha_2 \left(= \frac{\mu_2}{\mu_1} \right)$, with fixed $P = 1, \lambda_2 = 1, \lambda_3 = 1, \eta_1 = 1, \eta_2 = 1$ and $\alpha_1 = 0.5$

<table>
<thead>
<tr>
<th>$\lambda_3$</th>
<th>$\alpha_2 = 0.5$</th>
<th>$\alpha_2 = 1$</th>
<th>$\alpha_2 = 1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2.2757</td>
<td>1.8082</td>
<td>1.7043</td>
</tr>
<tr>
<td>0.8</td>
<td>2.3233</td>
<td>1.8471</td>
<td>1.7258</td>
</tr>
<tr>
<td>0.7</td>
<td>2.3684</td>
<td>1.8847</td>
<td>1.747</td>
</tr>
<tr>
<td>0.8</td>
<td>2.4111</td>
<td>1.921</td>
<td>1.7678</td>
</tr>
<tr>
<td>0.9</td>
<td>2.4515</td>
<td>1.9561</td>
<td>1.7882</td>
</tr>
<tr>
<td>1</td>
<td>2.49</td>
<td>1.99</td>
<td>1.8082</td>
</tr>
<tr>
<td>1.1</td>
<td>2.5266</td>
<td>2.0228</td>
<td>1.8278</td>
</tr>
<tr>
<td>1.2</td>
<td>2.5614</td>
<td>2.0545</td>
<td>1.8471</td>
</tr>
<tr>
<td>1.3</td>
<td>2.5947</td>
<td>2.0852</td>
<td>1.8661</td>
</tr>
<tr>
<td>1.4</td>
<td>2.6264</td>
<td>2.115</td>
<td>1.8847</td>
</tr>
<tr>
<td>1.5</td>
<td>2.6567</td>
<td>2.1438</td>
<td>1.903</td>
</tr>
</tbody>
</table>

The effects of Jeffrey parameter $\lambda_1$, with effect of ratio of viscosities $\alpha_1$, on the interface velocity $u_{0z}$ (between Region 1 and Region 2 at $y=1$), is shown in Table 5.8. It is observe that the interface velocity increases with the increment of Jeffrey parameter $\lambda_1$ with effect of ratio of viscosities $\alpha_2$. For a given Jeffrey parameter $\lambda_1$, the interface velocity decreases with increasing of ratio of viscosities $\alpha_1$. 

157
Fig. 5.2. Velocity profiles for different values of Pressure gradient $P$. 

153
Fig. 5.3. Velocity profiles for different values of ratio of viscosities $\alpha_1$. 

$\alpha_1=1$ 

$\alpha_1=0.5$ 

$\alpha_1=5$
Fig. 5.4. Velocity profiles for different values of ratio of viscosities $\alpha_2$. 

155
Fig. 5.5. Velocity profiles for different values of Jeffrey parameter $\lambda_1$. 

156
Fig. 5.6. Velocity profiles for different values of Jeffrey parameter $\lambda_2$. 
Fig. 5.7. Velocity profiles for different values of Jeffrey parameter $\lambda_3$. 
Fig. 5.8. Velocity profiles for different values of ratio of densities $\eta_1$. 
Fig. 5.9. Velocity profiles for different values of ratio of densities $\eta_2$. 
Fig. 5.10. Temperature profiles for different values of pressure gradient P.
Fig. 5.11. Temperature profiles for different values of Jeffrey parameter $\lambda_1$. 
Fig. 5.12. Temperature profiles for different values of Jeffrey parameter $\lambda_2$. 

163
Fig. 5.13. Temperature profiles for different values of Jeffrey parameter $\lambda_3$. 

164
Fig. 5.14. Temperature profiles for different values of ratio of viscosities $\alpha_1$. 

Region - 3 
JEFFREY FLUID 

$\alpha_1 = 0.2$ 
$\alpha_1 = 0.5$ 
$\alpha_1 = 1$ 

Region - 2 
JEFFREY FLUID 
Region - 1 
JEFFREY FLUID
Fig. 5.15. Temperature profiles for different values of ratio of viscosities $\alpha_2$. 
Fig. 5.16. Temperature profiles for different values of ratio of densities $\eta_l$. 

167
Fig. 5.17. Temperature profiles for different values of ratio of densities $\eta_2$. 

Fig. 5.18. Temperature profiles for different values of Eckert number Ec.
Fig. 5.19. Temperature profiles for different values of Prandtl number Pr.
FIG. 5.26. Temperature profiles for different values of thermal conductivity $K$. 

- Region 1
- Region 2
- Region 3

$K_1 = 1.5$
$K_1 = 1$
$K_1 = 2$
Fig. 5.21. Temperature profiles for different values of thermal conductivity $K_2$. 
Fig. 5.22. Temperature profiles for different values of thermal conductivity $K_3$. 

173