Introduction

Decision science has been emerging as a proficient area of research due to its ability to manage real world problems in a systematic manner. The process of decision making mainly involves four major steps, viz., problem identification, construction of the preferences, evaluation of the alternatives, and determination of the best alternative [184, 188, 344].

Among the various methods of finding the best alternatives, Mathematical programming (MP) models play a key role from the viewpoint of its capacity to solve different optimization models. The foundation of MP was first established due to the pioneer work of Von Neumann [269]. The major development in MP was made during the mid 1950s and 1960s by many researchers in the fields of Economics, Operations Research and Management Sciences. A book, written by Kumar and Bonnett [204], focuses the development of MP in the past century.

Decision making is extremely intuitive for solving a single criterion problem since the objective of the DM is only to choose that alternative which has the highest preference rating. But in the context of several criteria, DMs have to evaluate the available alternating based on weights of criteria, preference dependence, and conflicts among criteria which are a complicated task to the DMs. The area emerges for solving this kind of problems is known as multicriteria decision making (MCDM) arena.

To deal with MCDM problems the first step involves identification of the problems and to determine the number of attributes or criteria associated with the problems. Next to receive information about the preferences of the DMs to construct the preferences and to build a set of possible alternatives or strategies to reach the goal. Finally, an appropriate method is adopted to evaluate, rank and improve the possible alternatives or strategies for finding the best alternatives.

The MCDM was first introduced by Pareto [287] in 1906 and the mathematical model of MCDM was presented by Koopmans [197] in 1951. Many pioneer researchers [2, 3, 11, 46, 64, 104, 160, 194, 195, 221, 234, 250, 253, 297, 326, 337, 357, 359] extensively studied MCDM problems during 1950s and 1960s. Also the study on MCDM was enriched and well documented in the books written by Bana e Costa and Vincke [16], Cochrane and Zeleny [83], Hwang and Lin [153], Hwang and Yoon [155], Nijkamp and Spronk [271], Romero and Rehaman [314], Roy and
In the recent past survey works in the field of MCDM are done by [6, 26, 73, 77, 78, 82, 105, 144, 146, 148, 149, 201, 223, 227].

In order to facilitate systematic research in the area of MCDM, MCDM problems are classified into two main categories: multiattribute decision making (MADM) [76, 147, 155, 341, 376, 378, 379, 399, 403] and multiobjective decision making (MODM). MADM is basically applied in the evaluation phase, which is usually associated with a limited number of predetermined alternatives and discrete preference ratings. MODM is especially suitable for the design and planning steps and allows a user to achieve the optimal or aspired goals by considering the various interactions of the given constraints. A brief discussion on different aspects, developments and methodologies for solving MODM problems are presented in the following subsections.

1.1 Historical development of MODM

The objective of the study of MODM is to resolve optimal design problems in which several, conflicting, non commensurable objectives of the model have to be achieved simultaneously. Hence, it is naturally associated with MP methods for dealing with optimization problems.

A remarkable work in the area of MCDM with multiple objectives was introduced by Charnes and Cooper [62] in 1957. Thereafter, Blackwell [46], Briskin [48], Charnes and Cooper [64] did significant works in the field of MODM. Conventional methodologies for MODM problems have been systematically classified and discussed by Hwang and Masud [154]. Different approaches for solving MODM problems are presented in the books and monographs written by Goicoechea, et al. [123], Keeny and Raiffa [183], Sawaragi et al. [328], Steuer [358], Yu [402]. The applications of MODM problems in different real life problems are found due to the significant contribution of the pioneer researchers [13, 79, 80, 81, 107, 150, 183, 316, 377, 400, 408].

MODM problems are classified into two categories based on the linearity characters of the objectives as well as system constraints.
1.1.1 Linear Programming

Linear programming (LP) is a simple but efficient technique of MP and treats the problem of optimizing the objectives of one or more decision variables that are subject to a specified set of linear constraint.

A pioneer work to enrich the LP problems (LPPs), was first done by Kuhn and Tucker in 1956 [202]. A survey work on the development of LP was presented by Riley and Gass [305] in 1958. During the 1960s, the extensive study in the field of LPPs was done by Beale [21], Dantiz[94], Hadley [129], Hiller and Liberman [140] and Loomba [230]. The crucial event responsible for huge impact of LPPs in recent decades was computer revolution that made it possible for the simplex method to solve huge problems. In 1984, Karmarkar [174] of AT & T Bell laboratories published a landmark algorithm for solving huge LPPs. Also a dual- simplex implementation of a constraint selection algorithm for LP has been prepared by Myers [262].

The LPPs are classified into two categories: Integer Programming (IP) and Dynamic Programming (DP).

IP was first introduced by Dantzig [93]. The methodological development in the field of IP had been made by Balinski [15], Gomory [124], Smith and Pickard [352] and others. The IP was also extensively studied by Milano [252], Pochet&Wolsey [290] and Taha [365].

DP was connected with multistage decision making problem, introduced by Bellman [23], Wald [386] and developed by Bellman and Dreyfus [24], Nemhauser [268]. A survey work in the field of DP was done by Thomas and DaCosta [374] in 1979.

1.1.2 Nonlinear programming

In MP problems where the objectives and/or system constraints are nonlinear in nature are called NLP problems. In 1954 the solution technique for NLP problems was proposed by Charnes and Lemke [70] and in 1956 the different aspects of DP have been presented by Kuhn-Tucker [203].

The methodological enrichment of NLP was found in the text books of Avriel [12], Bazaraa and Shetty [19], Gill et al. [120], Hadley [130], and Ruszczynski [319].
Separable Programming, Fractional programming and Geometric Programming are MP problems nonlinear in nature.

1.2 Mathematical Formulation of MODM problems

The general multiobjective programming model for decision making is presented as:

\[
\text{Find } X(x_1, x_2, \ldots, x_n) \text{ so as to} \\
\text{Optimize } (\maximize / \minimize) z_j(X), \quad j = 1, 2, \ldots, r \\
\text{Subject to } f_i(X) \begin{cases} \geq & b_i, \ i = 1, 2, \ldots, m. \\ \leq & \end{cases} \quad (1.1)
\]

where \(X\) is a vector of \(n\) decision variables; \(z_j(X)\) is the \(j\)-th objective function; \(f_i(X)\) is the \(i\)-th structural constraints; \(b_i\) is the right hand side value of the \(i\)-th structural constraints.

For solving MODM problems, all the mathematical programming approaches developed so far, the Goal Programming (GP) has appeared as the most promising technique.

1.2.1 Goal programming

The terminology “Goal Programming” was originally proposed by Charnes and Cooper [64]. The root of GP lies in a paper authored by Charnes et al. in 1955 [68] in which they deal with the executive compensation method.

