Chapter 6

Coding Theory on Generalized Fibonacci $n$-Step Polynomials

6.1 Introduction

We know the Fibonacci $n$-step polynomials $F_k^{(n)}(x) \ (k = 0, \pm 1, \pm 2, \pm 3, \ldots)$ defined by the recurrence relation:

$$F_k^{(n)}(x) = x^{n-1}F_{k-1}^{(n)}(x) + x^{n-2}F_{k-2}^{(n)}(x) + \ldots + xF_{k-n+1}^{(n)}(x) + F_{k-n}^{(n)}(x) \quad (6.1)$$

with the initial terms $F_0^{(n)}(x) = F_1^{(n)}(x) = \ldots = F_{n-2}^{(n)}(x) = 0, F_{n-1}^{(n)}(x) = 1$ and $n = 1, 2, 3, \ldots$ [32].

In this chapter we introduce generalized Fibonacci $n$-step polynomials $F_{h,k}^{(n)}(x) \ (k = 0, \pm 1, \pm 2, \pm 3, \ldots)$ by generalizing the Fibonacci $n$-step polynomials $F_k^{(n)}(x)$ with the recurrence relation:

$$F_{h,k}^{(n)}(x) = h_1(x)F_{h,k-1}^{(n)}(x) + h_2(x)F_{h,k-2}^{(n)}(x) + \ldots + h_n(x)F_{h,k-n}^{(n)}(x), \ h_n(x) \neq 0 \quad (6.2)$$

and the initial terms $F_{h,0}^{(n)}(x) = F_{h,1}^{(n)}(x) = \ldots = F_{h,n-2}^{(n)}(x) = 0, F_{h,n-1}^{(n)}(x) = 1$ for $n = 1, 2, 3, \ldots$ and $h_1(x), h_2(x), \ldots, h_n(x)$ are polynomials with real coefficients.

Now we form a matrix $M_{h,n}(x)$ of order $n$ given by
and \( M_{h,n}(x) = I_n \), identity matrix of order \( n \)

Therefore, \( \det M_{h,n}(x) = (-1)^{n+1} h_n(x) \).

The inverse of \( M_{h,n}(x) \) is
6.2 Some Properties of the Matrix $M_{h,n}(x)$

The matrix $M_{h,n}(x)$ satisfies the following properties for $k, l = 0, \pm 1, \pm 2, \pm 3 \ldots$

**Property 6.2.1** $M_{h,n}(x)M_{h,n}^l(x) = M_{h,n}^l(x)M_{h,n}(x) = M_{h,n}^{k+l}(x)$.

**Property 6.2.2** $det\ M_{h,n}^k(x) = (det\ M_{h,n}(x))^k = ((-1)^{n+1}h_n(x))^k = (-1)^{(n+1)k}(h_n(x))^k$. 

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Theorem 6.2.1 \( M^k_{h,n}(x) = \)

\[
\begin{pmatrix}
F^{(n)}_{h,k+n-1}(x) & h_2(x)F^{(n)}_{h,k+n-2}(x) + h_3(x)F^{(n)}_{h,k+n-3}(x) + \ldots + h_n(x)F^{(n)}_{k}(x) \\
F^{(n)}_{h,k+n-2}(x) & h_2(x)F^{(n)}_{h,k+n-3}(x) + h_3(x)F^{(n)}_{h,k+n-4}(x) + \ldots + h_n(x)F^{(n)}_{h,k-1}(x) \\
& \vdots \\
F^{(n)}_{h,k+1}(x) & h_2(x)F^{(n)}_{h,k}(x) + h_3(x)F^{(n)}_{h,k-1}(x) + \ldots + h_n(x)F^{(n)}_{h,k-n+2}(x) \\
F^{(n)}_{h,k}(x) & h_2(x)F^{(n)}_{h,k-1}(x) + h_3(x)F^{(n)}_{h,k-2}(x) + \ldots + h_n(x)F^{(n)}_{h,k-n+1}(x)
\end{pmatrix}
\]

\[h_3(x)F^{(n)}_{h,k+n-2}(x) + h_4(x)F^{(n)}_{h,k+n-3}(x) + \ldots + h_n(x)F^{(n)}_{h,k+1}(x) \ldots \]

\[h_3(x)F^{(n)}_{h,k+n-3}(x) + h_4(x)F^{(n)}_{h,k+n-4}(x) + \ldots + h_n(x)F^{(n)}_{h,k}(x) \ldots \]

\[\vdots \]

\[h_3(x)F^{(n)}_{h,k+1}(x) + h_4(x)F^{(n)}_{h,k}(x) + \ldots + h_n(x)F^{(n)}_{h,k-n+3}(x) \ldots \]

\[h_3(x)F^{(n)}_{h,k}(x) + h_4(x)F^{(n)}_{h,k-1}(x) + \ldots + h_n(x)F^{(n)}_{h,k-n+2}(x) \ldots \]

\[h_3(x)F^{(n)}_{h,k-1}(x) + h_4(x)F^{(n)}_{h,k-2}(x) + \ldots + h_n(x)F^{(n)}_{h,k-n+1}(x)
\]

for \( k = 0, \pm 1, \pm 2, \pm 3, \ldots \).

**Proof:** Case 1: \( k \geq 0 \).

We have,

for \( k = 0 \)

\[
M^0_{h,n}(x) = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\]
\[ \begin{pmatrix}
F_{h,n-1}^{(n)}(x) & h_2(x)F_{h,n-2}^{(n)}(x) + h_3(x)F_{h,n-3}^{(n)}(x) + \ldots + h_n(x)F_{n-1}^{(n)}(x) \\
F_{h,n-2}^{(n)}(x) & h_2(x)F_{h,n-3}^{(n)}(x) + h_3(x)F_{h,n-4}^{(n)}(x) + \ldots + h_n(x)F_{h,n-1}^{(n)}(x) \\
& \ddots \ddots \ddots \\
F_{h,1}^{(n)}(x) & h_2(x)F_{h,0}^{(n)}(x) + h_3(x)F_{h,1}^{(n)}(x) + \ldots + h_n(x)F_{h,n-2}^{(n)}(x) \\
F_{h,0}^{(n)}(x) & h_2(x)F_{h,1}^{(n)}(x) + h_3(x)F_{h,2}^{(n)}(x) + \ldots + h_n(x)F_{h,n-1}^{(n)}(x)
\end{pmatrix} \]

for \( k = 1 \)

\[
M_{h,n}^{(n)}(x) = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
& \ldots & \ldots & \ldots & \ldots & \ldots \\
& \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
F_{h,n}^{(n)}(x) & h_2(x)F_{h,n-1}^{(n)}(x) + h_3(x)F_{h,n-2}^{(n)}(x) + \ldots + h_n(x)F_{h,1}^{(n)}(x) \\
F_{h,n-1}^{(n)}(x) & h_2(x)F_{h,n-2}^{(n)}(x) + h_3(x)F_{h,n-3}^{(n)}(x) + \ldots + h_n(x)F_{h,n-1}^{(n)}(x) \\
& \ddots \ddots \ddots \\
F_{h,2}^{(n)}(x) & h_2(x)F_{h,1}^{(n)}(x) + h_3(x)F_{h,0}^{(n)}(x) + \ldots + h_n(x)F_{h,n-2}^{(n)}(x) \\
F_{h,1}^{(n)}(x) & h_2(x)F_{h,0}^{(n)}(x) + h_3(x)F_{h,1}^{(n)}(x) + \ldots + h_n(x)F_{h,n-2}^{(n)}(x)
\end{pmatrix}
\]
Thus the theorem is true for \( k = 0 \) and 1.

Let the theorem be true for \( k = m \) then
Now using the property 6.2.1, we have

\[
M_{h,n}^m(x) = \begin{pmatrix}
F_{h,m+n-1}^{(n)}(x) & h_2(x)F_{h,m+n-2}^{(n)}(x) + h_3(x)F_{h,m+n-3}^{(n)}(x) + \ldots + h_n(x)F_{h,m}^{(n)}(x) \\
F_{h,m+n-2}^{(n)}(x) & h_2(x)F_{h,m+n-3}^{(n)}(x) + h_3(x)F_{h,m+n-4}^{(n)}(x) + \ldots + h_n(x)F_{h,m}^{(n)}(x) \\
\vdots & \ddots \\
F_{h,m+1}^{(n)}(x) & h_2(x)F_{h,m}^{(n)}(x) + h_3(x)F_{h,m-1}^{(n)}(x) + \ldots + h_n(x)F_{h,m-n+2}^{(n)}(x) \\
F_{h,m}^{(n)}(x) & h_2(x)F_{h,m-1}^{(n)}(x) + h_3(x)F_{h,m-2}^{(n)}(x) + \ldots + h_n(x)F_{h,m-n+1}^{(n)}(x)
\end{pmatrix}
\]

