Chapter 5

Coding Theory on Constant Coefficient Fibonacci $n$-Step Numbers

5.1 Introduction

This chapter presents constant coefficient Fibonacci $n$-step numbers. The constant coefficient Fibonacci $n$-step numbers $g_k^{(n)}$ satisfy the recurrence relation:

$$g_k^{(n)} = a_1 g_{k-1}^{(n)} + a_2 g_{k-2}^{(n)} + \ldots + a_n g_{k-n}^{(n)}, \quad a_n \neq 0 \quad (5.1)$$

with the initial terms $g_0^{(n)} = g_1^{(n)} = \ldots = g_{n-2}^{(n)} = 0, g_{n-1}^{(n)} = 1$ for $k = 0, \pm 1, \pm 2, \pm 3, \ldots, n = 1, 2, 3 \ldots$ and $a_1, a_2, \ldots, a_n$ are constants. $g_k^{(n)}$ coincides with $F_k^{(n)}$ for $a_1 = a_2 = \ldots = a_n = 1$.

At first we form a matrix $G_n$ given by
\[
G_n = \begin{pmatrix}
a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]

\[
g_{n-1}^{(n)} a_{2g_{n-2}}^{(n)} + \ldots + a_n g_{1}^{(n)} \\
g_{n-2}^{(n)} a_{2g_{n-3}}^{(n)} + \ldots + a_n g_{0}^{(n)} \\
\vdots \\
g_{1}^{(n)} a_{2g_{0}}^{(n)} + a_3 g_{-1}^{(n)} + \ldots + a_n g_{-n+2}^{(n)} \\
g_{0}^{(n)} a_{2g_{-1}}^{(n)} + a_3 g_{-2}^{(n)} + \ldots + a_n g_{-n+3}^{(n)} \\
\end{pmatrix}
\]

Therefore \( \det G_n = (-1)^{n+1}a_n \).

The inverse of \( G_n \) is

\[
G_n^{-1} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
g_{n-2}^{(n)} a_{2g_{n-3}}^{(n)} + a_3 g_{n-4}^{(n)} + \ldots + a_n g_{-1}^{(n)} \\
g_{n-3}^{(n)} a_{2g_{n-4}}^{(n)} + a_3 g_{n-5}^{(n)} + \ldots + a_n g_{-2}^{(n)} \\
\vdots \\
g_{1}^{(n)} a_{2g_{0}}^{(n)} + a_3 g_{-1}^{(n)} + \ldots + a_n g_{-n+2}^{(n)} \\
g_{0}^{(n)} a_{2g_{-1}}^{(n)} + a_3 g_{-2}^{(n)} + \ldots + a_n g_{-n+3}^{(n)} \\
\end{pmatrix}
\]
Next we develop a coding theory based on the constant coefficient Fibonacci $n$-step numbers. The interesting feature of this newly developed coding/decoding method is that correct ability of this method does not depend on the constant coefficients. Finally we compare the correct ability of this coding/decoding procedure with respect to the value of $n$.

5.2 Characteristic Equation of $g_k^{(n)}$

The characteristic equation of (5.1) is

$$x^n - a_1x^{n-1} - a_2x^{n-2} - \ldots - a_{n-1}x - a_n = 0$$

(5.2)

We consider $a_1, a_2, \ldots, a_n > 0$. So the equation (5.2) has $n$ roots of which exactly one is positive. We name the only one positive root $x = \alpha_n$ as $n$-anacci $(a_1, a_2, \ldots, a_n)$ constant.

For example, $\alpha_2 = (a_1 + \sqrt{a_1^2 + 4a_2})/2$,

$$\alpha_3 = \left\{ 2^{\frac{1}{3}}a_1 + (2a_1^3 + 9a_1a_2 + 27a_3 + \sqrt{-27a_1^2a_2^2 + 729a_3^3}486a_1a_2a_3 + 108a_1^3a_3 - 108a_2^3)^{\frac{1}{3}}
+ (2a_1^3 + 9a_1a_2 + 27a_3 - \sqrt{-27a_1^2a_2^2 + 729a_3^3}486a_1a_2a_3 + 108a_1^3a_3 - 108a_2^3)^{\frac{1}{3}} \right\}/(2^{\frac{1}{3}} \times 3).$$

When $a_1 = a_2 = \ldots = a_n = 1$, $\alpha_n$ coincides with $r_n$.

5.3 Some Properties of the Matrix $G_n$

$G_n$ satisfies the following properties for $k, l = 0, \pm 1, \pm 2, \pm 3 \ldots$.

Property 5.3.1 $G_n^k G_n^l = G_n^{k+l}$.

Property 5.3.2 $\det G_n^k = (\det G_n)^k = ((-1)^{n+1}a_n)^k = (-1)^{(n+1)k}a_n^k$. 

99
Theorem 5.3.1 \( G_n^k = \)
\[
\begin{pmatrix}
g^{(n)}_{k+n-1} & a_2g^{(n)}_{k+n-2} + a_3g^{(n)}_{k+n-3} + \ldots + a_ng^{(n)}_k \\
g^{(n)}_{k+n-2} & a_2g^{(n)}_{k+n-3} + a_3g^{(n)}_{k+n-4} + \ldots + a_ng^{(n)}_{k-1} \\
\vdots & \vdots \\
g^{(n)}_{k+1} & a_2g^{(n)}_k + a_3g^{(n)}_{k-1} + \ldots + a_ng^{(n)}_{k-n+2} \\
g^{(n)}_k & a_2g^{(n)}_{k-1} + a_3g^{(n)}_{k-2} + \ldots + a_ng^{(n)}_{k-n+1} \\
& \vdots \\
a_3g^{(n)}_{k+n-2} + a_4g^{(n)}_{k+n-3} + \ldots + a_ng^{(n)}_{k+1} & \ldots & a_ng^{(n)}_{k+n-2} \\
a_3g^{(n)}_{k+n-3} + a_4g^{(n)}_{k+n-4} + \ldots + a_ng^{(n)}_k & \ldots & a_ng^{(n)}_{k+n-3} \\
& \vdots \\
a_3g^{(n)}_k + a_4g^{(n)}_{k-1} + \ldots + a_ng^{(n)}_{k-n+3} & \ldots & a_ng^{(n)}_k \\
a_3g^{(n)}_{k-1} + a_4g^{(n)}_{k-2} + \ldots + a_ng^{(n)}_{k-n+2} & \ldots & a_ng^{(n)}_{k-1} \\
\end{pmatrix},
\]
for \( k = 0, \pm 1, \pm 2, \pm 3, \ldots \)

PROOF: Case 1: \( k \geq 0 \).

We have,
\[
G_n^0 = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\]
\[
\begin{align*}
\begin{pmatrix}
g_{n-1}^{(n)} & a_2 g_{n-2}^{(n)} + a_3 g_{n-3}^{(n)} + \ldots + a_n g_0^{(n)} \\
g_{n-2}^{(n)} & a_2 g_{n-3}^{(n)} + a_3 g_{n-4}^{(n)} + \ldots + a_n g_{n-1}^{(n)} \\
\vdots & \vdots \\
g_1^{(n)} & a_2 g_0^{(n)} + a_3 g_{-1}^{(n)} + \ldots + a_n g_{n+2}^{(n)} \\
g_0^{(n)} & a_2 g_{-1}^{(n)} + a_3 g_{-2}^{(n)} + \ldots + a_n g_{n+1}^{(n)}
\end{pmatrix}
& = \\
\begin{pmatrix}
a_3 g_{n-2}^{(n)} + a_4 g_{n-3}^{(n)} + \ldots + a_n g_1^{(n)} & \ldots & a_n g_{n-2}^{(n)} \\
a_3 g_{n-3}^{(n)} + a_4 g_{n-4}^{(n)} + \ldots + a_n g_0^{(n)} & \ldots & a_n g_{n-3}^{(n)} \\
\vdots & \vdots & \vdots \\
a_3 g_0^{(n)} + a_4 g_{-1}^{(n)} + \ldots + a_n g_{n+3}^{(n)} & \ldots & a_n g_{n}^{(n)} \\
a_3 g_{-1}^{(n)} + a_4 g_{-2}^{(n)} + \ldots + a_n g_{n+2}^{(n)} & \ldots & a_n g_{-1}^{(n)}
\end{pmatrix}
\end{align*}
\]

\[G_n^1 = \\
\begin{pmatrix}
a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]

\[
\begin{align*}
\begin{pmatrix}
g_1^{(n)} & a_2 g_{n-1}^{(n)} + a_3 g_{n-2}^{(n)} + \ldots + a_n g_1^{(n)} \\
g_{n-1}^{(n)} & a_2 g_{n-2}^{(n)} + a_3 g_{n-3}^{(n)} + \ldots + a_n g_{n-1}^{(n)} \\
\vdots & \vdots \\
g_2^{(n)} & a_2 g_1^{(n)} + a_3 g_0^{(n)} + \ldots + a_n g_{n+3}^{(n)} \\
g_1^{(n)} & a_2 g_0^{(n)} + a_3 g_{-1}^{(n)} + \ldots + a_n g_{n+2}^{(n)}
\end{pmatrix}
& = \\
\begin{pmatrix}
\end{align*}
\]
Let the theorem be true for \( k \).

Thus the theorem is true for \( k = 0 \) and 1.

