ABSTRACT

Graph theory is rapidly moving into the mainstream of mathematics mainly due to its applications in diverse fields which include biochemistry (genomics), electrical engineering (communications networks and coding theory), computer science (algorithms and computations) and operations research (scheduling). Although graph theory is one of the younger branches of mathematics, it is fundamental to a number of applied fields.

Few subjects in mathematics have as specified an origin as graph theory. Graph theory originated with the Konigsberg Bridge Problem, which Leonhard Euler (1707-1783) solved in 1736. Over the past sixty years, there has been a great deal of exploration in the area of graph theory. Its popularity has increased due to its many modern day applications and it has become the source of interest to many researchers. High-speed digital computer is one of the main reasons for the recent growth of interest in graph theory and its applications.

Graphs are really important and probably, they are more important than we think. Graphs are among the most ubiquitous models of both natural and human-made structures. They can be used to model many types of relations and process dynamics in Physical, Biological and Social systems.

Many problems of practical interest can be represented by graphs and can be solved using graph theory. In Architecture, bipartite graphs play an important role in finding minimum number of cross beams required to make a grid of beams rigid, when the joints provide no rigidity to the structure. This is discussed by Jenny A. Baglino and Jack E. Graver. The structure is rigid if and only if the corresponding bipartite graph is connected. But the smallest connected graph is a tree and the largest possible tree in the bipartite graph with m, n vertices has m+n-1 edges. Hence an m x n graph is rigid iff the corresponding bipartite graph is connected. The rigid bracing will have minimum cross beams iff the bipartite graph is a tree with m+n-1 cross beams.

So, one can confidently put forward that a mere act of thinking about a problem in terms of a graph will certainly suggest insights and probable solution methods.
The rules and the moves of the queen in the game of chess are well-known. A queen can go further any number of squares horizontally, vertically or diagonally (assuming no other chess piece lies in its way). Chess lovers of Europe in 1850 have hit upon the idea of determining the minimum number of queens, to be placed on the board so as to allow the queen to attack or occupy all the squares.

The main focus of the thesis is structured on the theory of domination numbers in graphs. The year 1850, for the first time witnessed a study of domination in graphs, where the problem is to place the minimum number of queens on an \( nxn \) chess board so that every square gets covered or dominated. It actually took over a decade, i.e., in 1960 researchers took up a serious and comprehensive study of the subject. For the first time in 1962, the concepts were entitled “dominating set” and “domination number” by Ore. In 1977, Cockayne and Hedetniemi conducted a commendable and broad survey on the outcomes of the existing concepts of dominating sets in graphs at that time. The notation \( \gamma(G) \) for the domination number of a graph, for the first time was applied by the pair and was accepted widely since then. The domination theory of graphs put forth by Ore and Berge has been the new area for the researchers, recently.

Dominating Set and Coloring are among the most fundamental problems in graph theory, algorithms and combinatorial optimization. Dominating Set asks for the minimum set of vertices such that every vertex of the graph not in this set has a neighbor in it.

In Coloring we are asked to color the vertices with as few colors as possible, so that no edge is monochromatic, that is, both the endpoints of each edge receive different colors. These are classical NP-hard problems and are well-studied from the point of view of approximation algorithms and parameterized complexity. Dominating Set and Coloring have a number of applications and this has led to the algorithmic study of numerous variants of these problems. Among the most well known ones are Connected Dominating Set, Independent Dominating Set, Perfect Code, List Coloring, Edge Coloring, Acyclic Edge Coloring and Choosability. Since both the problems and its variants are computationally hard problems, most of the research centers on algorithms in special classes of graphs like interval graphs, chordal graphs, planar graphs and H-minor free graphs.
The first coloring problem in graph theory is 150 years old and it is a famous four color problem. The problems in graph colorings that have received the most attention is coloring the vertices of a graph.

The problems in vertex colorings that have been studied most often are those referred to as a proper vertex coloring. The proper vertex coloring of a graph $G$ is an assignment of colors to the vertices of $G$, one color to each vertex, so that adjacent vertices are colored differently. When it is understood that we are dealing with a proper vertex coloring, we ordinarily refer to this more simply as a coloring of $G$. A graph $G$ is $k$-colorable if there exists a coloring of $G$ from a set of $k$-colors. The minimum positive integer $k$ for which $G$ is $k$-colorable is the chromatic number of $G$ and is denoted by $\chi(G)$. The chromatic number of $G$ is therefore the minimum number of independent sets into which the set of vertices can be partitioned. A graph $G$ with chromatic number $k$ is $k$-chromatic.

A dominator coloring of a graph $G$ is an assignment of colors to the vertices of $G$ such that it is a proper coloring (no edge is monochromatic) and every vertex dominates all vertices of at least one color class. The minimum number of colors required for a dominator coloring of $G$ is called the dominator chromatic number of $G$ and is denoted by $\chi_d(G)$.

Gera et al. [1] introduced the concept of dominator chromatic number, and a number of basic combinatorial and algorithmic results on dominator chromatic number have been obtained. For example, it was observed by Gera that dominator chromatic number is NP-complete on general graphs by a simple reduction from 3-Coloring. In a recent paper Chellali and Murray show that unlike 3-Coloring, one can decide in polynomial time if a graph has dominator chromatic number 3.

Virtual problems in life, when changed into graph problems exhibit some remarkable features, giving rise to special categories of graphs namely, Interval graphs, Circle graphs, Circular-arc graphs, Circular-arc overlap graphs etc.