Now using the property 6.2.1, we have

\[
M_{h,n}^{m+1}(x) = M_{h,n}^m(x) M_{h,n}^1(x) = \begin{pmatrix}
F_{h,m+n-1}^{(n)}(x) & h_2(x)F_{h,m+n-2}^{(n)}(x) + h_3(x)F_{h,m+n-3}^{(n)}(x) + \ldots + h_n(x)F_{h,m}^{(n)}(x) \\
F_{h,m+n-2}^{(n)}(x) & h_2(x)F_{h,m+n-3}^{(n)}(x) + h_3(x)F_{h,m+n-4}^{(n)}(x) + \ldots + h_n(x)F_{h,m}^{(n)}(x) \\
\vdots & \ddots \\
F_{h,m+1}^{(n)}(x) & h_2(x)F_{h,m}^{(n)}(x) + h_3(x)F_{h,m-1}^{(n)}(x) + \ldots + h_n(x)F_{h,m-n+2}^{(n)}(x) \\
F_{h,m}^{(n)}(x) & h_2(x)F_{h,m-1}^{(n)}(x) + h_3(x)F_{h,m-2}^{(n)}(x) + \ldots + h_n(x)F_{h,m-n+1}^{(n)}(x)
\end{pmatrix}
\]
Using the recurrence relation (6.2), we have

\[
M_{h,n}^{m+1}(x) = \\
\begin{pmatrix}
F_{h,m+n}(x) & h_2(x)F_{h,m+n-1}(x) + h_3(x)F_{h,m+n-2}(x) + \ldots + h_n(x)F_{h,m+n-2}(x) + h_n(x)F_{h,m+n-2}(x) \\
F_{h,m+n-1}(x) & h_2(x)F_{h,m+n-2}(x) + h_3(x)F_{h,m+n-3}(x) + \ldots + h_n(x)F_{h,m+n-3}(x) \\
\vdots & \vdots \\
F_{h,m+2}(x) & h_2(x)F_{h,m+1}(x) + h_3(x)F_{h,m}(x) + \ldots + h_n(x)F_{h,m+n-3}(x) \\
F_{h,m+1}(x) & h_2(x)F_{h,m}(x) + h_3(x)F_{h,m-1}(x) + \ldots + h_n(x)F_{h,m+n-2}(x)
\end{pmatrix}
\]
Similarly, we can prove for all Case 2:

\[
\begin{pmatrix}
  h_3(x)F_{h,m+n-1}^{(n)}(x) + h_4(x)F_{h,m+n-2}^{(n)}(x) + \ldots + h_n(x)F_{h,m+1}^{(n)}(x) & \ldots & h_n(x)F_{h,m+n-1}^{(n)}(x) \\
  h_3(x)F_{h,m+n-2}^{(n)}(x) + h_4(x)F_{h,m+n-3}^{(n)}(x) + \ldots + h_n(x)F_{h,m}^{(n)}(x) & \ldots & h_n(x)F_{h,m+n-2}^{(n)}(x) \\
  \vdots & \ddots & \vdots \\
  h_3(x)F_{h,m+1}^{(n)}(x) + h_4(x)F_{h,m}^{(n)}(x) + \ldots + h_n(x)F_{h,m-n+4}^{(n)}(x) & \ldots & h_n(x)F_{h,m+1}^{(n)}(x) \\
  h_3(x)F_{h,m}^{(n)}(x) + h_4(x)F_{h,m-1}^{(n)}(x) + \ldots + h_n(x)F_{h,m-n+3}^{(n)}(x) & \ldots & h_n(x)F_{m}^{(n)}(x)
\end{pmatrix}
\]

Hence by induction, we can write, for all \( k \geq 0 \)

\[
M_{h,n}^{(k)}(x) =
\begin{pmatrix}
  F_{h,k+n-1}^{(n)}(x) & h_2(x)F_{h,k+n-2}^{(n)}(x) + h_3(x)F_{h,k+n-3}^{(n)}(x) + \ldots + h_n(x)F_{h,k+1}^{(n)}(x) \\
  F_{h,k+n-2}^{(n)}(x) & h_2(x)F_{h,k+n-3}^{(n)}(x) + h_3(x)F_{h,k+n-4}^{(n)}(x) + \ldots + h_n(x)F_{h,k}^{(n)}(x) \\
  \vdots & \ddots & \vdots \\
  F_{h,k+1}^{(n)}(x) & h_2(x)F_{h,k}^{(n)}(x) + h_3(x)F_{h,k-1}^{(n)}(x) + \ldots + h_n(x)F_{h,k-2}^{(n)}(x) \\
  F_{h,k}^{(n)}(x) & h_2(x)F_{h,k-1}^{(n)}(x) + h_3(x)F_{h,k-2}^{(n)}(x) + \ldots + h_n(x)F_{h,k-n+1}^{(n)}(x)
\end{pmatrix}
\]

Case 2: \( k < 0 \).

Similarly, we can prove for all \( k < 0 \).
Hence, \( M_{h,n}^k(x) = \)

\[
\begin{pmatrix}
F_{h,k+n-1}^{(n)}(x) & h_2(x)F_{h,k+n-2}^{(n)}(x) + h_3(x)F_{h,k+n-3}^{(n)}(x) + \ldots + h_n(x)F_{h,k}^{(n)}(x) & \ldots & h_n(x)F_{h,k+n-2}^{(n)}(x) \\
F_{h,k+n-2}^{(n)}(x) & h_2(x)F_{h,k+n-3}^{(n)}(x) + h_3(x)F_{h,k+n-4}^{(n)}(x) + \ldots + h_n(x)F_{h,k-1}^{(n)}(x) & \ldots & h_n(x)F_{h,k+n-3}^{(n)}(x) \\
& \vdots & \ddots & \vdots \\
F_{h,k+1}^{(n)}(x) & h_2(x)F_{h,k}^{(n)}(x) + h_3(x)F_{h,k-1}^{(n)}(x) + \ldots + h_n(x)F_{h,k+n-3}^{(n)}(x) & \ldots & h_n(x)F_{h,k-2}^{(n)}(x) \\
F_{h,k}^{(n)}(x) & h_2(x)F_{h,k-1}^{(n)}(x) + h_3(x)F_{h,k-2}^{(n)}(x) + \ldots + h_n(x)F_{h,k-n}^{(n)}(x) & \ldots & h_n(x)F_{h,k-1}^{(n)}(x)
\end{pmatrix}
\]

(6.3)

for \( k = 0, \pm 1, \pm 2, \pm 3, \ldots \).

**Property 6.2.3** \( M_{h,n}^k(x) = h_1(x)M_{h,n}^{k-1}(x) + h_2(x)M_{h,n}^{k-2}(x) + \ldots + h_n(x)M_{h,n}^{k-n} \).

**Proof:** \( M_{h,n}^k(x) = \)

\[
\begin{pmatrix}
F_{h,k+n-1}^{(n)}(x) & h_2(x)F_{h,k+n-2}^{(n)}(x) + h_3(x)F_{h,k+n-3}^{(n)}(x) + \ldots + h_n(x)F_{h,k}^{(n)}(x) \\
F_{h,k+n-2}^{(n)}(x) & h_2(x)F_{h,k+n-3}^{(n)}(x) + h_3(x)F_{h,k+n-4}^{(n)}(x) + \ldots + h_n(x)F_{h,k-1}^{(n)}(x) \\
& \vdots & \ddots & \vdots \\
F_{h,k+1}^{(n)}(x) & h_2(x)F_{h,k}^{(n)}(x) + h_3(x)F_{h,k-1}^{(n)}(x) + \ldots + h_n(x)F_{h,k+n-3}^{(n)}(x) \\
F_{h,k}^{(n)}(x) & h_2(x)F_{h,k-1}^{(n)}(x) + h_3(x)F_{h,k-2}^{(n)}(x) + \ldots + h_n(x)F_{h,k-n}^{(n)}(x)
\end{pmatrix}
\]
Now using the recurrence relation (6.2), we can write

\[
\begin{pmatrix}
    h_3(x)F_{h,k+n-2}^{(n)}(x) & h_4(x)F_{h,k+n-3}^{(n)}(x) & \ldots & h_n(x)F_{h,k+1}^{(n)}(x) & \ldots & h_n(x)F_{h,k+n-2}^{(n)}(x) \\
    h_3(x)F_{h,k+n-3}^{(n)}(x) & h_4(x)F_{h,k+n-4}^{(n)}(x) & \ldots & h_n(x)F_{h,k}^{(n)}(x) & \ldots & h_n(x)F_{h,k+n-3}^{(n)}(x) \\
        & \ddots & \ddots & \ddots & \ddots & \ddots \\
    h_3(x)F_{h,k}^{(n)}(x) & h_4(x)F_{h,k-1}^{(n)}(x) & \ldots & h_n(x)F_{h,k-n+3}^{(n)}(x) & \ldots & h_n(x)F_{h,k}^{(n)}(x) \\
    h_3(x)F_{h,k-1}^{(n)}(x) & h_4(x)F_{h,k-2}^{(n)}(x) & \ldots & h_n(x)F_{h,k-n+2}^{(n)}(x) & \ldots & h_n(x)F_{h,k-1}^{(n)}(x)
\end{pmatrix}
\]

Now using the recurrence relation (6.2), we can write

\[
F_{h,k+n-1}^{(n)}(x) = h_1(x)F_{h,k+n-2}^{(n)}(x) + h_2(x)F_{h,k+n-3}^{(n)}(x) + \ldots + h_n(x)F_{h,k-1}^{(n)}(x),
\]

\[
F_{h,k+n-2}^{(n)}(x) = h_1(x)F_{h,k+n-3}^{(n)}(x) + h_2(x)F_{h,k+n-4}^{(n)}(x) + \ldots + h_n(x)F_{h,k-2}^{(n)}(x),
\]

\[
\ldots
\]

\[
F_{h,k-1}^{(n)}(x) = h_1(x)F_{h,k-2}^{(n)}(x) + h_2(x)F_{h,k-3}^{(n)}(x) + \ldots + h_n(x)F_{h,k-n+1}^{(n)}(x).
\]

Hence by using the property of matrix addition, we can write

\[
M_{h,n}^k(x) =
\begin{pmatrix}
    F_{h,k+n-2}^{(n)} & h_2(x)F_{h,k+n-3}^{(n)} + h_3(x)F_{h,k+n-4}^{(n)} + \ldots + h_n(x)F_{h,k-1}^{(n)} \\
    F_{h,k+n-3}^{(n)} & h_2(x)F_{h,k+n-4}^{(n)} + h_3(x)F_{h,k+n-5}^{(n)} + \ldots + h_n(x)F_{h,k-2}^{(n)} \\
        & \ddots & \ddots & \ddots & \ddots & \ddots \\
    F_{h,k}^{(n)} & h_2(x)F_{h,k-1}^{(n)} + h_3(x)F_{h,k-2}^{(n)} + \ldots + h_n(x)F_{h,k-n+1}^{(n)} \\
    F_{h,k-1}^{(n)} & h_2(x)F_{h,k-2}^{(n)} + h_3(x)F_{h,k-3}^{(n)} + \ldots + h_n(x)F_{h,k-n}^{(n)}
\end{pmatrix}
\]