Let the theorem be true for \( k = m \) then
\[
G_n^m = \begin{pmatrix}
\begin{array}{cccc}
g_{m+n-1}^{(n)} & a_2g_{m+n-2}^{(n)} + a_3g_{m+n-3}^{(n)} + \ldots + a_ng_m^{(n)} \\
g_{m+n-2}^{(n)} & a_2g_{m+n-3}^{(n)} + a_3g_{m+n-4}^{(n)} + \ldots + a_ng_{m-1}^{(n)} \\
& \ddots & \ddots & \ddots \\
g_{m+1}^{(n)} & a_2g_m^{(n)} + a_3g_{m-1}^{(n)} + \ldots + a_ng_{m-n+2}^{(n)} \\
g_m^{(n)} & a_2g_{m-1}^{(n)} + a_3g_{m-2}^{(n)} + \ldots + a_ng_{m-n+1}^{(n)} \\
\end{array}
\end{pmatrix}
\]

Now,
\[
G^{m+1} = G^m G^1 = \begin{pmatrix}
\begin{array}{cccc}
g_{m+n-1}^{(n)} & a_2g_{m+n-2}^{(n)} + a_3g_{m+n-3}^{(n)} + \ldots + a_ng_m^{(n)} \\
g_{m+n-2}^{(n)} & a_2g_{m+n-3}^{(n)} + a_3g_{m+n-4}^{(n)} + \ldots + a_ng_{m-1}^{(n)} \\
& \ddots & \ddots & \ddots \\
g_{m+1}^{(n)} & a_2g_m^{(n)} + a_3g_{m-1}^{(n)} + \ldots + a_ng_{m-n+2}^{(n)} \\
g_m^{(n)} & a_2g_{m-1}^{(n)} + a_3g_{m-2}^{(n)} + \ldots + a_ng_{m-n+1}^{(n)} \\
\end{array}
\end{pmatrix}
\]
Hence by induction, we can write, for all $k \geq 0$

\[
\begin{align*}
& a_{3g_{m+n-2}^{(n)} + a_{4g_{m+n-3}^{(n)} + a_n g_{m+1}^{(n)} + \ldots + a_n g_{m+n-2}^{(n)}}} \\
& a_{3g_{m+n-3}^{(n)} + a_{4g_{m+n-4}^{(n)} + a_n g_m^{(n)} + \ldots + a_n g_{m+n-3}^{(n)}}} \\
& \vdots \\
& a_{3g_m^{(n)} + a_{4g_{m-1}^{(n)} + a_n g_{m-n+3}^{(n)} + \ldots + a_n g_{m-2}^{(n)}}} \\
& a_{3g_{m-1} + a_{4g_{m-2} + a_n g_{m-n+2}^{(n)} + \ldots + a_n g_{m-1}^{(n)}}}
\end{align*}
\]

Using the recurrence relation (5.1), we have

\[
G^{m+1} = \begin{pmatrix}
0 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

Hence by induction, we can write, for all $k \geq 0$
Similarly, we can prove for all \( k < 0 \).

\[
G_n^k = \begin{pmatrix}
g_{k+n-1}^{(n)} & a_2g_{k+n-2}^{(n)} + a_3g_{k+n-3}^{(n)} + \ldots + a_ng_k^{(n)} \\
g_{k+n-2}^{(n)} & a_2g_{k+n-3}^{(n)} + a_3g_{k+n-4}^{(n)} + \ldots + a_ng_{k-1}^{(n)} \\
\vdots & \ddots \\
g_{k+1}^{(n)} & a_2g_k^{(n)} + a_3g_{k-1}^{(n)} + \ldots + a_ng_{k-n+2}^{(n)} \\
g_k^{(n)} & a_2g_{k-1}^{(n)} + a_3g_{k-2} + \ldots + a_ng_{k-n+2}^{(n)} \\
\end{pmatrix}
\]

Case 2: \( k < 0 \).

Similarly, we can prove for all \( k < 0 \).

\[
G_n^k = \begin{pmatrix}
g_{k+n-1}^{(n)} & a_2g_{k+n-2}^{(n)} + a_3g_{k+n-3}^{(n)} + \ldots + a_ng_k^{(n)} \\
g_{k+n-2}^{(n)} & a_2g_{k+n-3}^{(n)} + a_3g_{k+n-4}^{(n)} + \ldots + a_ng_{k-1}^{(n)} \\
\vdots & \ddots \\
g_{k+1}^{(n)} & a_2g_k^{(n)} + a_3g_{k-1}^{(n)} + \ldots + a_ng_{k-n+2}^{(n)} \\
g_k^{(n)} & a_2g_{k-1}^{(n)} + a_3g_{k-2} + \ldots + a_ng_{k-n+2}^{(n)} \\
\end{pmatrix}
\]
Proof:

Now using the recurrence relation (5.1), we can write

\[
\begin{pmatrix}
  a_{3}^{(n)}g_{k+n-2}^{(n)} + a_{4}^{(n)}g_{k+n-3}^{(n)} + \ldots + a_{n}^{(n)}g_{k+1}^{(n)} & \ldots & a_{n}^{(n)}g_{k+n-2}^{(n)} \\
  a_{3}^{(n)}g_{k+n-3}^{(n)} + a_{4}^{(n)}g_{k+n-4}^{(n)} + \ldots + a_{n}^{(n)}g_{k}^{(n)} & \ldots & a_{n}^{(n)}g_{k+n-3}^{(n)} \\
  \ldots & \ldots & \ldots \\
  a_{3}^{(n)}g_{k}^{(n)} + a_{4}^{(n)}g_{k-1}^{(n)} + \ldots + a_{n}^{(n)}g_{k-n}^{(n)} & \ldots & a_{n}^{(n)}g_{k}^{(n)} \\
  a_{3}^{(n)}g_{k-1}^{(n)} + a_{4}^{(n)}g_{k-2}^{(n)} + \ldots + a_{n}^{(n)}g_{k-n+1}^{(n)} & \ldots & a_{n}^{(n)}g_{k-1}^{(n)} \\
\end{pmatrix}
\]

(5.3)

for \( k = 0, \pm 1, \pm 2, \pm 3, \ldots \).

Property 5.3.3 \( G_{n}^{k} = a_{1}G_{n}^{k-1} + a_{2}G_{n}^{k-2} + \ldots + a_{n}G_{n}^{k-n} \).

Proof:

\[
G_{n}^{k} = \begin{pmatrix}
  g_{k+n-1}^{(n)} & a_{2}^{(n)}g_{k+n-2}^{(n)} + a_{3}^{(n)}g_{k+n-3}^{(n)} + \ldots + a_{n}^{(n)}g_{k}^{(n)} \\
  g_{k+n-2}^{(n)} & a_{2}^{(n)}g_{k+n-3}^{(n)} + a_{3}^{(n)}g_{k+n-4}^{(n)} + \ldots + a_{n}^{(n)}g_{k-1}^{(n)} \\
  \ldots & \ldots & \ldots \\
  g_{k}^{(n)} & a_{2}^{(n)}g_{k-1}^{(n)} + a_{3}^{(n)}g_{k-2}^{(n)} + \ldots + a_{n}^{(n)}g_{k-n+1}^{(n)} \\
\end{pmatrix}
\]

Now using the recurrence relation (5.1), we can write

\[
g_{k+n-1}^{(n)} = a_{1}^{(n)}g_{k+n-2}^{(n)} + a_{2}^{(n)}g_{k+n-3}^{(n)} + \ldots + a_{n}^{(n)}g_{k-1}^{(n)};
\]

\[
g_{k+n-2}^{(n)} = a_{1}^{(n)}g_{k+n-3}^{(n)} + a_{2}^{(n)}g_{k+n-4}^{(n)} + \ldots + a_{n}^{(n)}g_{k-2}^{(n)};
\]
$g_{k-1} = a_1 g^{(n)}_{k-2} + a_2 g^{(n)}_{k-3} + \ldots + a_n g^{(n)}_{k-n-1}$.

Hence by using the property of matrix addition, we can write

$$G_n^k = a_1$$

$$\begin{pmatrix}
  g^{(n)}_{k+n-2} & a_2 g^{(n)}_{k+n-3} + a_3 g^{(n)}_{k+n-4} + \ldots + a_n g^{(n)}_{k-1} \\
  g^{(n)}_{k+n-3} & a_2 g^{(n)}_{k+n-4} + a_3 g^{(n)}_{k+n-5} + \ldots + a_n g^{(n)}_{k-2} \\
  \vdots & \vdots & \vdots & \vdots \\
  g^{(n)}_{k-1} & a_2 g^{(n)}_{k-2} + a_3 g^{(n)}_{k-3} + \ldots + a_n g^{(n)}_{k-n} \\
  g^{(n)}_{k} & a_2 g^{(n)}_{k-1} + a_3 g^{(n)}_{k-2} + \ldots + a_n g^{(n)}_{k-n+1} \\
  a_3 g^{(n)}_{k+n-3} + a_4 g^{(n)}_{k+n-4} + \ldots + a_n g^{(n)}_{k} & \ldots & a_n g^{(n)}_{k+n-3} \\
  a_3 g^{(n)}_{k+n-4} + a_4 g^{(n)}_{k+n-5} + \ldots + a_n g^{(n)}_{k-1} & \ldots & a_n g^{(n)}_{k+n-4} \\
  \vdots & \vdots & \vdots & \vdots \\
  a_3 g^{(n)}_{k-1} + a_4 g^{(n)}_{k-2} + \ldots + a_n g^{(n)}_{k-n+2} & \ldots & a_n g^{(n)}_{k-1} \\
  a_3 g^{(n)}_{k-2} + a_4 g^{(n)}_{k-3} + \ldots + a_n g^{(n)}_{k-n+1} & \ldots & a_n g^{(n)}_{k-2}
\end{pmatrix}$$