The application of GP to different real life problems has been presented by Buckley and Hayashi [51], Fisk [112], Hannan [132], Nellayet al. [266], and others.

The pioneer researchers Cook [84], Crowder and Sposito [86], Gass [115], Ignizio [157, 158, 159], Masud and Hwang [249], Ogryczak [272], Olson [277] extensively studied and enriched the field of GP during 1980s. On the other hand Charnes&Storbeck [72], Dobbins and Mapp [97], Dryzan [98], Kvanli [206], Rehman and Romero [303] Taylor et al. [372], applied GP in different real life problems. Also the application of GP for solving different real life problems was surveyed by Lin [222], Romero [311] and Spronk [353].

GP is a widely used MCDM technique [312]. Although Schniederjans [330] has detected a decline in the life cycle of GP with regards to theoretical developments, but
still it is impressive, as shown in recent surveys by Romero [311, 312], Schniederjans [329], and Tamiz et al. [367].

The new methodological improvement of GP in the recent past was discussed by Caballero et al. [53], Pant and Shah [286], Sinha and Jha [348] and well documented by Michnik and Trzaskalil [251], Tanino et al. [370].

The various aspects of GP have been further studied by Aouni and Kettani [7], Mirrazaviet et al. [255], Romero [313] recent past. In 2006 Agha [4] applied GP in water quality management. Also Bal et al. [14], Leung et al. [214] applied GP in various real life problems recently.

GP models can be classified into two major subsets. In the first type the unwanted deviations are assigned weights according to their relative importance to the DM and minimized as an Archimedean sum. This is known as weighted GP (WGP). In the other major subset of GP the deviational variable are assigned into a number of priority structures where the goals in the same priority level are distinguished by putting relative weights. The goals with higher priority level will be achieved before lower priority goals and so on. In the field of GP, the priority based GP is the most powerful technique for solving multiple and conflicting goals in MODM environment.

The priority based GP model, first introduced by Ijiri [160] in 1965 and extended by Ignizio [156, 159], Lee [213], Steuer [357]. Tamiz et al. [367] shows that around 64% of GP application reported in the literature use priority based GP model.

The advancement of the theory and practice of GP is maintaining a steady rate. Some of the important GP approaches for solving different MODM problems are given by

Interactive GP [302, 348, 353]
Nonlinear GP [156, 320]
Integer GP [159, 247]
Fractional GP [132,198, 246, 286]
GP applied in different real life problems by [47, 52, 72, 69].

1.2.2 An overview of Goal programming formulation

In GP, instead of optimizing the objective function directly, a desired target value, termed as aspiration level, is introduced to each of the objectives.
In MP, an objective function is of the form:

Optimize (Maximize / Minimize) \( Z(X) \)

Then, in conjunction with the aspiration level, an objective function takes either of the forms \( Z(X) \geq b \); \( Z(X) = b \) or \( Z(X) \leq b \)

Depending on the decision situation, where “\( b \)” indicates aspiration level. The forms of the above inequalities are the same as that of the structural constraints in a programming problem.

The objectives with their aspiration levels along with the structural constraints are presented in a unified from as:

\[
Z_{j}(X) \begin{array}{c}
\geq \\
= \\
\leq
\end{array} b_{j}, \ j = 1, 2, ..., m
\]  \quad (1.2)

Traditionally, the above inequalities are termed as constraints, but in the literature of GP, the term goal, instead of constraints, is used.

The conceptual and technical difference between constraints and goals is classified in the manner that a constraint is a fixed requirement, which cannot be violated in any decision making situation, whereas goal is a fixed requirement, which is to be satisfied as closely as possible in a decision making environment.

In GP, it is not only sufficient to satisfy a goal as a constraint, but also it is necessary to satisfy a goal as a constraint in a best possible way.

In general, the mathematical form of inflexible goals appears as:

Satisfy \( Z_{j}(X) \begin{array}{c}
\geq \\
= \\
\leq
\end{array} b_{j}, \ j = 1, 2, ..., m \)

The logical variables which are termed as deviational variables (under- and over-deviational variables) are introduced to make the standard form of GP. The goals with the inclusion of deviational variables appear as:

\( Z_{j}(X) + d_{j}^{-} - d_{j}^{+} = b_{j}, \ \text{for} \ j = 1, 2, ..., m \)

where \( d_{j}^{-} (\geq 0) \) and \( d_{j}^{+} (\geq 0) \) represent the under- and over-deviational variables, respectively, with \( d_{j}^{-} \cdot d_{j}^{+} = 0 \). This form of a goal is called the flexible goal. The objective of the decision maker is to minimize the deviational variables so as to achieve the desired aspiration level to the extent possible.

In Table 1.1, a summary of the conversion of inflexible goals into the flexible goals and deviational variables to be minimized in the achievement function is presented.
<table>
<thead>
<tr>
<th>Inflexible Goal</th>
<th>Flexible Goal</th>
<th>Contribution to the achievement function (deviational variables to be minimized)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_j(X) \geq b_j$</td>
<td>$Z_j(X) + d^{-}_j - d^{+}_j = b_j$</td>
<td>$d^{-}_j$ is to be minimized</td>
</tr>
<tr>
<td>$Z_j(X) = b_j$</td>
<td>$Z_j(X) + d^{-}_j - d^{+}_j = b_j$</td>
<td>$d^{-}_j + d^{+}_j$ are to be minimized</td>
</tr>
<tr>
<td>$Z_j(X) \leq b_j$</td>
<td>$Z_j(X) + d^{-}_j - d^{+}_j = b_j$</td>
<td>$d^{+}_j$ is to be minimized</td>
</tr>
</tbody>
</table>

To deal with multiobjective decision making problems, there are two prominent approaches available in the literature as: (a) *Minsun* GP and (b) *Priority based* GP.

(a) **Minsun GP**

The GP model initially proposed by Charnes and Cooper [1961] is actually a *minsum* GP model and it is the simplest form of GP.

The general form of *minsum* GP model is given by

Find $X(x_1, x_2, \ldots, x_n)$ so as to

Minimize $D = \sum_{j=1}^{m} (w^{-}_j d^{-}_j + w^{+}_j d^{+}_j)$

and satisfy $Z_j(X) + d^{-}_j - d^{+}_j = b_j$  \hspace{1cm} (1.3)

$d^{-}_j, d^{+}_j \geq 0$ with $d^{-}_j, d^{+}_j = 0$, $j = 1, 2, \ldots, m$

where $Z_j(X)$ is the $j$-th goal constraint of the decision vector $X$; $b_j$ is the aspiration level of the $j$-th goal; and $D$ represent the achievement function consisting of the weighted deviational variables, where $d^{-}_j, d^{+}_j \geq 0$ represent the under and over – deviational variables of the $j$-th goal and $w^{-}_j (\geq 0)$ and $w^{+}_j (\geq 0)$ are the numerical weights of importance of achieving the goal in the decision making situation.