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\[
\begin{pmatrix}
 h_3(x) F^{(n)}_{h,k+n-3} + h_4(x) F^{(n)}_{h,k+n-4} + \ldots + h_n(x) F^{(n)}_{h,k} & \ldots & h_n(x) F^{(n)}_{h,k+n-3} \\
 h_3(x) F^{(n)}_{h,k+n-4} + h_4(x) F^{(n)}_{h,k+n-5} + \ldots + h_n(x) F^{(n)}_{h,k-1} & \ldots & h_n(x) F^{(n)}_{h,k+n-4} \\
 \vdots & \ddots & \vdots \\
 h_3(x) F^{(n)}_{h,k-1} + h_4(x) F^{(n)}_{h,k-2} + \ldots + h_n(x) F^{(n)}_{h,k-n+2} & \ldots & h_n(x) F^{(n)}_{h,k-1} \\
 h_3(x) F^{(n)}_{h,k-2} + h_4(x) F^{(n)}_{h,k-3} + \ldots + h_n(x) F^{(n)}_{h,k-n+1} & \ldots & h_n(x) F^{(n)}_{h,k-2} \\
\end{pmatrix}
\]

\[
+ h_2(x)
\begin{pmatrix}
 F^{(n)}_{h,k+n-3} & h_2(x) F^{(n)}_{h,k+n-4} + h_3(x) F^{(n)}_{h,k+n-5} + \ldots + h_n(x) F^{(n)}_{h,k-2} \\
 F^{(n)}_{h,k+n-4} & h_2(x) F^{(n)}_{h,k+n-5} + h_3(x) F^{(n)}_{h,k+n-6} + \ldots + h_n(x) F^{(n)}_{h,k-3} \\
 \vdots & \ddots & \vdots \\
 F^{(n)}_{h,k-1} & h_2(x) F^{(n)}_{h,k-2} + h_3(x) F^{(n)}_{h,k-3} + \ldots + h_n(x) F^{(n)}_{h,k-n} \\
 F^{(n)}_{h,k-2} & h_2(x) F^{(n)}_{h,k-3} + h_3(x) F^{(n)}_{h,k-4} + \ldots + h_n(x) F^{(n)}_{h,k-n-1} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
 h_3(x) F^{(n)}_{h,k+n-4} + h_4(x) F^{(n)}_{h,k+n-5} + \ldots + h_n(x) F^{(n)}_{h,k-1} & \ldots & h_n(x) F^{(n)}_{h,k+n-4} \\
 h_3(x) F^{(n)}_{h,k+n-5} + h_4(x) F^{(n)}_{h,k+n-6} + \ldots + h_n(x) F^{(n)}_{h,k-2} & \ldots & h_n(x) F^{(n)}_{h,k+n-5} \\
 \vdots & \ddots & \vdots \\
 h_3(x) F^{(n)}_{h,k-2} + h_4(x) F^{(n)}_{h,k-3} + \ldots + h_n(x) F^{(n)}_{h,k-n+1} & \ldots & h_n(x) F^{(n)}_{h,k-2} \\
 h_3(x) F^{(n)}_{h,k-3} + h_4(x) F^{(n)}_{h,k-4} + \ldots + h_n(x) F^{(n)}_{h,k-n} & \ldots & h_n(x) F^{(n)}_{h,k-4} \\
\end{pmatrix}
\]

+ \ldots

+ \ldots
**6.3 Coding/Decoding Method on Generalized Fibonacci n-Step Polynomials**

We consider \( h_n(x) = 1 \) so that \( det M^k_{h,n}(x) = (-1)^{(n+1)k} \) for developing the application of generalized Fibonacci \( n \)-Step polynomials in the coding theory.

We take the initial message \( P \) having at least \( n(n - 1) + 1 \) characters. Now we represent \( P \) in the form of square matrix \( P_n = (p_{ij})_{n \times n} \), where \( n \) is any positive integer.
\( p_{ij} (\geq 0) \ i, j = 1, 2, \ldots n, \) depends on the decision makers choice considering the fact that for each \( i \) at least one \( p_{ij} \neq 0 \). Otherwise this method is defunct one. We take the matrix \( M_{h,n}^k(x) \) as a coding matrix and its inverse matrix \( M_{h,n}^{-k}(x) \) as a decoding matrix for an arbitrary positive integer \( k \). We name the transformation \( P_n \times M_{h,n}^k(x) = E \) as coding, the transformation \( E \times M_{h,n}^{-k}(x) = P_n \) as decoding and define \( E \) as code matrix.

### 6.3.1 Determinant of the Code Matrix \( E \)

The code matrix \( E = P_n \times M_{h,n}^k(x) \).

Using the basic property of determinants, we have

\[
det E = det(P_n \times M_{h,n}^k(x)) = detP_n \times detM_{h,n}^k(x) = (-1)^{(n+1)k} \times detP_n. \tag{6.5}
\]

### 6.3.2 Example

We consider \( n = 3 \).

We represent the initial message \( P = p_{11}p_{12}p_{13}p_{21}p_{22}p_{23}p_{31}p_{32}p_{33} \) in the form of square matrix \( P_3 \) of order 3 as

\[
P_3 = \begin{pmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33}
\end{pmatrix} \tag{6.6}
\]

\( p_{ij} (\geq 0) \ i, j = 1, 2, 3, \) depends on the decision makers choice and for each \( i \) at least one \( p_{ij} \neq 0 \). We select for any value of \( k \), the matrix \( M_{h,3}^k(x) \) as the coding matrix.

Without any loss of generality, we assume that \( k = 3 \). Then by (6.3), we have
The inverse of \( M_{h,3}^3 \) is given by

\[
M_{h,3}^{-3} = \begin{pmatrix}
F_{h,-1}^{(3)}(x) & h_2(x)F_{h,-2}^{(3)}(x) + F_{h,-3}^{(3)}(x) & F_{h,-2}^{(3)}(x) \\
F_{h,-2}^{(3)}(x) & h_2(x)F_{h,-3}^{(3)}(x) + F_{h,-4}^{(3)}(x) & F_{h,-3}^{(3)}(x) \\
F_{h,-3}^{(3)}(x) & h_2F_{h,-4}^{(3)}(x) + F_{h,-5}^{(3)}(x) & F_{h,-4}^{(3)}(x)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & -2(h_2(x))^2 - h_1(x) & -h_2(x) \\
-h_2(x) & h_1(x)h_2(x) + 1 & -h_1(x) - (h_2(x))^2 \\
-h_1(x) - (h_2(x))^2 & (h_1(x))^2 + 2h_1(x)(h_2(x))^2 - h_2(x) & 1 + 2h_1(x)h_2(x) + (h_2(x))^3
\end{pmatrix}
\]

Then the coding of the message \( P \) consists of the multiplication of the matrix \( P_3 \) by the code matrix \( M_{h,3}^3 \) i.e.

\[
P_3 \times M_{h,3}^3(x)
\]

\[
= \begin{pmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33}
\end{pmatrix} \times
\]

\[
\begin{pmatrix}
(h_1(x))^3 + 2h_1(x)h_2(x) + 1 & (h_1(x))^2h_2(x) + (h_2(x))^2 + h_1(x) & (h_1(x))^2 + h_2(x) \\
(h_1(x))^2 + h_2(x) & h_1(x)h_2(x) + 1 & h_1(x) \\
h_1(x) & h_2(x) & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
e_{11} & e_{12} & e_{13} \\
e_{21} & e_{22} & e_{23} \\
e_{31} & e_{32} & e_{33}
\end{pmatrix} = E
\]

(6.8)

where

\[
e_{11} = ((h_1(x))^3 + 2h_1(x)h_2(x) + 1)p_{11} + ((h_1(x))^2 + h_2(x))p_{12} + h_1(x)p_{13},
\]
\[ e_{12} = ((h_1(x))^2h_2(x) + (h_2(x))^2 + h_1(x)p_{11} + (h_1(x)h_2(x) + 1)p_{12} + h_2(x)p_{13}, \]
\[ e_{13} = ((h_1(x))^2 + h_2(x))p_{11} + h_1(x)p_{12} + p_{13}, \]
\[ e_{21} = ((h_1(x))^3 + 2h_1(x)h_2(x) + 1)p_{21} + ((h_1(x))^2 + h_2(x))p_{22} + h_1(x)p_{23}, \]
\[ e_{22} = ((h_1(x))^2h_2(x) + (h_2(x))^2 + h_1(x))p_{21} + (h_1(x)h_2(x) + 1)p_{22} + h_2(x)p_{23}, \]
\[ e_{23} = ((h_1(x))^2 + h_2(x))p_{21} + h_1(x)p_{22} + p_{23}. \]
\[ e_{31} = ((h_1(x))^3 + 2h_1(x)h_2(x) + 1)p_{31} + ((h_1(x))^2 + h_2(x))p_{32} + h_1(x)p_{33}, \]
\[ e_{32} = ((h_1(x))^2h_2(x) + (h_2(x))^2 + h_1(x))p_{31} + (h_1(x)h_2(x) + 1)p_{32} + h_2(x)p_{33}, \]
\[ e_{33} = ((h_1(x))^2 + h_2(x))p_{31} + h_1(x)p_{32} + p_{33}. \]

Solving these nine equations for nine \( p_{ij} \)s, we have
\[ p_{11} = e_{11} - h_2(x)e_{12} - (h_1(x) + (h_2(x))^2)e_{13}, \]
\[ p_{12} = -(2(h_2(x))^2 + h_1(x))e_{11} + (1 + h_1(x)h_2(x))e_{12} + ((h_1(x))^2 + 2h_1(x))(h_2(x))^2 - h_2(x)e_{13}, \]
\[ p_{13} = -h_2(x)e_{11} - (h_1(x) + (h_2(x))^2)e_{12} + (1 + 2h_1(x)h_2(x) + (h_2(x))^2)e_{13}, \]
\[ p_{21} = e_{21} - h_2(x)e_{22} - (h_1(x) + (h_2(x))^2)e_{23}, \]
\[ p_{22} = -(2(h_2(x))^2 + h_1(x))e_{21} + (1 + h_1(x)h_2(x))e_{22} + ((h_1(x))^2 + 2h_1(x))(h_2(x))^2 - h_2(x)e_{23}, \]
\[ p_{23} = -h_2(x)e_{21} - (h_1(x) + (h_2(x))^2)e_{22} + (1 + 2h_1(x)h_2(x) + (h_2(x))^2)e_{23}, \]
\[ p_{31} = e_{31} - h_2(x)e_{32} - (h_1(x) + (h_2(x))^2)e_{33}, \]
\[ p_{32} = -(2(h_2(x))^2 + h_1(x))e_{31} + (1 + h_1(x)h_2(x))e_{32} + ((h_1(x))^2 + 2h_1(x))(h_2(x))^2 - h_2(x)e_{33}, \]
\[ p_{33} = -h_2(x)e_{31} - (h_1(x) + (h_2(x))^2)e_{32} + (1 + 2h_1(x)h_2(x) + (h_2(x))^2)e_{33}. \]