$$+ a_2$$

$$\begin{pmatrix}
  g^{(n)}_{k+n-3} & a_2 g^{(n)}_{k+n-4} + a_3 g^{(n)}_{k+n-5} + \ldots + a_n g^{(n)}_{k-2} \\
  g^{(n)}_{k+n-4} & a_2 g^{(n)}_{k+n-5} + a_3 g^{(n)}_{k+n-6} + \ldots + a_n g^{(n)}_{k-3} \\
  \vdots & \vdots & \vdots & \vdots \\
  g^{(n)}_{k-1} & a_2 g^{(n)}_{k-2} + a_3 g^{(n)}_{k-3} + \ldots + a_n g^{(n)}_{k-n} \\
  g^{(n)}_{k} & a_2 g^{(n)}_{k-1} + a_3 g^{(n)}_{k-2} + \ldots + a_n g^{(n)}_{k-n+1} \\
  \vdots & \vdots & \vdots & \vdots \\
  g^{(n)}_{k-2} & a_2 g^{(n)}_{k-3} + a_3 g^{(n)}_{k-4} + \ldots + a_n g^{(n)}_{k-n-1}
\end{pmatrix}$$
which is the explicit form of the matrix $G_n^k$.\[5.4\]
5.4 Constant Coefficient Fibonacci $n$-Step Coding/Decoding Method

The $G_n^k$ matrix develops a new coding theory when $a_n = 1$. We take the initial message $P$ having at least $n(n-1)+1$ characters. Now we represent $P$ in the form of square matrix $P_n = (p_{ij})_{n \times n}$, where $n$ is any positive integer. $p_{ij}$ ($\geq 0$) $i, j = 1, 2, \ldots n$, depends on the decision makers choice considering the fact that for each $i$ at least one $p_{ij} \neq 0$. We take the $G_n^k$ of order $n$ as a coding matrix and its inverse matrix $G_n^{-k}$ as a decoding matrix for an arbitrary positive integer $k$. We name the transformation $P_n \times G_n^k = E$ as coding, the transformation $E \times G_n^{-k} = P_n$ as decoding and define $E$ as code matrix.

5.4.1 Determinant of the Code Matrix $E$

We have the code matrix $E = P_n \times G_n^k$.

Using the basic property of determinants,

\[ detE = det(P_n \times G_n^k) = detP_n \times detG_n^k = detP_n \times (-1)^{(n+1)k} \]  

(5.5)

5.4.2 Example

We consider $n = 3$.

We represent the initial message $P = p_{11}p_{12}p_{13}p_{21}p_{22}p_{23}p_{31}p_{32}p_{33}$ in the form of square matrix $P_3$ of order 3 as

\[
P_3 = \begin{pmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33}
\end{pmatrix}
\]  

(5.6)

where $p_{ij} \geq 0$, $i, j = 1, 2, 3$ and for each $i$ at least one $p_{ij} \neq 0$ depends on the decision makers choice from the given message.
Without any loss of generality, we assume that \( k = 3 \). Then by (5.3), we have

\[
G_3^3 = \begin{pmatrix}
\begin{pmatrix} g_1^{(3)} \\ g_2^{(3)} \\ g_3^{(3)} \end{pmatrix} & \begin{pmatrix} a_2 g_4^{(3)} + g_3^{(3)} \\ a_2 g_3^{(3)} + g_2^{(3)} \\ a_2 g_2^{(3)} + g_1^{(3)} \end{pmatrix} \\
\end{pmatrix} = \begin{pmatrix}
\begin{pmatrix} a_1^3 + 2a_1 a_2 + 1 \\ a_1^2 + a_2 \\ a_1 \\
\end{pmatrix} & \begin{pmatrix} a_1^2 a_2 + a_2 + a_1 \\ a_1 a_2 + 1 \\ a_1 \end{pmatrix} \\
\end{pmatrix}
\]

(5.7)

The inverse of \( G_3^3 \) is given by

\[
G_3^{-3} = \begin{pmatrix}
\begin{pmatrix} g_{-1}^{(3)} \\ g_{-2}^{(3)} \\ g_{-3}^{(3)} \end{pmatrix} & \begin{pmatrix} a_2 g_{-2}^{(3)} + g_{-3}^{(3)} \\ a_2 g_{-3}^{(3)} + g_{-4}^{(3)} \\ a_2 g_{-4}^{(3)} + g_{-5}^{(3)} \end{pmatrix} \\
\end{pmatrix} = \begin{pmatrix}
\begin{pmatrix} 1 \\ -a_2 \\ -a_1 -a_2 \end{pmatrix} & \begin{pmatrix} -2a_2 - a_1 \\ a_1 a_2 + 1 \\ -a_1 - a_2 \end{pmatrix} \\
\end{pmatrix}
\]

Then the coding of the message \( P \) consists of the multiplication of the matrix \( P_3 \) by the code matrix \( G_3^3 \) i.e.

\[
P_3 \times G_3^3 = \begin{pmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33} \\
\end{pmatrix} \times \begin{pmatrix}
a_1^3 + 2a_1 a_2 + 1 \\ a_1^2 + a_2 \\ a_1 \\
\end{pmatrix} = \begin{pmatrix}
a_1^3 + 2a_1 a_2 + 1 + (a_1^2 + a_2)p_{11} + a_1 p_{13} \\
(a_1^3 + 2a_1 a_2 + 1)p_{21} + (a_1^2 + a_2)p_{22} + a_1 p_{23} \\
(a_1^3 + 2a_1 a_2 + 1)p_{31} + (a_1^2 + a_2)p_{32} + a_1 p_{33} \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
(a_1^2 a_2 + a_2 + a_1)p_{11} + (a_1 a_2 + 1)p_{12} + a_2 p_{13} \\ (a_1^2 a_2 + a_2 + a_1)p_{21} + (a_1 a_2 + 1)p_{22} + a_2 p_{23} \\ (a_1^2 a_2 + a_2 + a_1)p_{31} + (a_1 a_2 + 1)p_{32} + a_2 p_{33} \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
e_{11} & e_{12} & e_{13} \\
e_{21} & e_{22} & e_{23} \\
e_{31} & e_{32} & e_{33} \\
\end{pmatrix} = E
\]

(5.8)

where

\[
e_{11} = (a_1^3 + 2a_1 a_2 + 1)p_{11} + (a_1^2 + a_2)p_{12} + a_1 p_{13}, \quad e_{12} = (a_1^2 a_2 + a_2 + a_1)p_{11} + (a_1 a_2 + 1)p_{12} + a_2 p_{13}, \quad e_{13} = (a_1^2 + a_2)p_{11} + a_1 p_{12} + p_{13},
\]

110
We represent the initial message in the form of a square matrix.

\[
e_{21} = (a_1^3 + 2a_1a_2 + 1)p_{21} + (a_1^2 + a_2)p_{22} + a_1p_{23}, \quad e_{22} = (a_1^2a_2 + a_2^2 + a_1)p_{21} + (a_1a_2 + 1)p_{22} + a_2p_{23}, \quad e_{23} = (a_1^2 + a_2)p_{21} + a_1p_{22} + p_{23},
\]
\[
e_{31} = (a_1^3 + 2a_1a_2 + 1)p_{31} + (a_1^2 + a_2)p_{32} + a_1p_{33}, \quad e_{32} = (a_1^2a_2 + a_2^2 + a_1)p_{31} + (a_1a_2 + 1)p_{32} + a_2p_{33}, \quad e_{33} = (a_1^2 + a_2)p_{31} + a_1p_{32} + p_{33}.
\]

Solving these nine equations for nine \( p_{ij} \)s, we have
\[
\begin{align*}
p_{11} &= e_{11} - a_2 e_{12} - (a_1 + a_2^2)e_{13}, \\
p_{12} &= -(2a_2^2 + a_1)e_{11} + (1 + a_1a_2)e_{12} + (a_1^2 + 2a_1a_2^2 - a_2)e_{13}, \\
p_{13} &= -a_2 e_{11} - (a_1 + a_2^2)e_{12} + (1 + 2a_1a_2 + a_2^2)e_{13}, \\
p_{21} &= e_{21} - a_2 e_{22} - (a_1 + a_2^2)e_{23}, \\
p_{22} &= -(2a_2^2 + a_1)e_{21} + (1 + a_1a_2)e_{22} + (a_1^2 + 2a_1a_2^2 - a_2)e_{23}, \\
p_{23} &= -a_2 e_{21} - (a_1 + a_2^2)e_{22} + (1 + 2a_1a_2 + a_2^2)e_{23}, \\
p_{31} &= e_{31} - a_2 e_{32} - (a_1 + a_2^2)e_{33}, \\
p_{32} &= -(2a_2^2 + a_1)e_{31} + (1 + a_1a_2)e_{32} + (a_1^2 + 2a_1a_2^2 - a_2)e_{33}, \\
p_{33} &= -a_2 e_{31} - (a_1 + a_2^2)e_{32} + (1 + 2a_1a_2 + a_2^2)e_{33}.
\end{align*}
\]