(b) **Priority based GP**

In the *Priority based* GP procedure, the goals are ranked according to their priorities for achievement of the respective aspiration levels in the decision making situation.

The general *Priority based* GP model is presented as

Find $X(x_1, x_2, \ldots, x_n)$ so as to

Minimize $D = [P_{1}(d^{-}, d^{+}), P_{2}(d^{-}, d^{+}), \ldots, P_{k}(d^{-}, d^{+}), \ldots, P_{K}(d^{-}, d^{+})]$  \hspace{1cm} (1.4)

and satisfy $Z_j(X) + d^{-}_j - d^{+}_j = b_j$

$X \geq 0, d^{-}_j, d^{+}_j \geq 0$ with $d^{-}_j, d^{+}_j = 0$, $j = 1, 2, \ldots, m$
where $D$ is the $K$ priority achievement function and where $P_k(d^-,d^+) \text{ is of the }$
form $P_k(d^-,d^+) = \sum_{j=1}^{m}(w_{jk}^- d_{jk}^- + w_{jk}^+ d_{jk}^+); j = 1,2,\ldots,m; \ k = 1,2,\ldots,K; \text{ and}$

$P_1 >>> P_2 >>> \ldots >>> P_k >>> P_K$

where ‘>>>’ implies much greater than the previous priority level.

$w_{jk}^-,w_{jk}^+ \geq 0; \ d_{jk}^-,d_{jk}^+ \geq 0; \text{ with } d_{jk}^- d_{jk}^+ = 0, \ j = 1,2,\ldots,m; \ k = 1,2,\ldots,K; \ K \leq m.$

It is to be noted that the minsum GP is actually considered as a special case of priority based GP where no priority preference is given to the goals.

### 1.3 Uncertainty in decision making

Most of the information received from the real world problems is uncertain in nature. In the context of MODM problems, the uncertainties are frequently involved with the various parameters defining objectives and constraints. With the development of different computational models and scientific computing techniques, sophisticated optimization models [10, 279, 412] can now be solved efficiently. Yet there are many MODM problems and their applications in different real world problems which are affected by uncertainties involved with input data or in model relationships. In 1980 Dubois and Prade [100] also pointed out the importance of pondering uncertain quantification in complex systems. Uncertainty can be described in several ways depending on the availability of information. Among mathematical tools for coping with uncertainty, mention worst case scenario analysis, evidence theory, probability theory and fuzzy set theory, etc. Accessible accounts of these tools may be found in Liu [226], Sakawa [321], Shafer[334], Shiryaer [342]. If there is vague uncertainty in decision making then the probabilistic programming is used for MODM problems. On the other hand imprecision in decision making problems can be handled by fuzzy mathematical programming. The research has progressed at a steady pace in the fields of stochastic optimization [171, 172, 380, 385, 331] and fuzzy mathematical programming [29, 209, 238, 415] in the past decade.

#### 1.3.1 Probabilistic Decision Making

In mathematical programming the coefficients in the formulation are not always crisp in nature. Most of the real-life problems involve some level of uncertainty about the
values to be assigned to various parameters or about the layout of some of the problem components. To deal with uncertainty, the concepts and techniques of probability theory are usually employed.

In 1955 Dantzig [92] formulated linear programming under uncertainty. In 1960s, many other researchers [22, 27, 28, 111, 181, 387, 392, 393] used probability theory to capture the uncertainty in linear programming. Also in 1970s Bereanu [27], Prekopa [291], Thompson et al. [375] and Ziemba [414] enriched the uncertain LP with the assistance of probability theory.

Probabilistic programming is one of the most interesting and challenging development of MP. In LP problems, if the coefficients of the objectives of the DMs and/or one or more of the coefficients of system constraints are involved with random variables and/or occurrence of the system constraints are uncertain, general LP methodology fails to solve these types of problems. Usually, these types of problems are converted first into a deterministic equivalent and then solved using different methodologies developed in different MP environments.

1.3.1.1 Probability theory and its basic concepts

Different terms associated with the probability theory are presented below.

I. Experiment: An experiment denotes the act of performing something, the outcome of which is subject to uncertainty and is not known exactly. Let the outcome is denoted by \( \omega \), and the set of all possible outcomes is called a sample space denoted by \( \Omega \).

II. Event: An event represents the outcome of a single experiment.

III. Probability: The probability is defined in terms of the likelihood of a specific event. If \( E \) denotes an event, the probability of occurrence of the event \( E \) is usually denoted by \( P(E) \). The probability of occurrence depends on the number of observation or trials.

The probability of an event is given by

\[
P(E) = \lim_{n \to \infty} \frac{m}{n}
\]

Where \( m \) = number of successful occurrences of the event \( E \),
And \( n \) = total number of trials.

IV. \( \sigma \) – Field: A family \( \mathcal{B} \) of subsets of the sample space \( \Omega \) which has the following properties is called a \( \sigma \) – field:
i) $\Omega \in \mathcal{B}$;

ii) \( A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B} \)

iii) \( A_1, A_2, ... \in \mathcal{B} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{B} \)

Let $\mathcal{G}$ be a family of subsets of the sample space $\Omega$. The intersection of all $\sigma$–fields containing $\mathcal{G}$ is called the $\sigma$ – field generated by $\mathcal{G}$ and is denoted by $\sigma[\mathcal{G}]$.

V. Probability Measure: Let $\mathcal{B}$ be a $\sigma$ – field in the sample space $\Omega$. A real valued function $P$ defined on the $\sigma$ – field $\mathcal{B}$ in $\Omega$ satisfying the following conditions is called a probability measure

i) $0 \leq P(A) \leq 1$ for $A \in \mathcal{B}$

ii) $P(\Omega) = 1$;

iii) $A_1, A_2, ... \in \mathcal{B}$ and $A_1, A_2 ...$ is a disjoint sequence then, $P(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k)$

VI. Independent Event: If the occurrence of an event $E_1$ in no way affects the probability of occurrence of another event $E_2$, the events $E_1$ and $E_2$ are said to be statistically independent. In this case, the probability of simultaneous occurrence of both events is given by

$$P(E_1, E_2) = P(E_1)P(E_2).$$

VII. Random Variable: Let a random event be the measurement of a quantity $X$, which takes on various values in the range $-\infty$ to $\infty$. Such a quantity is called a random variable.