Then the code matrix \( E \) is sent to a channel. The following computation gives the decoding of the code message \( E \):
\[ E \times M_{h,n}^k(x) = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \times \]

\[
\begin{pmatrix}
1 & -2(h_2(x))^2 - h_1(x) & -h_2(x) \\
-h_2(x) & h_1(x)h_2(x) + 1 & -h_1(x) - (h_2(x))^2 \\
-h_1(x) - (h_2(x))^2 & (h_1(x))^2 + 2h_1(x)(h_2(x))^2 - h_2(x) & 1 + 2h_1(x)h_2(x) + (h_2(x))^3 
\end{pmatrix} = P_3.
\]

### 6.4 Relations among the Code Matrix Elements

In this section, we develop the relations among the code matrix elements considering the basic property that \( \det M_{h,n}^k(x) = (-1)^{(n+1)k} \) i.e. \( h_n(x) = 1 \). We write the code matrix \( E \) with the help of the initial matrix \( P_n \) and the coding matrix \( M_{h,n}^k(x) \) as

\[
E = P_n \times M_{h,n}^k(x)
\]

\[
\begin{pmatrix}
p_{11} & p_{12} & \ldots & p_{1n} \\
p_{21} & p_{22} & \ldots & p_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n1} & p_{n2} & \ldots & p_{nn} 
\end{pmatrix} \times
\]

\[
\begin{pmatrix}
F_{h,k+n-1}^{(n)}(x) & h_2(x)F_{h,k+n-2}^{(n)}(x) + h_3(x)F_{h,k+n-3}^{(n)}(x) + \ldots + F_{h,k}^{(n)}(x) \\
F_{h,k+n-2}^{(n)}(x) & h_2(x)F_{h,k+n-3}^{(n)}(x) + h_3(x)F_{h,k+n-4}^{(n)}(x) + \ldots + F_{h,k-1}^{(n)}(x) \\
\vdots & \vdots & \ddots & \vdots \\
F_{h,k+1}^{(n)}(x) & h_2(x)F_{h,k}^{(n)}(x) + h_3(x)F_{h,k-1}^{(n)}(x) + \ldots + F_{h,k-n+2}^{(n)}(x) \\
F_{h,k}^{(n)}(x) & h_2(x)F_{h,k-1}^{(n)}(x) + h_3(x)F_{h,k-2}^{(n)}(x) + \ldots + F_{h,k-n+1}^{(n)}(x)
\end{pmatrix}
\]
\[
\begin{pmatrix}
h_3(x)F_{h,k+n-2}^{(n)}(x) + h_4(x)F_{h,k+n-3}^{(n)}(x) + \ldots + F_{h,k+1}^{(n)}(x) & \cdots & F_{h,k+n-2}^{(n)}(x) \\
h_3(x)F_{h,k+n-3}^{(n)}(x) + h_4(x)F_{h,k+n-4}^{(n)}(x) + \ldots + F_{h,k}^{(n)}(x) & \cdots & F_{h,k+n-3}^{(n)}(x) \\
\vdots & \ddots & \vdots \\
h_3(x)F_{h,k}^{(n)}(x) + h_4(x)F_{h,k-1}^{(n)}(x) + \ldots + F_{h,k-n+3}^{(n)}(x) & \cdots & F_{h,k}^{(n)}(x) \\
h_3(x)F_{h,k-1}^{(n)}(x) + h_4(x)F_{h,k-2}^{(n)}(x) + \ldots + F_{h,k-n+2}^{(n)}(x) & \cdots & F_{h,k-1}^{(n)}(x)
\end{pmatrix}
\]

\[
\begin{pmatrix}
e_{11} & e_{12} & \ldots & e_{1n} \\
e_{21} & e_{22} & \ldots & e_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
e_{n1} & e_{n2} & \ldots & e_{nn}
\end{pmatrix}
\]

We choose \( k \) in such a manner that \( e_{ij} > 0 \), for all \( i, j \).

After decoding, we have

\[
P_n = E \times M_n^{-k}(x)
\]

\[
\begin{pmatrix}
e_{11} & e_{12} & \ldots & e_{1n} \\
e_{21} & e_{22} & \ldots & e_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
e_{n1} & e_{n2} & \ldots & e_{nn}
\end{pmatrix} \times
\]
We write the code matrix $E$ and the initial matrix $P_2$ as:

$$
E = P_2 \times M_{h,2}^k(x) = \begin{pmatrix}
    p_{11} & p_{12} & \cdots & p_{1n} \\
    p_{21} & p_{22} & \cdots & p_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    p_{n1} & p_{n2} & \cdots & p_{nn}
\end{pmatrix} \begin{pmatrix}
    F_{h,k+1}^{(2)}(x) & F_{h,k}^{(2)}(x) \\
    F_{h,k}^{(2)}(x) & F_{h,k-1}^{(2)}(x)
\end{pmatrix} = \begin{pmatrix}
    e_{11} & e_{12} \\
    e_{21} & e_{22}
\end{pmatrix}
$$

(6.9)

and

$$
P_2 = \begin{pmatrix}
    p_{11} & p_{12} \\
    p_{11} & p_{22}
\end{pmatrix} = E \times M_{h,2}^{-k}(x) = \begin{pmatrix}
    e_{11} & e_{12} \\
    e_{21} & e_{22}
\end{pmatrix} \times M_{h,2}^{-k}(x).
$$

(6.10)
**Case 1.1**: \( k \) is an odd integer. Then, we have

\[
\begin{pmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{pmatrix} = \begin{pmatrix}
e_{11} & e_{12} \\
e_{21} & e_{22}
\end{pmatrix} \times \begin{pmatrix}
-F_{h,k}^{(2)}(x) & F_{h,k}^{(2)}(x) \\
F_{h,k}^{(2)}(x) & -F_{h,k+1}^{(2)}(x)
\end{pmatrix}.
\] (6.11)

It follows from (6.11) that the elements of the matrix \( P_2 \) are

\[
p_{11} = -F_{h,k-1}^{(2)}(x)e_{11} + F_{h,k}^{(2)}(x)e_{12},
\] (6.12)

\[
p_{12} = F_{h,k}^{(2)}(x)e_{11} - F_{h,k+1}^{(2)}(x)e_{12},
\] (6.13)

\[
p_{21} = -F_{h,k-1}^{(2)}(x)e_{21} + F_{h,k}^{(2)}(x)e_{22}
\] (6.14)

and

\[
p_{22} = F_{h,k}^{(2)}(x)e_{21} - F_{h,k+1}^{(2)}(x)e_{22}.
\] (6.15)

We have

\[
-F_{h,k-1}^{(2)}(x)e_{11} + F_{h,k}^{(2)}(x)e_{12} \geq 0,
\] (6.16)

\[
F_{h,k}^{(2)}(x)e_{11} - F_{h,k+1}^{(2)}(x)e_{12} \geq 0,
\] (6.17)

\[
-F_{h,k-1}^{(2)}(x)e_{21} + F_{h,k}^{(2)}(x)e_{22} \geq 0
\] (6.18)

and

\[
F_{h,k}^{(2)}(x)e_{21} - F_{h,k+1}^{(2)}(x)e_{22} \geq 0
\] (6.19)

since \( p_{11}, p_{12}, p_{21} \) and \( p_{22} \) are positive integers.

From (6.16) and (6.17), we have

\[
\frac{F_{h,k+1}^{(2)}(x)}{F_{h,k}^{(2)}(x)} \leq \frac{e_{11}}{e_{12}} \leq \frac{F_{h,k}^{(2)}(x)}{F_{h,k-1}^{(2)}(x)}, \text{ since } e_{12} > 0.
\] (6.20)

Again from (6.18) and (6.19), we have

\[
\frac{F_{h,k+1}^{(2)}(x)}{F_{h,k}^{(2)}(x)} \leq \frac{e_{21}}{e_{22}} \leq \frac{F_{h,k}^{(2)}(x)}{F_{h,k-1}^{(2)}(x)}, \text{ since } e_{22} > 0.
\] (6.21)

**Case 1.2**: \( k \) is an even integer. Then, we have

\[
\begin{pmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{pmatrix} = \begin{pmatrix}
e_{11} & e_{12} \\
e_{21} & e_{22}
\end{pmatrix} \times \begin{pmatrix}
-F_{h,k-1}^{(2)}(x) & -F_{h,k}^{(2)}(x) \\
F_{h,k}^{(2)}(x) & F_{h,k+1}^{(2)}(x)
\end{pmatrix}.
\] (6.22)
It follows from (6.22) that the elements of the matrix $P_2$ are

$$p_{11} = F_{h,k-1}^{(2)}(x)e_{11} - F_{h,k}^{(2)}(x)e_{12}, \quad (6.23)$$

$$p_{12} = -F_{h,k}^{(2)}(x)e_{11} + F_{h,k+1}^{(2)}(x)e_{12}, \quad (6.24)$$

$$p_{21} = F_{h,k-1}^{(2)}(x)e_{21} - F_{h,k}^{(2)}(x)e_{22}, \quad (6.25)$$

and

$$p_{22} = -F_{h,k}^{(2)}(x)e_{21} + F_{h,k+1}^{(2)}(x)e_{22}. \quad (6.26)$$

We have

$$F_{h,k-1}^{(2)}(x)e_{11} - F_{h,k}^{(2)}(x)e_{12} \geq 0, \quad (6.27)$$

$$-F_{h,k}^{(2)}(x)e_{11} + F_{h,k+1}^{(2)}(x)e_{12} \geq 0, \quad (6.28)$$

$$F_{h,k-1}^{(2)}(x)e_{21} - F_{h,k}^{(2)}(x)e_{22} \geq 0 \quad (6.29)$$

and

$$-F_{h,k}^{(2)}(x)e_{21} + F_{h,k+1}^{(2)}(x)e_{22} \geq 0, \quad (6.30)$$

since $p_{11}, p_{12}, p_{21}$ and $p_{22}$ are positive integers.