Then the code matrix \( E \) is sent to a channel. After that the decoding of the code message \( E \) is performed by the following manner:
\[
E \times G_3^{-1} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \times \begin{pmatrix} 1 & -2a_2^2 - a_1 & -a_2 \\ -a_2 & a_1a_2 + 1 & -a_1 - a_2^2 \\ -a_1 - a_2^2 & a_1^2 + 2a_1a_2^2 - a_2 & 1 + 2a_1a_2 + a_2^2 \end{pmatrix}
\]
\[
= \begin{pmatrix} e_{11} - a_2 e_{12} - (a_1 + a_2^2)e_{13} - (2a_2^2 + a_1)e_{11} + (1 + a_1a_2)e_{12} + (a_1^2 + 2a_1a_2^2 - a_2)e_{13} \\ e_{21} - a_2 e_{22} - (a_1 + a_2^2)e_{23} - (2a_2^2 + a_1)e_{21} + (1 + a_1a_2)e_{22} + (a_1^2 + 2a_1a_2^2 - a_2)e_{23} \\ e_{31} - a_2 e_{32} - (a_1 + a_2^2)e_{33} - (2a_2^2 + a_1)e_{31} + (1 + a_1a_2)e_{32} + (a_1^2 + 2a_1a_2^2 - a_2)e_{33} \end{pmatrix}
\]
\[
- a_2 e_{11} - (a_1 + a_2^2)e_{12} + (1 + 2a_1a_2 + a_2^2)e_{13} - a_2 e_{21} - (a_1 + a_2^2)e_{22} + (1 + 2a_1a_2 + a_2^2)e_{23} - a_2 e_{31} - (a_1 + a_2^2)e_{32} + (1 + 2a_1a_2 + a_2^2)e_{33} = P_3.
\]

5.5 Relations among the Code Matrix Elements

In this section, we develop the relations among the code matrix elements taking \( a_n = 1 \). We represent the initial message in the form of a square matrix \( P_n = (p_{ij})_{n \times n} \), where \( n \) is
any positive integer. \( p_{ij} \ (\geq 0) \ i, j = 1, 2, \ldots n\), depends on the decision makers choice and for each \( i \) at least one \( p_{ij} \neq 0 \). We write the code matrix \( E \) with the help of the initial matrix \( P_n \) and the coding matrix \( G_n^k \) as

\[
E = P_n \times G_n^k
\]

\[
\begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1n} \\
p_{21} & p_{22} & \cdots & p_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n1} & p_{n2} & \cdots & p_{nn}
\end{pmatrix}
\begin{pmatrix}
g_{k+n-1}^{(n)} & a_2 g_{k+n-2}^{(n)} + a_3 g_{k+n-3}^{(n)} + \ldots + g_k^{(n)} \\
g_{k+n-2}^{(n)} & a_2 g_{k+n-3}^{(n)} + a_3 g_{k+n-4}^{(n)} + \ldots + g_{k-1}^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
g_{k+1}^{(n)} & a_2 g_k^{(n)} + a_3 g_{k-1}^{(n)} + \ldots + g_{k+n-2}^{(n)} \\
g_k^{(n)} & a_2 g_{k-1}^{(n)} + a_3 g_{k-2}^{(n)} + \ldots + g_{k-n+1}^{(n)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
a_3 g_{k+n-2}^{(n)} + a_4 g_{k+n-3}^{(n)} + \ldots + g_{k+1}^{(n)} \\
a_3 g_{k+n-3}^{(n)} + a_4 g_{k+n-4}^{(n)} + \ldots + g_k^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
a_3 g_k^{(n)} + a_4 g_{k-1}^{(n)} + \ldots + g_{k-n+3}^{(n)} \\
a_3 g_{k-1}^{(n)} + a_4 g_{k-2}^{(n)} + \ldots + g_{k-n+2}^{(n)} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
e_{11} & e_{12} & \cdots & e_{1n} \\
e_{21} & e_{22} & \cdots & e_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
e_{n1} & e_{n2} & \cdots & e_{nn}
\end{pmatrix}
\]

We choose \( k \) in such a manner that \( e_{ij} > 0 \), for all \( i, j \).

After decoding, we have

\[
P_n = E \times G_n^{-k}
\]
We write the code matrix $E$ and the initial matrix $P_2$ as:

$$E = P_2 \times G_k^2 = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} g_{k+1}^{(2)} \\ g_k^{(2)} \\ g_{k-1}^{(2)} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$$

and

$$P_2 = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = E \times G_{2}^{-k} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \times G_{2}^{-k}.$$

Case 1: $n = 2$.
**Case 1.1:** \( k \) is an even integer. Then, we have

\[
\begin{pmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{pmatrix} = \begin{pmatrix}e_{11} & e_{12} \\
e_{21} & e_{22}\end{pmatrix} \times \begin{pmatrix}g_{k-1}^{(2)} & -g_k^{(2)} \\
-g_k^{(2)} & g_{k+1}^{(2)}\end{pmatrix}.
\] (5.11)

It follows from (5.11) that the elements of the matrix \( P_2 \) can be calculated according to the following formulas:

\[
P_{11} = g_{k-1}^{(2)}e_{11} - g_k^{(2)}e_{12},
\] (5.12)

\[
P_{12} = -g_k^{(2)}e_{11} + g_{k+1}^{(2)}e_{12},
\] (5.13)

\[
P_{21} = g_{k-1}^{(2)}e_{21} - g_k^{(2)}e_{22}
\] (5.14)

and

\[
P_{22} = -g_k^{(2)}e_{21} + g_{k+1}^{(2)}e_{22}.
\] (5.15)

Since \( P_{11}, P_{12}, P_{21} \) and \( P_{22} \) are positive integers, we have

\[
g_{k-1}^{(2)}e_{11} - g_k^{(2)}e_{12} \geq 0,
\] (5.16)

\[
-g_k^{(2)}e_{11} + g_{k+1}^{(2)}e_{12} \geq 0,
\] (5.17)

\[
g_{k-1}^{(2)}e_{21} - g_k^{(2)}e_{22} \geq 0
\] (5.18)

and

\[
-g_k^{(2)}e_{21} + g_{k+1}^{(2)}e_{22} \geq 0.
\] (5.19)

From (5.16) and (5.17), we have

\[
\frac{g_{k+1}^{(2)}}{g_k^{(2)}} e_{12} \geq e_{11} \geq \frac{g_k^{(2)}}{g_{k-1}^{(2)}} e_{12}.
\] (5.20)

Therefore,

\[
\frac{g_{k+1}^{(2)}}{g_k^{(2)}} \geq \frac{e_{11}}{e_{12}} \geq \frac{g_k^{(2)}}{g_{k-1}^{(2)}}.
\] (5.21)

Again from (5.18) and (5.19), we have

\[
\frac{g_{k+1}^{(2)}}{g_k^{(2)}} e_{22} \geq e_{21} \geq \frac{g_k^{(2)}}{g_{k-1}^{(2)}} e_{22}
\] (5.22)
and

$$\frac{g_{k+1}}{g_k} \geq \frac{e_{21}}{e_{22}} \geq \frac{g_k}{g_{k-1}}.$$  (5.23)

**Case 1.2:** $k$ is an odd integer. Then, we have

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \times \begin{pmatrix} -g_{k-1} & g_k \\ g_k & -g_{k-1} \end{pmatrix}.$$  (5.24)

It follows from (5.24) that the elements of the matrix $P_2$ can be calculated according to the following formulas:

$$p_{11} = -g_{k-1} e_{11} + g_k e_{12},$$  (5.25)

$$p_{12} = g_k e_{11} - g_{k+1} e_{12},$$  (5.26)

$$p_{21} = -g_{k-1} e_{21} + g_k e_{22}$$  (5.27)

and

$$p_{22} = g_k e_{21} - g_{k+1} e_{22}.$$  (5.28)

Since $p_{11}, p_{12}, p_{21}$ and $p_{22}$ are positive integers, we have

$$-g_{k-1} e_{11} + g_k e_{12} \geq 0,$$  (5.29)

$$g_k e_{11} - g_{k+1} e_{12} \geq 0,$$  (5.30)

$$-g_{k-1} e_{21} + g_k e_{22} \geq 0$$  (5.31)

and

$$g_k e_{21} - g_{k+1} e_{22} \geq 0.$$  (5.32)

From (5.29) and (5.30), we have

$$\frac{g_{k+1}}{g_k} e_{12} \leq e_{11} \leq \frac{g_k}{g_{k-1}} e_{12}.$$  (5.33)

Therefore,

$$\frac{g_{k+1}}{g_k} \leq \frac{e_{11}}{e_{12}} \leq \frac{g_k}{g_{k-1}}.$$  (5.34)
Again from (5.31) and (5.32), we have
\[
\frac{g_k^{(2)}}{g_k} e_{22} \leq e_{21} \leq \frac{g_k^{(2)}}{g_k} e_{22} \tag{5.35}
\]
and
\[
\frac{g_k^{(2)}}{g_k} \leq \frac{e_{21}}{e_{22}} \leq \frac{g_k^{(2)}}{g_k} \tag{5.36}
\]
Hence from (5.21), (5.23), (5.34) and (5.36), we have
\[
\frac{g_k^{(2)}}{g_k} \geq e_{11} e_{12} \geq \frac{g_k^{(2)}}{g_k} \tag{5.37}
\]
and
\[
\frac{e_{21}}{e_{22}} \leq \frac{g_k^{(2)}}{g_k} \tag{5.38}
\]
For the large \( k \), we have
\[
\frac{e_{11}}{e_{12}} \approx \alpha_2 \tag{5.37}
\]
and
\[
\frac{e_{21}}{e_{22}} \approx \alpha_2 \tag{5.38}
\]
where
\[
\alpha_2 = \frac{a_1 + \sqrt{a_1^2 + 4}}{2}
\]

Case 2: \( n = 3 \).