If $\mathcal{B}$ be a $\sigma$ – field in a sample space $\Omega$ and $P$ is a probability measure on $\mathcal{B}$, the triple $\langle \Omega, \mathcal{B}, P \rangle$ is called a probability measure space or simply a probability space. For any arbitrary probability space $\langle \Omega, \mathcal{B}, P \rangle$, let $X(\omega)$ be a real valued function on $\Omega$. Then $X(\omega)$ is a random variable if

$$\{\omega | X(\omega) \leq x \} \in \mathcal{B}$$

holds for each real value $x$.

VIII. Discrete Random Variable: If a random variable is allowed to take only discrete values, is called a discrete random variable.

IX. Continuous Random Variable: If a random variable is allowed to take any real value in a specific range, it is called a continuous random variable.
X. Probability Distribution: For the random variable $X(\omega)$ the function

$$ F(x) = P\{\omega | X(\omega) \leq x \} $$

is called a probability distribution function, where $P\{\omega | X(\omega) \leq x \}$ is often simply denoted by $P(X \leq x)$. The distribution function has the following properties:

i) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$

ii) If $x < y$, then $F(x) \leq F(y)$.

XI. Probability Density Function: For the random variable $X(\omega)$ the function $f(x) = \frac{d}{dx} \{F(x)\}$ is called the probability density function with the property

$$ \int_{-\infty}^{+\infty} f(x) \, dx = 1. $$

1.3.1.2 Chance constrained programming

When the parameters in an MP model are presumed to be random variables rather than constants, the concept of stochastic programming (SP) arises. These types of problems involve risk if the probability distribution of the random variables is known or involve uncertainty if the distribution of at least one variable is unknown. The difficulties of dealing with risk and uncertainty in programming problems have been discussed in the literature since the 1950’s. In LP when the presumed level of the reliability of the constraints is less than one, then Chance constrained programming (CCP) can be employed as a means of describing that level of constraint violation.

CCP has been introduced into the literature through the works of Charnes and Cooper [63], and since then has been developed by Carbone [56], Hillier [139], Kataoka [181], Panne and Popp [285]. Also Seppala [333] constructed sets of uniformly tighter linear constraints that replace a single chance constraint. Panne and Popp [285] incorporated the concept of CCP in the cattle feed problem and the deterministic equivalent was solved using Zoutendijk’s methods [424] for NLP.

Formulation of CCP model

For decision problems under probabilistic uncertainty, Charnes and Cooper [66] proposed CCP technique which admits random data variations and permits constraint violations up to specified probability limits.

A set of chance constraints of a LPP can be presented as

\[ \{ x \in R^n \mid Pr \left[ \sum_{j=1}^{n} a_{ij} x_j \left( \sum_{\leq \infty} b_i \right) \geq 1 - p_i \right], i = 1, 2, ..., m \} \]  

(1.5)

Where \( \sum_{j=1}^{n} a_{ij} x_j \) is the vector of decision variables \( x_j (j = 1, 2, ..., n) \), \( a_{ij} \) and \( b_i \) are normally distributed random variables, \( 1 - p_i (0 \leq p_i \leq 1) \) is the satisfying probability level defined for the randomness occurs in \( i \)-th constraints.

Deterministic equivalent of chance constraints

In converting CCP constraints into their equivalent deterministic one the following three cases arise which are presented as follows:

(a) Only \( a_{ij} \) are random variables,

(b) Only \( b_i \) are random variables,

(c) \( a_{ij} \) and \( b_i \) are random variables,

The methodology for the conversion of deterministic equivalents of the above three cases considering ‘\( \leq \)’ restriction are presented independently in the following subsections.

(a) Only \( a_{ij} \) are random variables

Let \( E(a_{ij}) \) and \( Var(a_{ij}) \) be the mean and variance of the normally distributed random variable \( a_{ij} \), for all \( i \) and \( j \). Now, a random variable \( y_i \) is defined as

\[ y_i = \sum_{j=1}^{n} a_{ij} x_j \; ; \; i = 1, 2, ..., m. \]

It can be easily realized that \( y_i \) also follows normal distribution with the following mean and variance

\[ E(y_i) = \sum_{j=1}^{n} E(a_{ij}) x_j \; ; \; i = 1, 2, ..., m. \]

\[ Var(y_i) = X^T V_i X \; ; \; i = 1, 2, ..., m. \]

where \( V_i \) is the \( i \)-th covariance matrix defined as

\[
V_i = \begin{bmatrix}
\text{Var}(a_{i1}) & \text{Cov}(a_{i1}, a_{i2}) & \cdots & \text{Cov}(a_{i1}, a_{in}) \\
\text{Cov}(a_{i2}, a_{i1}) & \text{Var}(a_{i2}) & \cdots & \text{Cov}(a_{i2}, a_{in}) \\
\vdots & \vdots & \ddots & \vdots \\
\text{Cov}(a_{in}, a_{i1}) & \text{Cov}(a_{in}, a_{i2}) & \cdots & \text{Var}(a_{in})
\end{bmatrix}
\]
The probabilistic constraints in (1.5) take the form
\[ \Pr(y_i \leq b_i) \geq 1 - p_i; \ i = 1, 2, ..., m. \]
i.e.,
\[ \Pr\left(\frac{y_i - E(y_i)}{\sqrt{\text{var}(y_i)}} \leq \frac{b_i - E(y_i)}{\sqrt{\text{var}(y_i)}}\right) \geq 1 - p_i \]
where \( \frac{y_i - E(y_i)}{\sqrt{\text{var}(y_i)}} \) is a standard normal variate with mean zero and variance one.

Then \( \Pr(y_i \leq b_i) = \Phi\left(\frac{b_i - E(y_i)}{\sqrt{\text{var}(y_i)}}\right) \), where \( \Phi(.) \) represents the cumulative distribution function of a standard normal variate. Let \( K_{p_i} \) denote the value of the standard normal random variate at which \( \Phi(K_{p_i}) = 1 - p_i \).

Thus \( \Phi\left(\frac{b_i - E(y_i)}{\sqrt{\text{var}(y_i)}}\right) \geq \Phi(K_{p_i}), \ i = 1, 2, ..., m. \)
This inequality is satisfied only if
\[ \left(\frac{b_i - E(y_i)}{\sqrt{\text{var}(y_i)}}\right) \geq (K_{p_i}), \ i = 1, 2, ..., m. \]

It can be simplified as
\[ E(y_i) + K_{p_i}\sqrt{\text{var}(y_i)} \leq b_i, \ i = 1, 2, ..., m. \]
i.e.,
\[ \sum_{j=1}^{n} E(a_{ij})x_j + \Phi^{-1}(1 - p_i) \sqrt{X^T V X} \leq b_i, \ i = 1, 2, ..., m. \] (1.6)
These are the deterministic nonlinear constraints equivalent to the original linear probabilistic constraints. It is worthy to mention here that all the normally distributed random variables are independent then their covariance will be zero. So (1.6) can be simplified as:
\[ \sum_{j=1}^{n} E(a_{ij})x_j + \Phi^{-1}(1 - p_i) \sqrt{\sum_{j=1}^{n} \text{Var}(a_{ij})x_j^2} \leq b_i, \ i = 1, 2, ..., m. \] (1.7)
which is the deterministic equivalent in this context.