From (6.27) and (6.28), we have

$$\frac{F_{h,k+1}^{(2)}(x)}{F_{h,k}^{(2)}(x)} \geq \frac{e_{11}}{e_{12}} \geq \frac{F_{h,k}^{(2)}(x)}{F_{h,k-1}^{(2)}(x)}, \quad \text{since } e_{12} > 0. \quad (6.31)$$

Again from (6.29) and (6.30), we have

$$\frac{F_{h,k+1}^{(2)}(x)}{F_{h,k}^{(2)}(x)} \geq \frac{e_{21}}{e_{22}} \geq \frac{F_{h,k}^{(2)}(x)}{F_{h,k-1}^{(2)}(x)}, \quad \text{since } e_{22} > 0. \quad (6.32)$$

Hence, from (6.20), (6.21), (6.31) and (6.32), we have

\[
\frac{F_{h,k+1}^{(2)}(x)}{F_{h,k}^{(2)}(x)} \leq \frac{e_{11}}{e_{12}} \leq \frac{F_{h,k}^{(2)}(x)}{F_{h,k-1}^{(2)}(x)} \quad \text{and} \quad \frac{F_{h,k+1}^{(2)}(x)}{F_{h,k}^{(2)}(x)} \leq \frac{e_{21}}{e_{22}} \leq \frac{F_{h,k}^{(2)}(x)}{F_{h,k-1}^{(2)}(x)}
\]

or

\[
\frac{F_{h,k+1}^{(2)}(x)}{F_{h,k}^{(2)}(x)} \geq \frac{e_{11}}{e_{12}} \geq \frac{F_{h,k}^{(2)}(x)}{F_{h,k-1}^{(2)}(x)} \quad \text{and} \quad \frac{F_{h,k+1}^{(2)}(x)}{F_{h,k}^{(2)}(x)} \geq \frac{e_{21}}{e_{22}} \geq \frac{F_{h,k}^{(2)}(x)}{F_{h,k-1}^{(2)}(x)}.
\]
For large value of $k$, we have

$$\frac{e_{11}}{e_{12}} \approx r_{h,2}(x)$$

and

$$\frac{e_{21}}{e_{22}} \approx r_{h,2}(x)$$

where $r_{h,2}(x) = (h_1(x) + \sqrt{h_1^2(x) + 4})/2$.

**Case 2: $n = 3$.**

In this case,

$$E = P_3 \times M_{h,3}^k(x)$$

and

$$P_3 = E \times M_{h,3}^{-k}(x)$$

Now

$$\text{det} M_{h,3}^k(x) =$$

$$F_{h,k+2}^{(3)}(x)h_2^{(3)}(x)F_{h,k}^{(3)}(x)F_{h,k-1}^{(3)}(x) + (F_{h,k-1}^{(3)}(x))^2 - h_2^{(3)}(x)F_{h,k}^{(3)}(x)F_{h,k-1}^{(3)}(x) - F_{h,k}^{(3)}(x)F_{h,k-2}^{(3)}(x))$$

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\[+(h_2(x)F_{h,k+1}^{(3)}(x) + F_{h,k}^{(3)}(x))((F_{h,k}^{(3)}(x))^2 - F_{h,k+1}^{(3)}(x)F_{h,k-1}^{(3)}(x)) + F_{h,k+1}^{(3)}(x)(h_2(x)F_{h,k+1}^{(3)}(x))\]

\[
F_{h,k-1}^{(3)}(x) + F_{h,k+1}^{(3)}(x)F_{h,k-2}^{(3)}(x) - h_2(x)(F_{h,k}^{(3)}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k-1}^{(3)}(x) = 1 \quad (6.35)
\]

and

\[
M_{h,k}^{-1}(x) = \frac{1}{\det M_{h,k}(x)} \text{adj} M_{h,k}(x)
\]

\[
= \begin{pmatrix}
(F_{h,k-1}^{(3)}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k-2}^{(3)}(x) & F_{h,k+1}^{(3)}(x)F_{h,k-2}^{(3)}(x) - F_{h,k}^{(3)}(x)F_{h,k-1}^{(3)}(x)
(F_{h,k-1}^{(3)}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k-2}^{(3)}(x) & F_{h,k+1}^{(3)}(x)F_{h,k-2}^{(3)}(x) - F_{h,k}^{(3)}(x)F_{h,k-1}^{(3)}(x)
(F_{h,k+1}^{(3)}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k+2}^{(3)}(x) & F_{h,k+1}^{(3)}(x)F_{h,k+2}^{(3)}(x) - F_{h,k}^{(3)}(x)F_{h,k+3}^{(3)}(x)
(F_{h,k+1}^{(3)}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k+2}^{(3)}(x) & F_{h,k+1}^{(3)}(x)F_{h,k+2}^{(3)}(x) - F_{h,k}^{(3)}(x)F_{h,k+3}^{(3)}(x)
\end{pmatrix}
\]

Thus, \[
\begin{pmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33}
\end{pmatrix} = \begin{pmatrix}
e_{11} & e_{12} & e_{13} \\
e_{21} & e_{22} & e_{23} \\
e_{31} & e_{32} & e_{33}
\end{pmatrix} \times
\]

\[
\begin{pmatrix}
(F_{h,k-1}^{(3)}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k-2}^{(3)}(x) & F_{h,k+1}^{(3)}(x)F_{h,k-2}^{(3)}(x) - F_{h,k}^{(3)}(x)F_{h,k-1}^{(3)}(x)
(F_{h,k-1}^{(3)}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k-2}^{(3)}(x) & F_{h,k+1}^{(3)}(x)F_{h,k-2}^{(3)}(x) - F_{h,k}^{(3)}(x)F_{h,k-1}^{(3)}(x)
(F_{h,k+1}^{(3)}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k+2}^{(3)}(x) & F_{h,k+1}^{(3)}(x)F_{h,k+2}^{(3)}(x) - F_{h,k}^{(3)}(x)F_{h,k+3}^{(3)}(x)
(F_{h,k+1}^{(3)}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k+2}^{(3)}(x) & F_{h,k+1}^{(3)}(x)F_{h,k+2}^{(3)}(x) - F_{h,k}^{(3)}(x)F_{h,k+3}^{(3)}(x)
\end{pmatrix}
\]

Therefore,

\[
p_{11} = e_{11}((F_{h,k-1}^{(3)}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k-2}^{(3)}(x)) + e_{12}((F_{h,k}^{(3)}(x))^2 - F_{h,k+1}^{(3)}(x)F_{h,k-1}^{(3)}(x))
+ e_{13}((F_{h,k+1}^{(3)}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k+2}^{(3)}(x)) \geq 0, \quad (6.36)
\]

\[
p_{12} = e_{11}(F_{h,k-2}^{(3)}(x)F_{h,k+1}^{(3)}(x) - F_{h,k-1}^{(3)}(x)F_{h,k}^{(3)}(x)) + e_{12}(F_{h,k-1}^{(3)}(x)F_{h,k+2}^{(3)}(x) - F_{h,k}^{(3)}(x)F_{h,k+1}^{(3)}(x))
+ e_{13}(F_{h,k}^{(3)}(x)F_{h,k+3}^{(3)}(x) - F_{h,k+1}^{(3)}(x)F_{h,k+2}^{(3)}(x)) \geq 0, \quad (6.37)
\]
\[ p_{13} = e_{11}((F_{h,k}^{(3)}(x))^2 - F_{h,k-1}(x)F_{h,k+1}(x)) + e_{12}((F_{h,k+1}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k+2}(x)) + e_{13}((F_{h,k+2}(x))^2 - F_{h,k+1}(x)F_{h,k+3}(x)) \geq 0, \]

(6.38)

\[ p_{21} = e_{21}((F_{h,k-1}(x))^2 - F_{h,k}(x)F_{h,k-2}(x)) + e_{22}((F_{h,k}(x))^2 - F_{h,k+1}(x)F_{h,k-1}(x)) + e_{23}((F_{h,k+1}(x))^2 - F_{h,k}(x)F_{h,k+2}(x)) \geq 0, \]

(6.39)

\[ p_{22} = e_{21}(F_{h,k-2}(x)F_{h,k+1}(x) - F_{h,k-1}(x)F_{h,k}(x)) + e_{22}(F_{h,k-1}(x)F_{h,k+2}(x) - F_{h,k}(x)F_{h,k+1}(x)) + e_{23}(F_{h,k}(x)F_{h,k+3}(x) - F_{h,k+1}(x)F_{h,k+2}(x)) \geq 0, \]

(6.40)

\[ p_{23} = e_{21}((F_{h,k}^{(3)}(x))^2 - F_{h,k-1}(x)F_{h,k+1}(x)) + e_{22}((F_{h,k+1}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k+2}(x)) + e_{23}((F_{h,k+2}(x))^2 - F_{h,k+1}(x)F_{h,k+3}(x)) \geq 0, \]

(6.41)

\[ p_{31} = e_{31}((F_{h,k-1}(x))^2 - F_{h,k}(x)F_{h,k-2}(x)) + e_{32}((F_{h,k}(x))^2 - F_{h,k+1}(x)F_{h,k-1}(x)) + e_{33}((F_{h,k+1}(x))^2 - F_{h,k}(x)F_{h,k+2}(x)) \geq 0, \]

(6.42)

\[ p_{32} = e_{31}(F_{h,k-2}(x)F_{h,k+1}(x) - F_{h,k-1}(x)F_{h,k}(x)) + e_{32}(F_{h,k-1}(x)F_{h,k+2}(x) - F_{h,k}(x)F_{h,k+1}(x)) + e_{33}(F_{h,k}(x)F_{h,k+3}(x) - F_{h,k+1}(x)F_{h,k+2}(x)) \geq 0 \]

(6.43)

and

\[ p_{33} = e_{31}((F_{h,k}^{(3)}(x))^2 - F_{h,k-1}(x)F_{h,k+1}(x)) + e_{32}((F_{h,k+1}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k+2}(x)) + e_{33}((F_{h,k+2}(x))^2 - F_{h,k+1}(x)F_{h,k+3}(x)) \geq 0. \]

(6.44)

Dividing both sides of (6.36), (6.37) and (6.38) by \(e_{11} \ (> 0)\), we have

\[
((F_{h,k+1}^{(3)}(x))^2 - F_{h,k}^{(3)}(x)F_{h,k+2}(x))^\frac{\varepsilon_{11}}{e_{11}} \geq (F_{h,k+1}^{(3)}(x)F_{h,k-2}(x) - (F_{h,k}^{(3)}(x))^2)^\frac{e_{12}}{e_{11}}
\]

\[ + (F_{h,k}^{(3)}(x)F_{h,k-2}(x) - (F_{h,k-1}^{(3)}(x))^2), \]

(6.45)

\[
(F_{h,k+1}^{(3)}(x)F_{h,k+2}(x) - F_{h,k}^{(3)}(x)F_{h,k+3}(x))^\frac{\varepsilon_{11}}{e_{11}} \leq (F_{h,k+2}^{(3)}(x)F_{h,k-1}(x) - F_{h,k}^{(3)}(x)F_{h,k+1}(x))^\frac{e_{14}}{e_{11}}
\]

\[ + (F_{h,k+1}(x)F_{h,k-2}(x) - F_{h,k}^{(3)}(x)F_{h,k-1}(x)) \]

(6.46)
Let \( a = ((F_{h,k+1}(x))^2 - F_{h,k+1}(x)F_{h,k+3}(x)) \), \( b = (F_{h,k+1}(x)F_{h,k+2}(x) - F_{h,k}(x)F_{h,k+3}(x)) \), \( c = ((F_{h,k+2}(x))^2 - F_{h,k+1}(x)F_{h,k+3}(x)) \).