In this case,
\[
E = P_3 \times G_3^k
\]
\[
= \begin{pmatrix}
p_1 & p_2 & p_3 \\
p_4 & p_5 & p_6 \\
p_7 & p_8 & p_9
\end{pmatrix}
\begin{pmatrix}
g_{k+2}^{(3)} & a_2 g_{k+1}^{(3)} + g_k^{(3)} & g_k^{(3)} \\
g_{k+1}^{(3)} & a_2 g_k^{(3)} + g_{k-1}^{(3)} & g_k^{(3)} \\
g_k^{(3)} & a_2 g_{k-1}^{(3)} + g_{k-2}^{(3)} & g_{k-1}^{(3)}
\end{pmatrix}
\begin{pmatrix}
e_{11} & e_{12} & e_{13} \\
e_{21} & e_{22} & e_{23} \\
e_{31} & e_{32} & e_{33}
\end{pmatrix}
\]
and

\[ P_3 = E \times G_3^{-k} \]

\[
= \begin{pmatrix}
e_{11} & e_{12} & e_{13} \\
e_{21} & e_{22} & e_{23} \\
e_{31} & e_{32} & e_{33}
\end{pmatrix}
\begin{pmatrix}
g_{k+2}^{(3)} & a_2g_{k+1}^{(3)} + g_k^{(3)} & g_{k+1}^{(3)} \\
g_{k+1}^{(3)} & a_2g_k^{(3)} + g_{k-1}^{(3)} & g_k^{(3)} \\
g_k^{(3)} & a_2g_{k-1}^{(3)} + g_{k-2}^{(3)} & g_{k-1}^{(3)}
\end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33}
\end{pmatrix}
\]

We choose \( k \) in such a manner that \( e_{ij} > 0 \) for \( i, j = 1, 2, 3 \).

Now

\[ \det G_3^k = g_{k+1}^{(3)}(a_2g_k^{(3)}g_{k-1}^{(3)} + g_{k-1}^{(3)} - a_2g_k^{(3)}g_{k-1}^{(3)} - g_k^{(3)}g_{k-2}^{(3)}) + (a_2g_k^{(3)} + g_k^{(3)})(g_k^{(3)} - g_{k+1}g_{k-1}) + g_{k+1}^{(3)}(a_2g_{k+1}^{(3)}g_{k-1}^{(3)} + g_{k+1}^{(3)}g_{k-2}^{(3)} - a_2g_k^{(3)}g_{k-2}^{(3)} - g_k^{(3)}g_{k-1}^{(3)}) = 1. \]  \hspace{1cm} (5.39)

and

\[ G_3^{-k} = \frac{1}{\det G_3^k} \text{adj} G_3^k \]

\[
= \begin{pmatrix}
g_{k-1}^{(3)} & -g_k^{(3)} & g_{k-2}^{(3)} \\
g_k^{(3)} & -g_{k+1}g_{k-1}^{(3)} & g_{k+2}^{(3)} - g_k^{(3)}g_{k-1}^{(3)} \\
g_{k+1}^{(3)} - g_k^{(3)}g_{k-2}^{(3)} & -g_{k+1}g_{k-2}^{(3)} & g_{k+3}^{(3)} - g_k^{(3)}g_{k-1}^{(3)}
\end{pmatrix}
\]

Thus,

\[
\begin{pmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33}
\end{pmatrix}
= \begin{pmatrix}
\begin{pmatrix}
e_{11} & e_{12} & e_{13} \\
e_{21} & e_{22} & e_{23} \\
e_{31} & e_{32} & e_{33}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}g_{k-1}^{(3)} & -g_k^{(3)} & g_{k-2}^{(3)} \\
g_k^{(3)} & -g_{k+1}g_{k-1}^{(3)} & g_{k+2}^{(3)} - g_k^{(3)}g_{k-1}^{(3)} \\
g_{k+1}^{(3)} - g_k^{(3)}g_{k-2}^{(3)} & -g_{k+1}g_{k-2}^{(3)} & g_{k+3}^{(3)} - g_k^{(3)}g_{k-1}^{(3)}
\end{pmatrix}
\end{pmatrix}
\]

Therefore,

\[ p_{11} = e_{11}(g_{k-1}^{(3)} - g_k^{(3)}g_{k-2}^{(3)}) + e_{12}(g_k^{(3)} - g_{k+1}g_{k-1}^{(3)}) + e_{13}(g_{k+1}^{(3)} - g_k^{(3)}g_{k+2}^{(3)}) \geq 0, \]  \hspace{1cm} (5.40)
\[ p_{12} = e_{11}(g_{k-2}g_{k+1} - g_{k-1}g_k) + e_{12}(g_{k-1}g_{k+2} - g_k g_{k+1}) + \]
\[ e_{13}(g_k g_{k+3} - g_{k+1}g_{k+2}) \geq 0, \quad (5.41) \]
\[ p_{13} = e_{11}(g_k^2 - g_{k-1}g_{k+1}) + e_{12}(g_{k+1}^2 - g_k g_{k+2}) + e_{13}(g_{k+2}^2 - g_{k+1}g_{k+3}) \geq 0, \quad (5.42) \]
\[ p_{21} = e_{21}(g_k^2 - g_k g_{k-2}) + e_{22}(g_k^2 - g_k g_{k-1}) + e_{23}(g_{k+1}^2 - g_{k+1}g_{k+2}) \geq 0, \quad (5.43) \]
\[ p_{22} = e_{21}(g_{k-2}g_{k+1} - g_{k-1}g_k) + e_{22}(g_{k-1}g_{k+2} - g_k g_{k+1}) + \]
\[ e_{23}(g_k g_{k+3} - g_{k+1}g_{k+2}) \geq 0, \quad (5.44) \]
\[ p_{23} = e_{21}(g_k^2 - g_{k-1}g_{k+1}) + e_{22}(g_{k+1}^2 - g_k g_{k+2}) + e_{23}(g_{k+2}^2 - g_{k+1}g_{k+3}) \geq 0, \quad (5.45) \]
\[ p_{31} = e_{31}(g_k^2 - g_k g_{k-2}) + e_{32}(g_k^2 - g_k g_{k-1}) + e_{33}(g_{k+1}^2 - g_k g_{k+2}) \geq 0, \quad (5.46) \]
\[ p_{32} = e_{31}(g_{k-2}g_{k+1} - g_{k-1}g_k) + e_{32}(g_{k-1}g_{k+2} - g_k g_{k+1}) + \]
\[ e_{33}(g_k g_{k+3} - g_{k+1}g_{k+2}) \geq 0 \quad (5.47) \]

and

\[ p_{33} = e_{31}(g_k^2 - g_{k-1}g_{k+1}) + e_{32}(g_{k+1}^2 - g_k g_{k+2}) + e_{33}(g_{k+2}^2 - g_{k+1}g_{k+3}) \geq 0. \quad (5.48) \]

Dividing both sides of (5.40), (5.41) and (5.42) by \( e_{11} \) (\( > 0 \)), we have

\[ \frac{(g_{k+1}^2 - g_k g_{k+2})e_{13}}{e_{11}} \geq \frac{(g_{k+1} g_{k-2} - g_k^2)e_{12}}{e_{11}} + \frac{(g_k^2 - g_{k-1}^2)e_{13}}{e_{11}}, \quad (5.49) \]

\[ \frac{(g_{k+1} g_{k+2} - g_k g_{k+3})e_{13}}{e_{11}} \leq \frac{(g_{k+2} g_{k-1} - g_k^3)e_{12}}{e_{11}} + \frac{(g_{k+1} g_{k-2} - g_k g_{k-1})e_{13}}{e_{11}} \quad (5.50) \]

and

\[ \frac{(g_{k+2} g_{k+2} - g_k g_{k+3})e_{13}}{e_{11}} \geq \frac{(g_{k+2} g_k - g_{k+1}^2)e_{12}}{e_{11}} + \frac{(g_{k+1} g_{k+2} - g_k g_{k+3})e_{13}}{e_{11}} \quad (5.51) \]

Let \( a = (g_{k+1}^2 - g_k g_{k+2}), b = (g_{k+1} g_{k+2} - g_k g_{k+3}), c = (g_{k+2} g_{k+2} - g_k g_{k+3}). \)

Now \( 3^3 = 27 \) cases arise for \( a \geq 0, b \geq 0 \) and \( c \geq 0 \). Here we discuss some of the 27 cases.