(b) Only \( b_i \) is a random variable
Let \( E(b_i) \) and \( \text{Var}(b_i) \) be the mean and variance of the normally distributed random variables \( b_i \).

Then the constraints in (1.5) can be written as:
\[ \Pr\left[\frac{b_i - E(b_i)}{\sqrt{\text{var}(b_i)}} \geq \frac{\sum_{j=1}^{n} a_{ij}x_j - E(b_i)}{\sqrt{\text{var}(b_i)}}\right] \geq q_i, \ i = 1, 2, ..., m \] (1.8)
Where \( q_i = 1 - p_i \) and \( \frac{b_i - E(b_i)}{\sqrt{\text{var}(b_i)}} \) is a standard normal random variable. Then the inequality (1.8) can be written as
If $K_{q_i}$ represents the value of standard normal random variable at which $\Phi(K_{q_i}) = 1 - q_i$ then the constraint (1.9) can be expressed as:

$$\Phi \left( \frac{\sum_{j=1}^{n} a_{ij} \bar{x}_j - E(b_i)}{\sqrt{\text{var}(b_i)}} \right) = \Phi(K_{q_i}), \quad i = 1, 2, \ldots, m.$$  

This inequality will be satisfied only if

$$\left( \sum_{j=1}^{n} a_{ij} \bar{x}_j - E(b_i) \right) = K_{q_i}, \quad i = 1, 2, \ldots, m. \quad (1.10)$$

Hence $\sum_{j=1}^{n} a_{ij} \bar{x}_j \leq E(b_i) + K_{q_i} \sqrt{\text{var}(b_i)}, \quad i = 1, 2, \ldots, m. \quad (1.11)$

(c) $a_{ij}$ and $b_i$ are random variables.

Here $a_{ij}$ and $b_i$ both follow the normal distribution.

Then let $Y_i = \sum_{j=1}^{n} a_{ij} \bar{x}_j - b_i$

It can be easily realize that $Y_i$ also follows normal distribution. Then the probabilistic constraints in (1.5) take the form

$$\Pr(Y_i \leq 0) \geq 1 - p_i$$

This expression can be generalized as

$$\Pr \left[ \frac{Y_i - E(Y_i)}{\sqrt{\text{var}(Y_i)}} \leq \frac{-E(Y_i)}{\sqrt{\text{var}(Y_i)}} \right] \geq 1 - p_i \quad (1.12)$$

Where $\frac{Y_i - E(Y_i)}{\sqrt{\text{var}(Y_i)}}$ is a standard normal variate, therefore the deterministic equivalent of (1.12) appears as

$$\frac{-E(Y_i)}{\sqrt{\text{var}(Y_i)}} \geq K_{pi}$$

i.e., $E(Y_i) + K_{pi} \sqrt{\text{var}(Y_i)} \leq 0, \quad i = 1, 2, \ldots, m \quad (1.13)$

Similarly, the deterministic constraints for the expression ‘$\geq$’ type restriction can be obtained.

1.3.1.3 Two- stage Stochastic programming

The journey towards stochastic linear programming (SLP) was started with the pioneering works of Beale [20] and Dantzig [92] in the form of two-stage (SLP). Afterwards, several models for solving SLP were developed by Wets [392, 394], Madansky [244], Sengupta [332], Kall [171] and so many authors.

In the process of two stage stochastic programming (SP) the problem is first converted into a deterministic problem. Unlike CCP, two-stage programming does not allow any constraints to be violated. In two-stage programming under uncertainty a
decision is first taken without knowing the values of the random variables. After occurring the random events and identifying the values of the random variables second stage is executed in order to minimize the penalties that may appear due to infeasibility in the first phase. The conversion of stochastic model to deterministic one can be performed through the following processes:

(a) The maximization of the mean value (called E-model)
(b) The minimization of the variance (called V-model)
(c) The minimization of the mean value with a constraint on the variance
(d) The minimization of the second order moment.

Maximization of the probability that the value of the function exceeds a given level $k$ (one is satisfied if the objective function takes at least value $k$) (called P-model).

(e) Maximization of the $\alpha$-fractile of the cumulative distribution of the objective function with the value of $\alpha$ preassigned by the DM known as a fractile criterion model. [116].

1.3.1.4 Multiobjective Chance Constrained Programming

Most of the real world optimization problems contain multiple, non commensurable, conflicting objectives which should have considered simultaneously. If the CCP problems discussed in the Subsection 1.3.1.2 earlier having multiplicity of objectives are known as multiobjective CCP (MOCCP) problems. In MOCCP the DM’s have to take optimal compromise decision with probabilistic set of system constraints. In 1977 Charnes& Copper [67] used GP for multiobjective optimization. Also in 1982 Ignizio [158] discussed LP problems with single and multiple objective functions. Many other researchers [229, 397, 398] discussed MOCCP problems in recent years.

1.3.1.5 Multiobjective Stochastic Programming

In a multiobjective CCP problem if the coefficients of the objectives are also random variables with known distribution function is known as multiobjective SP problem. When a probabilistic description of unknown elements is available, one is naturally led to stochastic linear programming problem [66, 109, 117, 171, 172, 304, 380].

In 1980 Lau (1980) considers the problem with three objective functions (a) maximizing the expected profit, (b) maximizing expected utility, (c) Maximizing the probability of achieving a budget profit. In this way the SP with multiple objective functions are considered in several contexts. Application of probabilistic goal
programming have been presented by Charnes and Stedry [71], and also studied by Contini [85], Geoffrion [116], Kataoka [181], Louveaux [232], and Stancu-Minasian and Tigan [354, 355].

1.3.2 Possibilistic Decision Making

Very often, in the context of decision making problems, the various parameters associated with the objectives and system constraints and the priorities/weighting structure to reflect the relative importance are fuzzy in nature. To take the situation of fuzziness, the fuzzy set theory initially introduced by Zadeh [404], deals with imprecisely defined data. Decision making in fuzzy environment was developed by Bellman and Zadeh [25]. Fuzzy mathematical programming offers a powerful means of handling optimization problems involving multiple objectives. Zimmermann [416, 417] presented a pioneering work in the field of fuzzy linear programming with multiple objectives. Leberling [211] proved that solution obtained by fuzzy linear programming are always a compromise solution of the original multiobjective problem.