Interval graphs, their importance over the years can be seen in the increasing number of researchers trying to explore the field. Their sincere efforts are observed in establishing the
relevance of the subject in practicality in coherence with reality. Further, the authenticity of the subject is seen in modeling the problems in terms of interval graphs. Interval graphs as a part of graph theory led to an extensive study paving a way for the enthusiastic researchers to carry out further investigations on the subject. The reason for the major concern on the interval graphs is due to their simplicity and clarity of structure leading towards many a number of properties. Besides, the subject also demands for a multitude of applications, which include the arenas of database, artificial intelligence, biology, archaeology, genetics, traffic control, computer scheduling storage information retrieval and electronic circuit design etc.

Circular-arc graphs are a new class of intersection graphs, defined for a set of arcs on a circle. A graph is a circular-arc graph, if it is the intersection graph of a finite set of arcs on a circle. That is, there exists one arc for each vertex of G and two vertices in G are adjacent in G, if and only if the corresponding arcs intersect. A vertex is said to dominate another vertex if there is an edge between the two vertices. If we bend the arc into a line, then the family of arcs is transformed into a family of intervals. Therefore, every interval graph is a CAG, where the opposite is always not true. However, these days CAG as well as interval graphs are being patronized very much. The combinatorial structures in CAG are varied and extensive, where it finds an application in many other fields such as biology, genetics, traffic control, computer science and particularly useful in cyclic scheduling and computer storage allocation problems etc.

Circular-arc overlap graphs (Figure.3) are a new class of overlap graphs introduced by Kashiwabara and Masuda [2], defined for a set of arcs on a circle. They obtained many results. A representation of a graph with arcs helps in the solving of combinatorial problems on the graph. A graph is a circular arc graph, if it is the intersection graph of a finite set of arcs on a circle. A circular-arc overlap graph is a specific enclosure of circular arc graph; it is an overlap graph defined for a set of arcs on a circle. There is one arc for each vertex of G and two vertices in G are adjacent in G, if and only if the corresponding arcs intersect and one is not contained in the other.
This thesis elucidates the aforesaid concepts of interval graphs, Circular-arc graphs, and Circular-arc overlap graphs and further, proceeds to disclose certain graph theoretic aspects of all these graphs.

**FORMAT OF THE THESIS:**

In this thesis certain significant graph theoretic aspects related to interval graphs, circular-arc graphs and circular-arc overlap graphs are presented. The study of these concepts may serve as a beacon light for graph theory enthusiasts and may drive them to explore new graph theoretic aspects related to these three types of graphs.

The thesis is organized as seven chapters.

The first chapter deals with introduction and survey of literature on domination number and dominator chromatic number of graphs. It also deals with the necessary graph theoretic preliminaries.

Connected graphs characterized as paths and star graphs by S. Beena in [3] inspired in characterizing an interval graph as various graphs in second chapter. In this chapter, the conditions required for characterizing an interval graph as a connected graph are discussed
and connected interval graphs of order \( n \), size \( m \) and degree sequence \( d_1, d_2, \ldots, d_n \) are characterized as path graphs with the help of the inequality \( \sum_{i=1}^{n} d_i^2 < 4m \).

Apart from that, emphasis is given for specifying the conditions under which a connected interval graph befits a generalized 3-star, generalized double 3-star, generalized 4-ssstar. Furthermore, the above said star graphs of order \( n \), size \( m \) and degree sequence \( d_1, d_2, d_3, d_4, \ldots, d_n \) are characterized in terms of the equality \( \sum_{i=1}^{n} d_i^2 = 4n-x \), where \( x=4,2,0 \) respectively.

The third chapter puts forth the findings that characterize circular-arc overlap graphs (CAOG) as various complete partite graphs namely, \( k_{m,n}, k_{m,n,p}, \ldots \), \( k_{m_1,m_2,\ldots,m_p} \).

An attempt also has been made to promote the characterization of circular-arc overlap graphs, comprising a set consisting of any pair of adjacent vertices of the graph as a minimal dominating set, entailing the neighborhood of the vertices of the graph. Simultaneously, the chapter proposes an additional feature of algorithms that checks whether the given CAOG has every pair of adjacent vertices as a dominating set or not. Maheswari, B and Sudhakaraiah, A [4] proposed a formula to determine the domination number of a circular-arc overlap graph. Whereas in this chapter, CAOG'S having any pair of adjacent vertices as a dominating set are characterized.

In the fourth chapter, emphasis is given for characterizing circular-arc overlap graphs (CAOG) as \( k_{2,2,2} \) graph and complete bipartite graphs with partites of equal cardinality. Moreover, circular-arc overlap graphs having any pair of vertices of the CAOG as a minimal dominating set are characterized in terms of the neighborhood of the vertices of the graph. Apart from that, an algorithm is presented to check the same.
Dominator chromatic number of certain classes of interval graphs, circular-arc graphs and circular-arc overlap are studied in the remaining three chapters. A number of basic combinatorial and algorithmic results obtained on dominator chromatic number in [5] and [6] formed the basis of the study.

The fifth chapter concentrates on the theory of dominator coloring in graphs and focuses on resolving the dominator chromatic number of interval graphs. Some categorized interval graphs are selected in this process of study. To facilitate the study and to establish the results, emphasis is given to the analogy between the nature and lucidity of the intervals, which in turn played an essential role in determining the dominator chromatic number of the interval graphs.

The aim of the sixth chapter is to study the problem of dominator coloring that encompasses two classical problems, namely coloring and domination in circular-arc graphs. The chapter confers the revelations related to dominator coloring of some special classes of circular-arc graphs. The resolutions are reached at by opting the analogy between the nature and the coherence of the arcs and relation between the chromatic number and dominator chromatic number.

The seventh chapter brings to the fore, the findings related to dominator chromatic number of some special classes of circular-arc overlap graphs, upholding the concepts on the bounds of dominator chromatic number and the relation between the chromatic number and the dominator chromatic number.