Case 2.1: \( a > 0, b > 0, c > 0. \)
Then from (5.49), we have

\[
\frac{e_{13}}{e_{11}} \geq u,
\]

where

\[
u = \frac{e_{12}}{e_{11}} \left( \frac{(g_{k+1}g_{k-2} - g_k)(3)}{g_{k+1}^2 - g_k} \right) + \frac{g_k}{g_{k+1}} \frac{(3)}{g_{k+2} - g_k}.
\]

From (5.50), we have

\[
\frac{e_{13}}{e_{11}} \leq v,
\]

where

\[
v = \frac{e_{12}}{e_{11}} \left( \frac{(g_{k+2}g_{k-1} - g_k)(3)}{g_{k+1}g_{k+2} - g_k} \right) + \frac{g_k}{g_{k+1}g_{k+2} - g_k}.
\]

From (5.51), we have

\[
\frac{e_{13}}{e_{11}} \geq w,
\]

where

\[
w = \frac{e_{12}}{e_{11}} \left( \frac{(3)}{g_{k+2} - g_k} \right) + \frac{g_k}{g_{k+1}g_{k+2} - g_k}.
\]

From (5.52) and (5.53), we have

\[
\frac{e_{11}}{e_{12}} \geq \min\left\{ \frac{g_k(3)}{a_2g_{k+1} + g_k(3)}, \frac{g_k(3)}{a_2g_k + g_{k-1}(3)}, \frac{g_k(3)}{a_2g_{k-1} + g_{k-2}(3)} \right\}, \quad \text{using (5.39).}
\]

From (5.52) and (5.54), we have

\[
\frac{e_{11}}{e_{12}} \leq \max\left\{ \frac{g_k(3)}{a_2g_{k+1} + g_k(3)}, \frac{g_k(3)}{a_2g_k + g_{k-1}(3)}, \frac{g_k(3)}{a_2g_{k-1} + g_{k-2}(3)} \right\}, \quad \text{using (5.39).}
\]

From (5.55) and (5.56), we have

\[
\min\left\{ \frac{g_k(3)}{a_2g_{k+1} + g_k(3)}, \frac{g_k(3)}{a_2g_k + g_{k-1}(3)}, \frac{g_k(3)}{a_2g_{k-1} + g_{k-2}(3)} \right\} \leq \frac{e_{11}}{e_{12}} \leq \max\left\{ \frac{g_k(3)}{a_2g_{k+1} + g_k(3)}, \frac{g_k(3)}{a_2g_k + g_{k-1}(3)}, \frac{g_k(3)}{a_2g_{k-1} + g_{k-2}(3)} \right\}.
\]

Similarly, we have

\[
\min\left\{ \frac{a_2g_{k+1} + g_k(3)}{g_{k+1}(3)}, \frac{a_2g_k + g_{k-1}(3)}{g_k(3)}, \frac{a_2g_{k-1} + g_{k-2}(3)}{g_{k-1}(3)} \right\} \leq \frac{e_{12}}{e_{13}}.
\]
\[
\leq \max\left\{ \frac{a_2g_{k+1} + g_k}{g_{k+1}}, \frac{a_2g_k + g_{k-1}}{g_k}, \frac{a_2g_{k-1} + g_{k-2}}{g_{k-1}} \right\}
\]

and
\[
\min\left\{ \frac{g_{k+2}}{g_{k+1}}, \frac{g_{k+1}}{g_k}, \frac{g_k}{g_{k-1}} \right\} \leq \frac{e_{11}}{e_{12}} \leq \max\left\{ \frac{g_{k+2}}{g_{k+1}}, \frac{g_{k+1}}{g_k}, \frac{g_k}{g_{k-1}} \right\}.
\]

**Case 2.2: \(a = 0, b > 0, c > 0\).**

From (5.49), we have
\[
\frac{e_{11}}{e_{12}} \geq \min\left\{ \frac{g_{k+2}}{a_2g_{k+1} + g_k}, \frac{g_{k+1}}{a_2g_k + g_{k-1}}, \frac{g_k}{a_2g_{k-1} + g_{k-2}} \right\}, \text{ since } a = 0.
\] (5.58)

From (5.50), (5.51), (5.39) and using \(a = 0\), we have
\[
\frac{e_{11}}{e_{12}} \leq \max\left\{ \frac{g_{k+2}}{a_2g_{k+1} + g_k}, \frac{g_{k+1}}{a_2g_k + g_{k-1}}, \frac{g_k}{a_2g_{k-1} + g_{k-2}} \right\}.
\] (5.59)

From (5.58) and (5.59), we have
\[
\min\left\{ \frac{g_{k+2}}{a_2g_{k+1} + g_k}, \frac{g_{k+1}}{a_2g_k + g_{k-1}}, \frac{g_k}{a_2g_{k-1} + g_{k-2}} \right\} \leq \frac{e_{11}}{e_{12}}
\]
\[
\leq \max\left\{ \frac{g_{k+2}}{a_2g_{k+1} + g_k}, \frac{g_{k+1}}{a_2g_k + g_{k-1}}, \frac{g_k}{a_2g_{k-1} + g_{k-2}} \right\}.
\] (5.60)

**Case 2.3: \(a < 0, b < 0, c < 0\).**

Then from (5.49)
\[
\frac{e_{13}}{e_{11}} \leq u,
\] (5.61)

where
\[
u = \frac{e_{12}}{e_{11}} \left( g_{k+1}g_{k-2} - g_k^2 \right) + \frac{g_k^2}{g_{k+1}^2 - g_k^2} \frac{g_{k-2}^2}{g_{k+2}^2 - g_k^2}.
\]

From (5.50), we have
\[
\frac{e_{13}}{e_{11}} \geq v,
\] (5.62)

where
\[
v = \frac{e_{12}}{e_{11}} \left( \frac{g_{k+1}g_{k-1} - g_k^2}{g_{k+1}g_{k+2} - g_k^2} + \frac{g_{k-2}g_{k+2} - g_k^2}{g_{k+1}g_{k+2} - g_k^2} \right).
\]
From (5.61), we have
\[ \frac{e_{13}}{e_{11}} \leq w, \]  
(5.63)

where
\[ w = \frac{e_{12}}{e_{11}} \left( \frac{g_k}{g_{k+2}} \right)^2 - \frac{g_k}{g_{k+1}g_{k+3}} + \frac{g_k}{g_{k+2} - g_{k+1}g_{k+3}}. \]

From (5.61) and (5.62), we have
\[ \frac{e_{11}}{e_{12}} \leq \max \left\{ \frac{g_{k+2}}{a_2g_{k+1} + g_k}, \frac{g_{k+1}}{a_2g_k + g_{k-1}}, \frac{g_k}{a_2g_{k-1} + g_{k-2}} \right\}, \]  
(5.64)

From (5.61) and (5.63), we have
\[ \frac{e_{11}}{e_{12}} \geq \min \left\{ \frac{g_{k+2}}{a_2g_{k+1} + g_k}, \frac{g_{k+1}}{a_2g_k + g_{k-1}}, \frac{g_k}{a_2g_{k-1} + g_{k-2}} \right\}, \]  
(5.65)

From (5.64) and (5.65), we have
\[ \min \left\{ \frac{g_{k+2}}{a_2g_{k+1} + g_k}, \frac{g_{k+1}}{a_2g_k + g_{k-1}}, \frac{g_k}{a_2g_{k-1} + g_{k-2}} \right\} \leq \frac{e_{11}}{e_{12}} \leq \max \left\{ \frac{g_{k+2}}{a_2g_{k+1} + g_k}, \frac{g_{k+1}}{a_2g_k + g_{k-1}}, \frac{g_k}{a_2g_{k-1} + g_{k-2}} \right\}. \]  
(5.66)

Similarly, we have
\[ \min \left\{ \frac{a_2g_{k+1} + g_k}{g_{k+1}}, \frac{a_2g_k + g_{k-1}}{g_k}, \frac{a_2g_{k-1} + g_{k-2}}{g_{k-1}} \right\} \leq \frac{e_{12}}{e_{13}} \leq \frac{e_{11}}{e_{13}}, \]

and
\[ \min \left\{ \frac{g_{k+2}}{g_{k+1}}, \frac{g_{k+1}}{g_k}, \frac{g_k}{g_{k-1}} \right\} \leq \frac{e_{11}}{e_{13}} \leq \max \left\{ \frac{g_{k+2}}{g_{k+1}}, \frac{g_{k+1}}{g_k}, \frac{g_k}{g_{k-1}} \right\}. \]

Similarly, we can prove the other cases.

Thus, we have
\[ \min \left\{ \frac{g_{k+2}}{a_2g_{k+1} + g_k}, \frac{g_{k+1}}{a_2g_k + g_{k-1}}, \frac{g_k}{a_2g_{k-1} + g_{k-2}} \right\} \leq \frac{e_{11}}{e_{12}} \leq \frac{e_{11}}{e_{13}} \leq \max \left\{ \frac{g_{k+2}}{g_{k+1}}, \frac{g_{k+1}}{g_k}, \frac{g_k}{g_{k-1}} \right\}. \]
Similarly, we can prove the above types of inequalities for $i = 2, 3$.

Hence, we have

\[
\min\left\{ \frac{g_{k+2}}{a_2g_{k+1} + g_k}, \frac{g_{k+1}}{a_2g_k + g_{k-1}}, \frac{g_{k}}{a_2g_{k-1} + g_{k-2}} \right\} \leq \frac{e_{i1}}{e_{i2}} \leq \max\left\{ \frac{g_{k+2}}{a_2g_{k+1} + g_k}, \frac{g_{k+1}}{a_2g_k + g_{k-1}}, \frac{g_{k}}{a_2g_{k-1} + g_{k-2}} \right\}
\]

and

\[
\min\left\{ \frac{g_{k+2}}{g_{k+1} + g_k}, \frac{g_{k+1}}{g_k + g_{k-1}}, \frac{g_{k}}{g_{k-1} + g_{k-2}} \right\} \leq \frac{e_{i1}}{e_{i3}} \leq \max\left\{ \frac{g_{k+2}}{g_{k+1} + g_k}, \frac{g_{k+1}}{g_k + g_{k-1}}, \frac{g_{k}}{g_{k-1} + g_{k-2}} \right\}
\]

For the large value of $k$, we have

\[
\frac{e_{i1}}{e_{i2}} \approx \frac{\alpha_3^2}{1 + a_2\alpha_3}, \quad \frac{e_{i2}}{e_{i3}} \approx \frac{1 + a_2\alpha_3}{\alpha_3}, \quad \frac{e_{i1}}{e_{i3}} \approx \alpha_3, \quad \text{for } i = 1, 2, 3
\]

where,

\[
\alpha_3 = \left\{ 2^\frac{4}{3} a_1 + (2a_1^3 + 9a_1a_2 + 27a_3 + \sqrt{-27a_1^2a_2^2 + 729a_1^3486a_1a_2a_3 + 108a_1^3a_3 - 108a_2^3})^\frac{1}{3} \right. \\
\left. + (2a_1^3 + 9a_1a_2 + 27a_3 - \sqrt{-27a_1^2a_2^2 + 729a_1^3486a_1a_2a_3 + 108a_1^3a_3 - 108a_2^3})^\frac{1}{3} \right\} / (2^\frac{1}{3} \times 3).
\]
Case 3: For any value of $n$.