1.3.2.1 Basic concepts of fuzzy sets

Definition of Fuzzy set:

Let $X$ denote a universal set. Then, a fuzzy set $\tilde{A}$ is defined by the membership function

$$ \mu_{\tilde{A}} : X \rightarrow [0, 1] $$

The membership function $\mu_{\tilde{A}}$ assigns to each element $x \in X$ a real number $\mu_{\tilde{A}}$ in the interval $[0, 1]$, and the value of $\mu_{\tilde{A}}(x)$ represents the grade of membership of $x$ in $\tilde{A}$. A fuzzy set $\tilde{A}$ is represented by a pair of an element $x$ and its grade $\mu_{\tilde{A}}(x)$, and thus it often written as

$$ \tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\} $$

It is evident that a fuzzy set $\tilde{A}$ is a natural extension of an ordinary set $A$. 
**Definition of α-level set:**

For a given $\alpha \in [0, 1]$, the $\alpha$-level set of a fuzzy set $\tilde{A}$ is defined as a ordinary set $A_\alpha$ of elements $x$ such that the membership function value $\mu_{\tilde{A}}(x)$ of $x$ exceeds $\alpha$.

i.e., $A_\alpha = \{x | \mu_{\tilde{A}}(x) \geq \alpha\}$

Several operators for fuzzy sets have been developed and widely circulated in the literature.

Some of the basic operations by Zadeh [404] are presented as follows.

**Union of Fuzzy set:**

The union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ on $X$, denoted by $\tilde{A} \cup \tilde{B}$, is defined by

$$\mu_{\tilde{A} \cup \tilde{B}} = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \text{ for all } x \in X.$$

**Intersection of Fuzzy set:**

The intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ on $X$, denoted by $\tilde{A} \cap \tilde{B}$, is defined by

$$\mu_{\tilde{A} \cap \tilde{B}} = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \text{ for all } x \in X.$$

**Complement of Fuzzy set:**

The complement of $\tilde{A}$ on $X$, denoted by $\overline{\tilde{A}}$, is defined by $\mu_{\overline{\tilde{A}}}(x) = 1 - \mu_{\tilde{A}}(x)$ for all $x \in X$.

Several set-theoretic operations for fuzzy sets and their application to practical problems have been studied by Dubois and Prade [100], Gupta et al. [128], Jones et al. [169], Kandle [173], Kaufmann [182], Klir and Folger [191], Klir and Yuan [193], Wang and Chang [388], Zimmermann [418], among others in the past.

To deal with mathematical programming under uncertainty, among fuzzy sets, fuzzy numbers which are linguistically-expressed such as “approximately m” or “about n” play very significant role.

**Convex normalized fuzzy sets:**

A fuzzy set $\tilde{A}$ is said to be convex if any $\alpha$-level set $A_\alpha$ of $\tilde{A}$ is convex, and a fuzzy set $\tilde{A}$ is said to be normal if there is $x$ such that $\mu_{\tilde{A}}(x) = 1$. 
Fuzzy numbers:

A fuzzy number is a convex normalized fuzzy set of the real line $\mathbb{R}$ whose membership function is continuous or piecewise continuous. Nowadays for the sake of computational efficiency and ease of data acquisition, the fuzzy numbers with triangular, one sided or trapezoidal shaped membership functions are often used.

**Triangular fuzzy number:**

A fuzzy number always represents vague data. For instance, A vague data “close to $a$” can be represented by a triangular fuzzy number, which is represented by a triple of three real numbers as $\tilde{a} = (a^l, a, a^R)$ The membership function of the triangular fuzzy number is of the form

$$
\mu_{\tilde{a}}(x) = \begin{cases} 
0 & \text{if } x < a^l, x > a^R \\
\frac{x - a^l}{a - a^l} & \text{if } a^l \leq x \leq a \\
\frac{a^R - x}{a^R - a} & \text{if } a \leq x \leq a^R 
\end{cases} \quad (1.14)
$$

Where $a^l$ and $a^R$ denote, respectively, the left and right tolerance values of the fuzzy number $\tilde{a}$. It is represented by the following figure:

![Figure 1.1: The triangular fuzzy number](image)

**One sided fuzzy number:**

There are two variations of fuzzy numbers, viz. right sided fuzzy number and left sided fuzzy number.

A right sided fuzzy number $\tilde{b}$ with tolerance limit $\beta$ is represented by the following membership function as

$$
\mu_{\tilde{b}}(x) = \begin{cases} 
\frac{1}{\beta} & \text{if } x \leq b \\
\frac{(b + \beta) - x}{\beta} & \text{if } b \leq x \leq b + \beta \\
0 & \text{if } x \geq b + \beta 
\end{cases} \quad (1.15)
$$
Diagrammatically it is represented by the following figure

![Diagram](image1.png)

**Figure 1.2: Right sided fuzzy number**

Similarly, a left sided fuzzy number \( \tilde{b} \) with tolerance limit \( \delta \) is represented by the following membership function as

\[
\mu_{\tilde{b}}(x) = \begin{cases} 
0 & \text{if } x \leq b - \delta \\
\frac{x-(b-\delta)}{\delta} & \text{if } b - \delta \leq x \leq b \\
1 & \text{if } x \geq b 
\end{cases}
\]  

(1.16)

Diagrammatically it is represented by the following figure

![Diagram](image2.png)

**Figure 1.3: Leftsided fuzzy number**

**First Decomposition Theorem on Fuzzy Sets [192]:**

Every fuzzy set \( A \) defined on \( S \), the universal set of discourse, can be represented in the form \( A = \bigcup_{\alpha \in [0,1]} A_{\alpha} \), where the special fuzzy set \( A_{\alpha} \) is defined by the membership values \( A_{\alpha}(s) = \alpha \cdot \alpha A(s) \), where \( \bigcup \) is considered as the standard fuzzy union and \( A_{\alpha} \) represents the \( \alpha \)-cut of \( A \).

**1.3.2.2 Fuzzy decision**

Let \( X \) be a given set of possible alternatives, contains the solution of a decision problem under consideration.

Then a fuzzy goal \( \tilde{G} \) is a fuzzy set on \( X \) characterized by the membership function
A fuzzy constraint $\tilde{C}$ is a fuzzy set on $X$ characterized by the membership function
\[ \mu_{\tilde{C}} : X \to [0, 1]. \]

Realizing that both the fuzzy goal and fuzzy constraints are desired to be satisfied simultaneously in making decision, Bellman and Zadeh [25] defined the fuzzy decision resulting from the goal $\tilde{G}$ and constraint $\tilde{C}$ as the intersection of $\tilde{G}$ and $\tilde{C}$.