Now \( 3^3 = 27 \) cases arise for \( a \geq 0, b \geq 0, c \geq 0 \). Here we discuss some of the 27 cases.

**Case 2.1:** \( a > 0, b > 0, c > 0 \).

Then from (6.45), we have

\[
\frac{e_{13}}{e_{11}} \geq u, \tag{6.48}
\]

where

\[
u = \frac{e_{12}}{e_{11}} \left( \frac{F_{h,k+1}(x)F_{h,k-2}(x) - (F_{h,k}(x))^2}{(F_{h,k+1}(x))^2 - F_{h,k}(x)F_{h,k+2}(x)} + \frac{F_{h,k}(x)F_{h,k-2}(x) - (F_{h,k-1}(x))^2}{(F_{h,k+1}(x))^2 - F_{h,k}(x)F_{h,k+2}(x)} \right).
\]

From (6.46), we have

\[
\frac{e_{13}}{e_{11}} \leq v, \tag{6.49}
\]

where

\[
v = \frac{e_{12}}{e_{11}} \left( \frac{F_{h,k+1}(x)F_{h,k-1}(x) - F_{h,k}(x)F_{h,k+1}(x)}{F_{h,k+1}(x)F_{h,k+2}(x) - F_{h,k}(x)F_{h,k+3}(x)} + \frac{F_{h,k+1}(x)F_{h,k-2}(x) - F_{h,k}(x)F_{h,k+2}(x)}{F_{h,k+1}(x)F_{h,k+2}(x) - F_{h,k}(x)F_{h,k+3}(x)} \right).
\]

From (6.47), we have

\[
\frac{e_{13}}{e_{11}} \geq w, \tag{6.50}
\]

where

\[
w = \frac{e_{12}}{e_{11}} \left( \frac{F_{k+2}(x)F_{k}(x) - (F_{k+1}(x))^2}{(F_{k+2}(x))^2 - F_{k+1}(x)F_{k+3}(x)} + \frac{F_{k+1}(x)F_{k-1}(x) - (F_{k}(x))^2}{(F_{k+2}(x))^2 - F_{k+1}(x)F_{k+3}(x)} \right).
\]

Using (6.35), from (6.48) and (6.49), we have

\[
\frac{e_{11}}{e_{12}} \geq \min \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\}. \tag{6.51}
\]
Using (6.35), from (6.49) and (6.50), we have

$$
\frac{e_{11}}{e_{12}} \leq \max \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\}.
$$

From (6.51) and (6.52), we have

$$
\min \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\} \leq \frac{e_{11}}{e_{12}}
$$

$$
\leq \max \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\}.
$$

Similarly, we have

$$
\min \left\{ \frac{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}{F_{h,k+1}(x)}, \frac{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}{F_{h,k}(x)}, \frac{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)}{F_{h,k-1}(x)} \right\} \leq \frac{e_{12}}{e_{13}}
$$

$$
\leq \max \left\{ \frac{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}{F_{h,k+1}(x)}, \frac{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}{F_{h,k}(x)}, \frac{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)}{F_{h,k-1}(x)} \right\}
$$

and

$$
\min \left\{ \frac{F_{h,k+2}(x)}{F_{h,k+1}(x)}, \frac{F_{h,k+1}(x)}{F_{h,k}(x)}, \frac{F_{h,k}(x)}{F_{h,k-1}(x)} \right\} \leq \frac{e_{11}}{e_{13}} \leq \max \left\{ \frac{F_{h,k+2}(x)}{F_{h,k+1}(x)}, \frac{F_{h,k+1}(x)}{F_{h,k}(x)}, \frac{F_{h,k}(x)}{F_{h,k-1}(x)} \right\}.
$$

**Case 2.2:** $a = 0$, $b > 0$, $c > 0$.

Since $a = 0$, from (6.45) we have

$$
\frac{e_{11}}{e_{12}} \geq \min \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\}.
$$

Using (6.35) and $a = 0$, from (6.46) and (6.47) we have

$$
\frac{e_{11}}{e_{12}} \leq \max \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\}.
$$
From (6.54) and (6.55), we have

\[
\min \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\} \leq \frac{e_{11}}{e_{12}}
\]

\leq \max \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\},

\tag{6.56}

**Case 2.3:** \(a < 0, b < 0, c < 0\).

Then from (6.45), we have

\[
\frac{e_{13}}{e_{11}} \leq u,
\]

where

\[
u = \frac{e_{12}}{e_{11}} \left( \frac{F_{h,k+1}(x)F_{h,k-2}(x) - (F_{h,k}(x))^2}{(F_{h,k+1}(x))^2 - F_{h,k}(x)F_{h,k+2}(x)} + \frac{F_{h,k}(x)F_{h,k-2}(x) - (F_{h,k-1}(x))^2}{(F_{h,k+1}(x))^2 - F_{h,k}(x)F_{h,k+2}(x)} \right).
\]

From (6.46), we have

\[
\frac{e_{13}}{e_{11}} \geq v,
\]

where

\[
v = \frac{e_{12}}{e_{11}} \left( \frac{F_{h,k+2}(x)F_{h,k-1}(x) - F_{h,k}(x)F_{h,k+1}(x)}{F_{h,k+1}(x)F_{h,k+2}(x) - F_{h,k}(x)F_{h,k+3}(x)} + \frac{F_{h,k+1}(x)F_{h,k-2}(x) - F_{h,k}(x)F_{h,k-1}(x)}{F_{h,k+1}(x)F_{h,k+2}(x) - F_{h,k}(x)F_{h,k+3}(x)} \right).
\]

From (6.47), we have

\[
\frac{e_{13}}{e_{11}} \leq w,
\]

where

\[
w = \frac{e_{12}}{e_{11}} \left( \frac{F_{h,k+2}(x)F_{h,k}(x) - (F_{h,k+1}(x))^2}{(F_{h,k+2}(x))^2 - F_{h,k+1}(x)F_{h,k+3}(x)} + \frac{F_{h,k+1}(x)F_{h,k-1}(x) - (F_{h,k}(x))^2}{(F_{h,k+2}(x))^2 - F_{h,k+1}(x)F_{h,k+3}(x)} \right).
\]

Using (6.35), from (6.57) and (6.58) we have

\[
\frac{e_{11}}{e_{12}} \leq \max \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\},
\]

\tag{6.60}
Using (6.35), from (6.58) and (6.59) we have
\[
\frac{e_{11}}{e_{12}} \geq \min \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\}.
\]

From (6.60) and (6.61), we have
\[
\min \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\} \leq \frac{e_{11}}{e_{12}}
\]
\[
\leq \max \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\}.
\]

Similarly, we have
\[
\min \left\{ \frac{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}{F_{h,k+1}(x)}, \frac{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}{F_{h,k}(x)}, \frac{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)}{F_{h,k-1}(x)} \right\} \leq \frac{e_{12}}{e_{13}}
\]
\[
\leq \max \left\{ \frac{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}{F_{h,k+1}(x)}, \frac{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}{F_{h,k}(x)}, \frac{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)}{F_{h,k-1}(x)} \right\}
\]
and
\[
\min \left\{ \frac{F_{h,k+2}(x)}{F_{h,k+1}(x)}, \frac{F_{h,k+1}(x)}{F_{h,k}(x)}, \frac{F_{h,k}(x)}{F_{h,k-1}(x)} \right\} \leq \frac{e_{11}}{e_{13}} \leq \max \left\{ \frac{F_{h,k+2}(x)}{F_{h,k+1}(x)}, \frac{F_{h,k+1}(x)}{F_{h,k}(x)}, \frac{F_{h,k}(x)}{F_{h,k-1}(x)} \right\}.
\]

Similarly, we can prove the rest of the cases.