The generalized relations among the code matrix elements are

$$
\min \left\{ \frac{g_{k+r}^{(n)}}{a_2 g_{k+r-1}^{(n)} + a_3 g_{k+r-2}^{(n)} + \ldots + g_{k+r-n+1}^{(n)}} : r = 0, 1, \ldots, n - 1 \right\} \leq \frac{e_1}{e_2} \leq \max \left\{ \frac{g_{k+r}^{(n)}}{a_2 g_{k+r-1}^{(n)} + a_3 g_{k+r-2}^{(n)} + \ldots + g_{k+r-n+1}^{(n)}} : r = 0, 1, \ldots, n - 1 \right\},
$$

$$
\min \left\{ \frac{g_{k+r}^{(n)}}{a_3 g_{k+r-1}^{(n)} + a_4 g_{k+r-2}^{(n)} + \ldots + g_{k+r-n+2}^{(n)}} : r = 0, 1, \ldots, n - 1 \right\} \leq \frac{e_1}{e_3} \leq \max \left\{ \frac{g_{k+r}^{(n)}}{a_3 g_{k+r-1}^{(n)} + a_4 g_{k+r-2}^{(n)} + \ldots + g_{k+r-n+2}^{(n)}} : r = 0, 1, \ldots, n - 1 \right\},
$$

$$
\min \left\{ \frac{g_{k+r}^{(n)}}{g_{k+r-1}^{(n)}} : r = 0, 1, \ldots, n - 1 \right\} \leq \frac{e_1}{e_4} \leq \max \left\{ \frac{g_{k+r}^{(n)}}{g_{k+r-1}^{(n)}} : r = 0, 1, \ldots, n - 1 \right\},
$$

$$
\min \left\{ \frac{a_2 g_{k+r-1}^{(n)} + a_3 g_{k+r-2}^{(n)} + \ldots + g_{k+r-n+1}^{(n)}}{a_3 g_{k+r-1}^{(n)} + a_4 g_{k+r-2}^{(n)} + \ldots + g_{k+r-n+2}^{(n)}} : r = 0, 1, \ldots, n - 1 \right\} \leq \frac{e_2}{e_5} \leq \max \left\{ \frac{a_2 g_{k+r-1}^{(n)} + a_3 g_{k+r-2}^{(n)} + \ldots + g_{k+r-n+1}^{(n)}}{a_3 g_{k+r-1}^{(n)} + a_4 g_{k+r-2}^{(n)} + \ldots + g_{k+r-n+2}^{(n)}} : r = 0, 1, \ldots, n - 1 \right\},
$$

$$
\min \left\{ \frac{a_2 g_{k+r-1}^{(n)} + a_3 g_{k+r-2}^{(n)} + \ldots + g_{k+r-n+1}^{(n)}}{g_{k+r-1}^{(n)}} : r = 0, 1, \ldots, n - 1 \right\} \leq \frac{e_2}{e_6} \leq \max \left\{ \frac{a_2 g_{k+r-1}^{(n)} + a_3 g_{k+r-2}^{(n)} + \ldots + g_{k+r-n+1}^{(n)}}{g_{k+r-1}^{(n)}} : r = 0, 1, \ldots, n - 1 \right\},
$$

123
$$\min \left\{ \frac{a_{3g_{k+r-1}} + a_{4g_{k+r-2}} + \ldots + g_{k+r-n+2}}{a_{4g_{k+r-1}} + a_{5g_{k+r-2}} + \ldots + g_{k+r-n+3}} : r = 0, 1, \ldots, n - 1 \right\} \leq$$

$$\frac{e_{i3}}{e_{i4}} \leq \max \left\{ \frac{a_{3g_{k+r-1}} + a_{4g_{k+r-2}} + \ldots + g_{k+r-n+2}}{a_{4g_{k+r-1}} + a_{5g_{k+r-2}} + \ldots + g_{k+r-n+3}} : r = 0, 1, \ldots, n - 1 \right\},$$

$$\min \left\{ \frac{a_{3g_{k+r-1}} + a_{4g_{k+r-2}} + \ldots + g_{k+r-n+2}}{a_{5g_{k+r-1}} + a_{6g_{k+r-2}} + \ldots + g_{k+r-n+r}} : r = 0, 1, \ldots, n - 1 \right\} \leq$$

$$\frac{e_{i3}}{e_{i5}} \leq \max \left\{ \frac{a_{3g_{k+r-1}} + a_{4g_{k+r-2}} + \ldots + g_{k+r-n+2}}{a_{5g_{k+r-1}} + a_{6g_{k+r-2}} + \ldots + g_{k+r-n+r}} : r = 0, 1, \ldots, n - 1 \right\},$$

$$\ldots$$

$$\min \left\{ \frac{a_{3g_{k+r-1}} + a_{4g_{k+r-2}} + \ldots + g_{k+r-n+2}}{g_{k+r-1}} : r = 0, 1, \ldots, n - 1 \right\} \leq$$

$$\frac{e_{i3}}{e_{i4}} \leq \max \left\{ \left\{ \frac{a_{3g_{k+r-1}} + a_{4g_{k+r-2}} + \ldots + g_{k+r-n+2}}{g_{k+r-1}} : r = 0, 1, \ldots, n - 1 \right\} \right\},$$

$$\ldots$$

$$\min \left\{ \frac{a_{n-1}g_{k+r-1} + g_{k+r-2}}{g_{k+r-1}} : r = 0, 1, \ldots, n - 1 \right\} \leq$$

$$\frac{e_{in-1}}{e_{in}} \leq \max \left\{ \frac{a_{n-1}g_{k+r-1} + g_{k+r-2}}{g_{k+r-1}} : r = 0, 1, \ldots, n - 1 \right\},$$

for $i = 1, 2, 3, \ldots, n$.

For large value of $k$, we have

$$\frac{e_{i1}}{e_{i2}} \approx \frac{\alpha_n^{n-1}}{1 + a_{n-1}\alpha_n + a_{n-2}\alpha_n^2 + \ldots + a_n\alpha_n^n},$$

$$\frac{e_{i1}}{e_{i3}} \approx \frac{\alpha_{n-2}}{1 + a_{n-1}\alpha_n + a_{n-2}\alpha_n^2 + \ldots + a_n\alpha_n^n},$$

$$\ldots$$

$$\frac{e_{i1}}{e_{in}} \approx \alpha_n,$$
\[
e_{i2} \approx \frac{1 + a_{n-1} \alpha_n + a_{n-2} \alpha_n^2 + \ldots + a_2 \alpha_n^{n-2}}{a_n + a_{n-1} \alpha_n^2 + a_{n-2} \alpha_n^3 + \ldots + a_3 \alpha_n^{n-3}},
\]
\[
e_{i3} \approx \frac{1 + \alpha_n^2 + \ldots + \alpha_n^{n-2}}{\alpha_n^2 + \alpha_n^3 + \alpha_n^4 + \ldots + \alpha_n^{n-2}},
\]
\[
e_{i4} \approx \frac{1 + \alpha_n + \alpha_n^2 + \ldots + \alpha_n^{n-2}}{\alpha_n^2 + \alpha_n^3 + \alpha_n^4 + \ldots + \alpha_n^{n-2}},
\]
\[
e_{i5} \approx \frac{1 + \alpha_n + \alpha_n^2 + \ldots + \alpha_n^{n-2}}{\alpha_n^2 + \alpha_n^3 + \alpha_n^4 + \ldots + \alpha_n^{n-2}},
\]
\[
\vdots 
\]
\[
\epsilon_{in} \approx \frac{1 + a_{n-1} \alpha_n + a_{n-2} \alpha_n^2 + \ldots + a_2 \alpha_n^{n-2}}{\alpha_n^{n-3}},
\]
\[
\vdots 
\]
\[
\epsilon_{in} \approx \frac{1 + a_{n-1} \alpha_n + a_{n-2} \alpha_n^2 + \ldots + a_3 \alpha_n^{n-3}}{\alpha_n^{n-3}},
\]
\[
\epsilon_{in} \approx \frac{1 + a_{n-1} \alpha_n + a_{n-2} \alpha_n^2 + \ldots + a_4 \alpha_n^{n-3}}{\alpha_n^{n-3}},
\]
\[
\epsilon_{in} \approx \frac{1 + a_{n-1} \alpha_n + a_{n-2} \alpha_n^2 + \ldots + a_5 \alpha_n^{n-3}}{\alpha_n^{n-3}},
\]
\[
\vdots
\]
\[
\epsilon_{in} \approx \frac{1 + a_{n-1} \alpha_n + a_{n-2} \alpha_n^2 + \ldots + a_n \alpha_n^{n-1}}{\alpha_n^{n-1}},
\]
\[
\epsilon_{in} \approx \frac{1 + a_{n-1} \alpha_n + a_{n-2} \alpha_n^2 + \ldots + a_n \alpha_n^{n-1}}{\alpha_n^{n-1}},
\]
\[
\epsilon_{in} \approx \frac{1 + a_{n-1} \alpha_n + a_{n-2} \alpha_n^2 + \ldots + a_n \alpha_n^{n-1}}{\alpha_n^{n-1}},
\]
\[
\vdots
\]
where \( \alpha_n \) is the constant coefficient n-anacci constant.