Explicitly the fuzzy decision $\tilde{D}$ on $X$ is defined as $\tilde{D} = \tilde{G} \cap \tilde{C}$, and the membership function is
\[ \mu_{\tilde{D}}(X) = \min(\mu_{\tilde{G}}(X), \mu_{\tilde{C}}(X)), \ x \in X. \]

The maximizing type decision is then defined as
\[ \maximize_{x \in X} \mu_{\tilde{D}}(X) = \min_{x \in X} \mu_{\tilde{G}}(X), \mu_{\tilde{C}}(X)), \ x \in X. \]

In general, the fuzzy decision $\tilde{D}$ resulting from the $K$ fuzzy goals $\tilde{G}_1, \tilde{G}_2, ..., \tilde{G}_K$ and $M$ fuzzy constraints $\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_M$ is defined by
\[ \tilde{D} = \tilde{G}_1 \cap \tilde{G}_2 \cap ... \cap \tilde{G}_K \cap \tilde{C}_1 \cap \tilde{C}_2 \cap ... \cap \tilde{C}_M; \]

and is characterized by the membership function
\[ \mu_{\tilde{D}}(X) = \min\{\mu_{\tilde{G}_1}(X), \mu_{\tilde{G}_2}(X), ..., \mu_{\tilde{G}_K}(X), \mu_{\tilde{C}_1}(X), \mu_{\tilde{C}_2}(X), ..., \mu_{\tilde{C}_M}(X)\} \ x \in X. \]

In the above expression for decision $\tilde{D}$ it is observed that, the goal $\tilde{G}_K, k = 1, 2, ..., K$ and constraints $\tilde{C}_m, m = 1, 2, ..., M$ are included in $\tilde{D}$ in the same way.

That is there is no longer a difference between a fuzzy goal and fuzzy constraint. The mathematical aspects of fuzzy decision are well discussed in the books presented by Fodor and Roubens [113], Kickert [185], Li and Yen [215], Zhang et al. [411].

Now, the literature on applications of fuzzy set theory on different real world decision making problems is briefly discussed in the following section.

### 1.3.2.3 Decision Making in Fuzzy Environment

The application of fuzzy set theory introduced by Zadeh [404], in various MP problems studied by the researchers in different fields has been circulated in the literature.

The fuzzy MP (FMP) is generally used in the area of study of operational research or Management Sciences. But, the term fuzzy programming (FP) is widely used in the literature for the use of fuzzy set theory to various types of decision making problems in the field of MP.
There are two types of inexactness faced by the decision maker for establishment of a model in imprecise environment.

(i) DM’s ambiguity of understanding the nature of parameters associated with the problem formulation process.

(ii) The fuzzy goals of the DM for the objective function and constraints.

In 1974, Tanaka et al. [369] first proposed the concept of FP for general MP problems. Many other researchers [51, 89, 90, 99, 101, 102, 103, 209, 315, 383, 417] studied the FP approaches in MP problems. The FP approaches in MP problems, in programming and planning environment, involving single objective as well as multiobjective have been studied in the past.

In the following section, a brief review of the methodological developments and application of FP are presented for both types of decision making problems.

1.3.2.4 Fuzzy Single-Objective Programming

The FP approaches to single-objective optimization problems have been extensively studied by many active researchers during 1980s.

The early developments of FP approaches to both linear and nonlinear optimization problems have been reviewed thoroughly and classified systematically in the monographs prepared by Zimmermann [419], and others.

The general single-objective optimization problem, viz., the LP problem is of the form:

\[
\begin{align*}
\text{Find } X(x_1, x_2, \ldots, x_n) \\
\text{so as to } & \maximize z = \sum_{j=1}^{n} c_j x_j \\
\text{subject to } & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \\
x_j & \geq 0
\end{align*}
\]

where \(c_j, x_j\) and \(a_{ij}, b_i \in \mathbb{R}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n.\)

In many practical situations, it is not reasonable to require that the parameters of constraints or the objective in LP problems be specified in precise, crisp terms. In such situations, it is desirable to express these parameters in terms of fuzzy numbers and the update model is known as fuzzy LP (FLP)

The most general type of FLP is formulated as
Find $X(x_1, x_2, ..., x_n)$  
so as to Maximize $z = \tilde{c}_j x_j$  
subject to $\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i$,  
$x_j \geq 0$ \hspace{1cm} (1.18)

where $\tilde{c}_j$, $\tilde{a}_{ij}$, and $\tilde{b}_i$ are fuzzy numbers, and $x_j$ are variables whose states are fuzzy numbers.

Instead of discussing this general type FLP, two special types of FLP are discussed in the following cases.

**Case 1**: FLP problem where the right sided parameter $\tilde{b}_i$ are fuzzy numbers:

Find $X(x_1, x_2, ..., x_n)$  
so as to Maximize $z = \sum_{j=1}^{n} c_j x_j$  
subject to $\sum_{j=1}^{n} a_{ij} x_j \leq \tilde{b}_i$  
$x_j \geq 0$ \hspace{1cm} (1.19)

In general FLP is first converted into equivalent crisp problem and solved by standard method. In this case, fuzzy number $\tilde{b}_i$ is considered as right sided fuzzy numbers with membership functions as defined in (1.15) with the tolerance limits $\beta_i$.

Now to find the fuzzy set of optimal values, firstly the lower and upper bounds of the optimal values are to be determined from the following two sub problems.

The lower bound of the optimal values, $z^L$, is obtained by solving the standard LPP:

Find $X(x_1, x_2, ..., x_n)$  
so as to Maximize $z = \sum_{j=1}^{n} c_j x_j$  
subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$  
$x_j \geq 0$ \hspace{1cm} (1.19a)

The upper bound of the optimal values, $z^U$, is obtained by solving the LPP

Find $X(x_1, x_2, ..., x_n)$  
so as to Maximize $z = \sum_{j=1}^{n} c_j x_j$  
subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i + \beta_i$  
$x_j \geq 0$ \hspace{1cm} (1.19b)

Then the membership functions for the objectives are appeared as
Considering the membership function the following FLP model is developed as
\[
\mu_z(x) = \begin{cases} 
1 & \text{if} \quad z^U \leq x \\
\frac{x - z^L}{z^U - z^L} & \text{if} \quad z^L \leq x \leq z^U \\
0 & \text{if} \quad z^L \geq x 
\end{cases}
\]

Finding the maximum of
\[
\sum_{i=1}^{n} \alpha_i x_i
\]
subject to
\[
\sum_{j=1}^{n} a_{ij} x_j \leq b_i + \beta_i \\
\lambda x_i \geq 0
\]

The problem in (1.20) is solved by conventional single objective programming method.