Thus, we have
\[
\min \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\} \leq \frac{e_{11}}{e_{12}}
\]
\[
\leq \max \left\{ \frac{F_{h,k+2}(x)}{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}, \frac{F_{h,k+1}(x)}{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}, \frac{F_{h,k}(x)}{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)} \right\},
\]
\[
\min \left\{ \frac{h_2(x)F_{h,k+1}(x) + F_{h,k}(x)}{F_{h,k+1}(x)}, \frac{h_2(x)F_{h,k}(x) + F_{h,k-1}(x)}{F_{h,k}(x)}, \frac{h_2(x)F_{h,k-1}(x) + F_{h,k-2}(x)}{F_{h,k-1}(x)} \right\} \leq \frac{e_{12}}{e_{13}}.
\]
\[
\leq \max\left\{ \frac{h_2(x)F_{h,k+1}^{(3)}(x) + F_{h,k}^{(3)}(x)}{F_{h,k+1}^{(3)}(x)}, \frac{h_2(x)F_{h,k}^{(3)}(x) + F_{h,k-1}^{(3)}(x)}{F_{h,k}^{(3)}(x)}, \frac{h_2(x)F_{h,k}^{(3)}(x) + F_{h,k-2}^{(3)}(x)}{F_{h,k-1}^{(3)}(x)} \right\},
\]

\[
\min\left\{ \frac{F_{h,k+2}^{(3)}(x)}{F_{h,k+1}^{(3)}(x)}, \frac{F_{h,k+1}^{(3)}(x)}{F_{h,k}^{(3)}(x)}, \frac{F_{h,k}^{(3)}(x)}{F_{h,k-1}^{(3)}(x)} \right\} \leq \frac{e_{i1}}{e_{i2}} \leq \frac{e_{i2}}{e_{i3}},
\]

\[
\leq \max\left\{ \frac{F_{h,k+2}^{(3)}(x)}{h_2(x)F_{h,k+1}^{(3)}(x) + F_{h,k}^{(3)}(x)}, \frac{F_{h,k+1}^{(3)}(x)}{h_2(x)F_{h,k}^{(3)}(x) + F_{h,k-1}^{(3)}(x)}, \frac{F_{h,k}^{(3)}(x)}{h_2(x)F_{h,k-1}^{(3)}(x) + F_{h,k-2}^{(3)}(x)} \right\},
\]

\[
\min\left\{ \frac{h_2(x)F_{h,k+1}^{(3)}(x) + F_{h,k}^{(3)}(x)}{F_{h,k+1}^{(3)}(x)}, \frac{h_2(x)F_{h,k}^{(3)}(x) + F_{h,k-1}^{(3)}(x)}{F_{h,k}^{(3)}(x)}, \frac{h_2(x)F_{h,k}^{(3)}(x) + F_{h,k-2}^{(3)}(x)}{F_{h,k-1}^{(3)}(x)} \right\} \leq \frac{e_{i2}}{e_{i3}},
\]

and

\[
\min\left\{ \frac{F_{h,k+2}^{(3)}(x)}{F_{h,k+1}^{(3)}(x)}, \frac{F_{h,k+1}^{(3)}(x)}{F_{h,k}^{(3)}(x)}, \frac{F_{h,k}^{(3)}(x)}{F_{h,k-1}^{(3)}(x)} \right\} \leq \frac{e_{i1}}{e_{i3}} \leq \max\left\{ \frac{F_{h,k+2}^{(3)}(x)}{F_{h,k+1}^{(3)}(x)}, \frac{F_{h,k+1}^{(3)}(x)}{F_{h,k}^{(3)}(x)}, \frac{F_{h,k}^{(3)}(x)}{F_{h,k-1}^{(3)}(x)} \right\},
\]

for \(i = 1, 2, 3\).

Thus for large \(k\), we have

\[
\frac{e_{i1}}{e_{i2}} \approx \frac{(r_{h,3}(x))^2}{1 + h_2(x)r_{h,3}(x)},
\]

\[
\frac{e_{i2}}{e_{i3}} \approx \frac{1 + h_2(x)r_{h,3}(x)}{r_{h,3}(x)},
\]

\[
\frac{e_{i1}}{e_{i3}} \approx r_{h,3}(x),
\]

for \(i = 1, 2, 3\).
where, \( r_{h,3}(x) = \{2^{\frac{1}{3}}h_1(x) + (2(h_1(x))^3 + 9h_1(x)h_2(x) + 27 + \sqrt{-27(h_1(x))^2(h_2(x))^2 + 729 + 486h_1(x)h_2(x) + 108(h_1(x))^3 - 108(h_2(x))^3})^{\frac{1}{3}} + (2(h_1(x))^3 + 9h_1(x)h_2(x) + 27 - \sqrt{-27(h_1(x))^2(h_2(x))^2 + 729 + 486h_1(x)h_2(x) + 108(h_1(x))^3 - 108(h_2(x))^3})^{\frac{1}{3}} \}/(2^{\frac{1}{3}} \times 3) \)

**Case 3:** For any value of \( n \).

The generalized relations among the code matrix elements are:

\[
\min \left\{ \frac{F_{h,k+r}^{(n)}}{h_2(x)F_{h,k+r-1}^{(n)}(x) + h_3(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\} \leq \frac{e_{i1}}{e_{i2}}
\]

\[
\leq \max \left\{ \frac{F_{h,k+r}^{(n)}}{h_2(x)F_{h,k+r-1}^{(n)}(x) + h_3(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\},
\]

\[
\min \left\{ \frac{F_{h,k+r}^{(n)}}{h_3(x)F_{h,k+r-1}^{(n)}(x) + h_4(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\} \leq \frac{e_{i1}}{e_{i3}}
\]

\[
\leq \max \left\{ \frac{F_{h,k+r}^{(n)}}{h_3(x)F_{h,k+r-1}^{(n)}(x) + h_4(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\},
\]

\[
\min \left\{ \frac{h_2(x)F_{h,k+r-1}^{(n)}(x) + h_3(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+1}^{(n)}(x)}{h_3(x)F_{h,k+r-1}^{(n)}(x) + h_4(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+2}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\} \leq \frac{e_{i2}}{e_{i3}}
\]

\[
\leq \max \left\{ \{h_2(x)F_{h,k+r-1}^{(n)}(x) + h_3(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+1}^{(n)}(x) : r = 0, 1, \ldots, n-1 \} \leq \frac{e_{i2}}{e_{i3}}
\]

\[
\min \left\{ \frac{h_2(x)F_{h,k+r-1}^{(n)}(x) + h_3(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+1}^{(n)}(x)}{h_4(x)F_{h,k+r-1}^{(n)}(x) + h_5(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+3}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\} \leq \frac{e_{i2}}{e_{i4}}
\]

\[
\leq \max \left\{ \frac{h_2(x)F_{h,k+r-1}^{(n)}(x) + h_3(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+1}^{(n)}(x)}{h_4(x)F_{h,k+r-1}^{(n)}(x) + h_5(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+3}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\},
\]

\[
\ldots
\]
\[
\min \left\{ \frac{h_2(x)F_{h,k+r-1}^{(n)}(x) + h_3(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+1}^{(n)}(x)}{F_{h,k+r-1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\} \leq \frac{e_{i2}}{e_{in}}
\]

\[
\leq \max \left\{ \frac{h_2(x)F_{h,k+r-1}^{(n)}(x) + h_3(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+1}^{(n)}(x)}{F_{h,k+r-1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\},
\]

\[
\min \left\{ \frac{h_3(x)F_{h,k+r-1}^{(n)}(x) + h_4(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+2}^{(n)}(x)}{F_{h,k+r-1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\} \leq \frac{e_{i3}}{e_{i4}}
\]

\[
\leq \max \left\{ \frac{h_3(x)F_{h,k+r-1}^{(n)}(x) + h_4(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+2}^{(n)}(x)}{F_{h,k+r-1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\},
\]

\[
\min \left\{ \frac{h_3(x)F_{h,k+r-1}^{(n)}(x) + h_4(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+2}^{(n)}(x)}{F_{h,k+r-1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\} \leq \frac{e_{i3}}{e_{i5}}
\]

\[
\leq \max \left\{ \frac{h_3(x)F_{h,k+r-1}^{(n)}(x) + h_4(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+2}^{(n)}(x)}{F_{h,k+r-1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\},
\]

\[
\min \left\{ \frac{h_3(x)F_{h,k+r-1}^{(n)}(x) + h_4(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+2}^{(n)}(x)}{F_{h,k+r-1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\} \leq \frac{e_{i3}}{e_{in}}
\]

\[
\leq \max \left\{ \frac{h_3(x)F_{h,k+r-1}^{(n)}(x) + h_4(x)F_{h,k+r-2}^{(n)}(x) + \ldots + F_{h,k+r-n+2}^{(n)}(x)}{F_{h,k+r-1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\},
\]

\[
\min \left\{ \frac{h_{n-1}(x)F_{h,k+r-1}^{(n)}(x) + F_{h,k+r-2}^{(n)}(x)}{F_{h,k+r-1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\} \leq \frac{e_{in-1}}{e_{in}}
\]

\[
\leq \max \left\{ \frac{h_{n-1}(x)F_{h,k+r-1}^{(n)}(x) + F_{h,k+r-2}^{(n)}(x)}{F_{h,k+r-1}^{(n)}(x)} : r = 0, 1, \ldots, n-1 \right\},
\]

for \(i = 1, 2, 3, \ldots, n\).

Hence the large value of \(k\), we have

\[
\frac{e_{i1}}{e_{i2}} \approx 1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_2(x)(r_{h,n}(x))^{n-2},
\]

\[
\frac{e_{i1}}{e_{i3}} \approx 1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_3(x)(r_{h,n}(x))^{n-3},
\]

\[
\ldots
\]

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In this section we show that the error detection and correction depending fully on the
elements of the code matrix $E$ is possible by choosing

\[
e_{i1} \approx r_{h,n}(x),
\]

\[
e_{i2} \approx \frac{1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_2(x)(r_{h,n}(x))^{n-2}}{r_{h,n} + h_{n-1}(x)(r_{h,n}(x))^2 + h_{n-2}(x)(r_{h,n}(x))^3 + \ldots + h_3(x)(r_{h,n}(x))^{n-2}},
\]

\[
e_{i2} \approx \frac{1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_2(x)(r_{h,n}(x))^{n-2}}{(r_{h,n}(x))^2 + h_{n-1}(x)(r_{h,n}(x))^3 + h_{n-2}(x)(r_{h,n}(x))^4 + \ldots + h_4(x)(r_{h,n}(x))^{n-2}},
\]

\[
e_{i2} \approx \frac{1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_2(x)(r_{h,n}(x))^{n-2}}{(r_{h,n}(x))^{n-2}},
\]

\[
e_{i3} \approx \frac{1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_3(x)(r_{h,n}(x))^{n-3}}{r_{h,n}(x) + h_{n-1}(x)(r_{h,n}(x))^2 + h_{n-1}(x)(r_{h,n}(x))^3 + \ldots + h_4(x)(r_{h,n}(x))^{n-3}},
\]

\[
e_{i3} \approx \frac{1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_3(x)(r_{h,n}(x))^{n-3}}{(r_{h,n}(x))^2 + h_{n-1}(x)(r_{h,n}(x))^3 + h_{n-2}(x)(r_{h,n}(x))^4 + \ldots + h_5(x)(r_{h,n}(x))^{n-3}},
\]

\[
e_{i3} \approx \frac{1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_3(x)(r_{h,n}(x))^{n-3}}{(r_{h,n}(x))^{n-3}},
\]