5.6 Error Detection and Correction

The error detection and correction of a new coding/decoding method is most essential for its applications. In this section, we discuss the correct ability of this method.

We know that the error detection and correction fully depend on the equation

\[
det E = det P_n \times (-1)^{(n+1)k}. \tag{5.72}
\]

We show that the error detection and correction is possible for this coding/decoding method, if we choose \( k \) in such a manner that \( e_{ij} > 0 \) for \( i, j = 1, 2, 3, \ldots, n \). At first we calculate the determinant of the initial matrix \( P_n \) and then we determine the code matrix elements of the code matrix \( E \) and send it to a communication channel. We treat \( Det P_n \) as the checking elements of the code matrix \( E \) received from the communication channel. We calculate the determinant of the matrix \( E \) received and compare it with the given
$detP_n$ by the relation (5.72). If they satisfy the relation (5.72), then we conclude that the elements of the code matrix $E$ received are without errors, otherwise there are errors. If there are errors, then we try to correct these errors using the relations (5.72) and (5.71).

**Case 1: $n = 2$.**

Our first hypothesis is that we have the case of “single error” in the code matrix $E$ received from the communication channel. It is clear that there are four variants of the “single error” in the code matrix $E$ received:

\[
\begin{align*}
(a) & \begin{pmatrix} x_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}, \\
(b) & \begin{pmatrix} e_{11} & x_{12} \\ e_{21} & e_{22} \end{pmatrix}, \\
(c) & \begin{pmatrix} e_{11} & e_{12} \\ x_{21} & e_{22} \end{pmatrix} \text{ and } (d) \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & x_{22} \end{pmatrix},
\end{align*}
\]

where $x_{11}, x_{12}, x_{21}$ and $x_{22}$ are possible destroyed elements.

For checking the possible “single error”, we can write the following algebraic equations based on the “checking relation” (5.72):

\[
\begin{align*}
x_{11}e_{22} - e_{12}e_{31} &= detP_2 \times (-1)^{3k} \text{ ("single error" occurs in the (1, 1) cell)}, \\
e_{11}e_{22} - x_{12}e_{31} &= detP_2 \times (-1)^{3k} \text{ ("single error" occurs in the (1, 2) cell)}, \\
e_{11}e_{22} - e_{12}x_{31} &= detP_2 \times (-1)^{3k} \text{ ("single error" occurs in the (2, 1) cell)} \\
\text{and} \\
e_{11}x_{22} - e_{12}e_{31} &= detP_2 \times (-1)^{3k} \text{ ("single error" occurs in the (2, 2) cell)}.
\end{align*}
\]

From (5.73) - (5.76), the four variants of the possible “single error” are

\[
\begin{align*}
x_{11} &= \frac{(-1)^{3k} + e_{12}e_{21}}{e_{22}}, \\
x_{12} &= \frac{(-1)^{3k} + e_{11}e_{22}}{e_{21}}, \\
x_{21} &= \frac{(-1)^{3k} + e_{11}e_{22}}{e_{12}} \\
\text{and} \\
x_{22} &= \frac{(-1)^{3k} + e_{12}e_{21}}{e_{11}}.
\end{align*}
\]
To obtain the correct variant, we have to choose the integer solutions among the solutions of $x_{11}$, $x_{12}$, $x_{21}$ and $x_{22}$ satisfying the additional “checking relations” (5.37) and (5.38). If this fails then we have to conclude that our hypothesis about “single error” is incorrect or in other words there are more than one error in the code matrix $E$ received.

We consider one of the “double errors” cases in the code matrix $E$ received as

$$E = \begin{pmatrix} x_{11} & x_{12} \\ e_{21} & e_{22} \end{pmatrix}. \tag{5.81}$$

Using the “checking relation” (5.72), we have the following algebraic equation for the matrix (5.81):

$$x_{11}e_{22} - x_{12}e_{31} = det P_2 \times (-1)^k. \tag{5.82}$$

It is important to emphasize that equation (5.82) is “Diophantine” equation in two variables. As the “Diophantine” equation has many solutions, we must select such solutions $x_{11} = e_{11}$ and $x_{12} = e_{12}$ among them which satisfy the additional “checking relations” (5.37) and (5.38). In this way we can correct all possible “double errors” in the code matrix $E$ received which satisfy the “checking relations” (5.72), (5.37) and (5.38). Otherwise, there may be “triple errors” in the code matrix $E$ received and we try to correct all possible “triple errors” by the similar approach.

Let one form of the incorrect code matrices $E$ received having “triple errors” be

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & e_{22} \end{pmatrix}. \tag{5.81}$$

Thus, we use the “checking relations” (5.72), (5.37) and (5.38) to correct the errors in the code matrix $E$ received based on different hypotheses along with the integral elements of the code matrix $E$ received.

Code matrix $E$ received is “erroneous” or the case of “fourfold errors” if at least one of the previous solutions is not integer. In this case we reject the code matrix $E$ received.

Thus we have

$$\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4 - 1 = 15$$

cases of errors in the code matrix $E$ received.
Our method allows correcting all possible “single error”, “double errors” and “triple errors” i.e. 14 cases among them. Hence the correct ability of errors of the method is \( \frac{14}{15} = 0.9333 = 93.33\% \) which does not depend on the value of the coefficient \( a_1 \) but \( a_2 = 1 \).

**Case 2: \( n = 3 \).**

At first, we consider “single error” in the code matrix \( E \) received. It is clear that there are nine variants of “single error” in the code matrix \( E \) received. For example, one of them is

\[
\begin{pmatrix}
x_{11} & e_{12} & e_{13} \\
e_{21} & e_{22} & e_{23} \\
e_{31} & e_{32} & e_{33}
\end{pmatrix}
\]  
(5.83)

where \( x_{11} \) is the possible destroyed element in the \((1,1)\) cell.

Using the relation (5.72), we have the algebraic equation of the matrix (5.83):

\[
x_{11}(e_{22}e_{33} - e_{23}e_{32}) + e_{12}(e_{23}e_{31} - e_{21}e_{33}) + e_{13}(e_{21}e_{32} - e_{22}e_{31}) = \text{det}P_3.
\]  
(5.84)

There are nine equations similar to (5.84) for nine possible variants of “single error” \( x_{ij}, \) \( i, j = 1, 2, 3 \). But we have to select the correct variant only among these cases of the integer solutions \( x_{ij}, i, j = 1, 2, 3 \) satisfying the relations (5.70). If there is no integer solution, we conclude that our hypothesis about “single error” is incorrect or we have more than one error in the code matrix \( E \) received.

Now we check all hypotheses of “double errors” in the code matrix \( E \) received. We consider that the “double errors” code matrix \( E \) received is

\[
\begin{pmatrix}
x_{11} & x_{12} & e_{13} \\
e_{21} & e_{22} & e_{23} \\
e_{31} & e_{32} & e_{33}
\end{pmatrix}
\]  
(5.85)

Using the relation (5.72), the algebraic equation for the matrix (5.85) is

\[
x_{11}(e_{22}e_{33} - e_{23}e_{32}) + x_{12}(e_{23}e_{31} - e_{21}e_{33}) = e_{13}(e_{22}e_{31} - e_{21}e_{32}) + \text{det}P_3.
\]  
(5.86)

Equation (5.86) is a “Diophantine” equation in two variables. So it has many solutions. We choose the integer solutions of \( x_{11} \) and \( x_{12} \) satisfying the relation (5.70).
It is clear that there are \( \binom{9}{2} \) variants of “double errors” in the code matrix \( E \) received and by using similar approach, we correct all “double errors” in the code matrix \( E \) received.

Similarly, we try to correct all possible triple, fourfold, . . . , eightfold errors in the code matrix \( E \) received using this approach. There are

\[
\binom{9}{1} + \binom{9}{2} + \ldots + \binom{9}{9} = 2^3 - 1 = 511
\]

possible cases of errors in the code matrix \( E \) received.

We know that it is not possible to correct the “ninefold errors” in the code matrix \( E \) received. Hence the correct ability of errors in this method is \( \frac{510}{511} \approx 0.9980 = 99.80\% \) which does not depend on the values of the coefficients \( a_1, a_2 \) but \( a_3 = 1 \).

Case 3: \( n = m \), where \( m \) is large.

Similarly, we can show that the correct ability of this method is \( \frac{2^{n^2} - 2}{2^{n^2} - 1} \) which does not depend on the values of the coefficients \( a_1, a_2, \ldots, a_{m-1} \) but \( a_m = 1 \).

The interesting feature of this method is that for large value of \( n \) the correct ability of errors is

\[
\frac{2^{n^2} - 2}{2^{n^2} - 1} \approx 1 = 100\%
\]

which does not depend on the values of the coefficients \( a_1, a_2, \ldots, a_{n-1} \) but \( a_n = 1 \).