\[
e_{i4} \approx \frac{1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_4(x)(r_{h,n}(x))^{n-2}}{r_{h,n}(x) + h_{n-1}(x)(r_{h,n}(x))^2 + h_{n-1}(x)(r_{h,n}(x))^3 + \ldots + h_5(x)(r_{h,n}(x))^{n-3}},
\]

\[
e_{i5} \approx \frac{1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_5(x)(r_{h,n}(x))^{n-3}}{(r_{h,n}(x))^{n-3}},
\]

\[
e_{i3} \approx \frac{1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_5(x)(r_{h,n}(x))^{n-3}}{(r_{h,n}(x))^{n-3}},
\]

\[
e_{i4} \approx \frac{1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_4(x)(r_{h,n}(x))^{n-2}}{(r_{h,n}(x))^{n-3}},
\]

\[
e_{i5} \approx \frac{1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_5(x)(r_{h,n}(x))^{n-3}}{(r_{h,n}(x))^{n-3}},
\]

\[
e_{i6} \approx \frac{1 + h_{n-1}(x)r_{h,n}(x) + h_{n-2}(x)(r_{h,n}(x))^2 + \ldots + h_6(x)(r_{h,n}(x))^{n-2}}{(r_{h,n}(x))^{n-3}},
\]

\[
\ldots
\]

\[
\text{for } i = 1, 2, 3, \ldots, n
\]

where $r_{h,n}(x) = \lim_{k \to \infty} \frac{f_{h,k}^{(n)}(x)}{f_{h,k-1}^{(n)}(x)}$, generalized Fibonacci $n$-step polynomial constant.

### 6.5 Error Detection and Correction

In this section we show that the error detection and correction depending fully on the equation

\[
det E = det(P_n \times M_{h,n}^k(x)) = detP_n \times detM_{h,n}^k(x) = detP_n \times (-1)^{n+1}k,
\]

is possible by choosing $k$ in such a manner that $e_{ij} > 0$ for $i, j = 1, 2, 3, \ldots, n$. At first we calculate the determinant of the initial matrix $P_n$, then we determine the code matrix elements of the code matrix $E$ and send it to a communication channel. We treat $DetP_n$ as the checking elements of the code matrix $E$ received from the communication channel.
We calculate the determinant of the matrix $E$ received and compare it with the given $det P_n$ by the relation (6.70). If they satisfy the relation (6.70), then we conclude that the elements of the code matrix $E$ received are without errors otherwise there are errors. If there are errors, then we try to correct these errors using the relations (6.70) and (6.69).

**Case 1: $n = 2$.**

Our first hypothesis is that there is a case of “single error” in the code matrix $E$ received from the communication channel. It is clear that there are four variants of the “single error” in the code matrix $E$ received such as

$$(a) \begin{pmatrix} x_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}, (b) \begin{pmatrix} e_{11} & x_{12} \\ e_{21} & e_{22} \end{pmatrix}, (c) \begin{pmatrix} e_{11} & e_{12} \\ x_{21} & e_{22} \end{pmatrix} \text{ and (d) } \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & x_{22} \end{pmatrix},$$

where $x_{11}, x_{12}, x_{21}$ and $x_{22}$ are possible destroyed elements.

For checking the possible “single error”, we can write the following algebraic equations based on the “checking relation” (6.70):

$$x_{11}e_{22} - e_{12}e_{31} = det P_2 \times (-1)^{3k} \text{ (“single error” occurs in the (1, 1) cell), (6.71)}$$

$$e_{11}e_{22} - x_{12}e_{31} = det P_2 \times (-1)^{3k} \text{ (“single error” occurs in the (1, 2) cell), (6.72)}$$

$$e_{11}e_{22} - e_{12}x_{31} = det P_2 \times (-1)^{3k} \text{ (“single error” occurs in the (2, 1) cell) (6.73)}$$

and

$$e_{11}x_{22} - e_{12}e_{31} = det P_2 \times (-1)^{3k} \text{ (“single error” occurs in the (2, 2) cell). (6.74)}$$

From (6.71) - (6.74), the four variants of the possible “single error” are

$$x_{11} = \frac{(-1)^{3k} + e_{12}e_{21}}{e_{22}},$$

$$x_{12} = \frac{(-1)^{3k} + e_{11}e_{22}}{e_{21}},$$

$$x_{21} = \frac{(-1)^{3k} + e_{11}e_{22}}{e_{12}},$$

and

$$x_{22} = \frac{(-1)^{3k} + e_{12}e_{21}}{e_{11}}.$$
To obtain the correct variant, we have to choose the integer solutions of \( x_{11}, x_{12}, x_{21} \) and \( x_{22} \) satisfying the additional “checking relations” (6.33) and (6.34). If this fails then we have to conclude that our hypothesis about “single error” is incorrect or in other words there is more than one error in the code matrix \( E \) received.

Now, we consider one of the “double errors” in the code matrix \( E \) received as

\[
\begin{pmatrix}
  x_{11} & x_{12} \\
  e_{21} & e_{22}
\end{pmatrix}.
\]

(6.79)

We have the following algebraic equation for the matrix (6.79), using the “checking relation” (6.70):

\[
x_{11}e_{22} - x_{12}e_{31} = \text{det}P_2 \times (-1)^{3k}.
\]

(6.80)

It is important to emphasize that the equation (6.80) is “Diophantine” equation in two variables. As the “Diophantine” equation has many solutions, we must select such solutions of \( x_{11} \) and \( x_{12} \) which satisfy the additional “checking relations” (6.33) and (6.34). In this way we can correct all possible “double errors” in the code matrix \( E \) received which satisfy the “checking relations” (6.70), (6.33) and (6.34). Otherwise, there may be “triple errors” in the code matrix \( E \) received and we try to correct all possible “triple errors” by the similar approach.

Let one form of the incorrect code matrices \( E \) received having “triple errors” be

\[
\begin{pmatrix}
  x_{11} & x_{12} \\
  x_{21} & e_{22}
\end{pmatrix}.
\]

Thus, we try to correct the errors in the code matrix \( E \) received based on different hypotheses “single error”, “double errors” and “triple errors” using the “checking relations” (6.70), (6.33) and (6.34) together with considering the integral elements of the code matrix \( E \) received.

Code matrix \( E \) received is “erroneous” or the case of “fourfold errors” if at least one of the previous solutions is not integer. In this case we reject the code matrix \( E \) received.

Thus we have

\[
\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^{2^2} - 1 = 15
\]

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cases of errors in the code matrix $E$ received.

Our method shows that it allows correcting all possible “single error”, “double errors” and “triple errors” i.e. 14 cases among them. Hence the correct ability of errors of this method is $\frac{14}{15} = 0.9333 = 93.33\%$ which does not depend on the polynomial coefficients excepting $h_2(x) = 1$.

**Case 2: $n = 3$.**

At first, we consider “single error” in the code matrix $E$ received. It is clear that there are nine variants of “single error” in the code matrix $E$ received. For example, one of them is

$$
\begin{pmatrix}
x_{11} & e_{12} & e_{13} \\
e_{21} & e_{22} & e_{23} \\
e_{31} & e_{32} & e_{33}
\end{pmatrix}.
$$

(6.81)

where $x_{11}$ is the possible destroyed element in the $(1,1)$ cell.

Using relation (6.70), we have the algebraic equation of the matrix (6.81):

$$
x_{11}(e_{22}e_{33} - e_{23}e_{32}) + e_{12}(e_{23}e_{31} - e_{21}e_{33}) + e_{13}(e_{21}e_{32} - e_{22}e_{31}) = detP_3.
$$

(6.82)

There are nine equations similar to (6.82) for nine possible variants of “single error” $x_{ij}$, $i, j = 1, 2, 3$. But we have to select the correct variant only among these cases of the integer solutions of $x_{ij}$, $i, j = 1, 2, 3$ satisfying the relations (6.66), (6.67) and (6.68). If there is no integer solution, we conclude that our hypothesis about “single error” is incorrect or we have more than one error in the code matrix $E$ received.

Now we check all hypotheses of “double errors” in the code matrix $E$ received. We consider one of the “double errors” cases in the code matrix $E$ received as:

$$
\begin{pmatrix}
x_{11} & x_{12} & e_{13} \\
e_{21} & e_{22} & e_{23} \\
e_{31} & e_{32} & e_{33}
\end{pmatrix}.
$$

(6.83)

Using the relation (6.5), we have the algebraic equation for the matrix (6.83) as:

$$
x_{11}(e_{22}e_{33} - e_{23}e_{32}) + x_{12}(e_{23}e_{31} - e_{21}e_{33}) = e_{13}(e_{22}e_{31} - e_{21}e_{32}) + detP_3.
$$

(6.84)
Equation (6.84) is “Diophantine” equation in two variables. As the “Diophantine” equation has many solutions, we have to choose the integer solutions of \( x_{11} \) and \( x_{12} \) which satisfy the relation \( x_{11} \approx \frac{(r_{h,3}(x))^2}{1+h_2(x)r_{h,3}(x)}x_{12} \) according to the relation (6.66).

It is clear that there are \( \binom{9}{2} = 36 \) variants of “double errors” in the code matrix \( E \) received and by using similar approach, we try to correct all “double errors” in the code matrix \( E \) received.

Now there are
\[
\binom{9}{1} + \binom{9}{2} + \ldots + \binom{9}{9} = 2^3 - 1 = 511
\]
possible cases of errors in the code matrix \( E \) received. We try to correct all possible triple, fourfold, . . . , eightfold errors in the code matrix \( E \) received using this approach. But we know that it is not possible to correct the “ninefold errors” in the code matrix \( E \) received.

Hence the correct ability of errors in this method is \( \frac{510}{511} = 0.9980 = 99.80\% \) which does not depend on the polynomial coefficients excepting \( h_3(x) = 1 \).

**Case 3:** \( n = m \), where \( m \) is large.

In this case, the correct ability of this method is \( \frac{2n^2 - 2}{2n^2 - 1} \) which depends on the value of \( n = m \) but not on the polynomial coefficients excepting \( h_m(x) = 1 \).

The interesting feature of this method is that for large value of \( n \) the correct ability of errors is
\[
\frac{2n^2 - 2}{2n^2 - 1} \approx 1 = 100\%
\]
which does not depend on polynomial coefficients excepting \( h_n(x) = 1 \).