**Case 2:** FLP problem where \( \tilde{a}_{ij} \) and \( \tilde{b}_i \) are fuzzy numbers:

Find \( X(x_1, x_2, ..., x_n) \)
so as to Maximize \( z = \sum_{j=1}^{n} c_j x_j \)
subject to \( \sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i \)
\[
x_j \geq 0
\]

In this case, let \( \tilde{a}_{ij} \) and \( \tilde{b}_i \) are triangular fuzzy numbers as defined in (1.14). Therefore the problem in (1.21) is written as

Find \( X(x_1, x_2, ..., x_n) \)
so as to Maximize \( z = \sum_{j=1}^{n} c_j x_j \)
subject to
\[
\sum_{j=1}^{n} \left\{ \alpha a_{ij} + (a_{ij} - a_{ij}^L) \alpha \right\} x_j \leq \left\{ b_i^u + (b_i - b_i^L) \alpha \right\} \\
\sum_{j=1}^{n} \left\{ \alpha a_{ij}^R - (a_{ij}^R - a_{ij}) \alpha \right\} x_j \leq \left\{ b_i^R + (b_i^R - b_i) \alpha \right\} \\
0 < \alpha \leq 1, \quad x_j \geq 0
\]

However, since all numbers involved in (1.22) are all real numbers, this is a classical LPP and can be solved conventionally.

The mathematical framework of FLP problem was introduced by Zimmermann [415]. Hamacher et al. [131], gives the different frameworks of FLP problems. The FLP problems have been studied by the pioneer researchers [54, 59, 95, 96, 210, 267, 368, 383, 384] in the past. Recently the Zimmermann’s FLP approach has been applied successfully to several real world problems [106; 145, 307,
In 1998 Rommelfanger and Slowinski [315] has also been surveyed the application potential of FLP.

### 1.3.2.5 Fuzzy Multiobjective Programming

FP approaches to MP problems with multiplicity of objectives was first proposed by Zimmermann [416] in 1978. After that the research in the field of fuzzy MODM (FMODM) has been prolific in the area of MP.

The literature on FMODM has been systematically classified and discussed by Fedrizzi, *et al.* (1991); Lootsma, *et al.* [231], Sakawa and Kato [322]; Tanaka, *et al.* [369]; Zimmermann [418, 419], Zimmermann, *et al.* [421]. The methodological development in the field of FMODM has been extensively surveyed by Carlsson and Fuller [57], Inguichiet *et al.* [162], Luhandjula [238], Zimmermann [420]. However, most of the FMODM methods [75, 133, 236, 391, 422] developed so far are mainly the extensions of Zimmermann’s [416] modeling concept for multiobjective decision analysis.

There are two most prominently used methodologies for solving conventional FMODM problems are available in the literature as;

i) Fuzzy GP (FGP) approach [136, 265, 335, 339]

ii) Interactive FP approach [23, 108]

Between the above two multiobjective FP approaches, the literature on FGP has grown tremendously in the recent past from the viewpoint of its potential use in real world decision making context. For solving MODM problems by FGP method was first discussed by Narasimhan [265] in 1980. The same approach has been further extended by Hannan [134, 135], Narasimhan [265], Rubin and Narasimhan [318], and others. Most of these approaches are actually extended version of the Zimmermann’s [416] approach. The FGP approach has been briefly studied by Ignizio [158].

The various aspects of FGP formulation have also investigated by Abd El-wahed and Abo Sinha [1], Chen and Tasi [74], Iskander [165], Kim and Whang [186], Kuwano [205], Martel and Aouni [248], Mohanty and Vijayadraghavan [260], Ramik [299], Rao et al. [300, 301], Pal and Moitra [282] and many others. However, the FGP approaches developed in the past have been applied to different real life problems [8, 30, 44, 152, 212, 288, 289, 327, 347, 362, 363].
1.3.2.5.1 Fuzzy Goal Programming Model

An MODM problem with fuzzy goals and crisp constraints can be presented as

\[
\text{Find } X \\
\text{So as to satisfy } f_k(X) = \begin{cases} 
\leq & g_k, k = 1, 2, ..., K \\
\geq & g_k 
\end{cases} \\
\text{Subject to } AX = \begin{cases} 
\leq & b \\
\geq & b 
\end{cases} \\
X \geq 0 \quad (1.23)
\]

where \( X \) is the vector of decision variable, and \( \leq, \geq \) indicates the fuzziness of the aspiration levels. Let \( g_k^l \) and \( g_k^u \) be the lower and upper tolerance limits, respectively, for achievement of the aspired level \( g_k \) of the k-th fuzzy objective goal. Then the membership function of the fuzzy goal \( f_k(X) \) can be characterized as follows:

For \( \leq \) type restriction,

\[
\mu_k(X) = \begin{cases} 
1 & \text{if } f_k(X) \leq g_k \\
\frac{g_k^u - f_k(X)}{g_k^u - g_k} & \text{if } g_k < f_k(X) \leq g_k^u \\
0 & \text{if } f_k(X) > g_k^u
\end{cases} \quad (1.24)
\]

For \( \geq \) type restriction,

\[
\mu_k(X) = \begin{cases} 
1 & \text{if } f_k(X) \geq g_k \\
\frac{f_k(X) - g_k^l}{g_k - g_k^l} & \text{if } g_k^l \leq f_k(X) < g_k \\
0 & \text{if } f_k(X) < g_k^l
\end{cases} \quad (1.25)
\]

For \( \equiv \) type restriction,

\[
\mu_k(X) = \begin{cases} 
1 & \text{if } f_k(X) = g_k \\
\frac{(f_k(X) - g_k)}{(g_k - g_k^l)} & \text{if } g_k^l \leq f_k(X) < g_k \\
\frac{g_k^u - f_k(X)}{g_k^u - g_k} & \text{if } g_k < f_k(X) \leq g_k^u \\
0 & \text{if } f_k(X) > g_k^u \text{ or } f_k(X) < g_k^l
\end{cases} \quad (1.26)
\]

Now, achievement of the highest degree (unity) of a membership function means absolute achievement of the aspired level of the associated fuzzy goal. So, the
membership goal corresponding to the \( k \)-th membership function with unity as the aspiration level can be presented as [281]:
\[
\mu_k(X) + d^+_k - d^-_k = 1,
\]
\( d^-_k, d^+_k \geq 0 \) with \( d^-_k, d^+_k = 0, k = 1, 2, ..., K \)

Then the priority based FGP model is

Find \( X \)

Minimize \( Z = \{P_1(d^-), P_2(d^-), ..., P_n(d^-), ..., P_N(d^-)\} \)

and satisfy \( \mu_k(X) + d^-_k - d^+_k = 1 \)

subject to the system constraints in (1.23)
\[
d^-_k, d^+_k \geq 0 \text{ with } d^-_k, d^+_k = 0, k = 1, 2, ..., K \quad (1.27)
\]

Where \( Z \) represent the vector of \( N \) priority achievement functions consisting of the under deviational variables of the goals for minimizing them on the basis of the priorities of achieving the aspired levels of the associated goals, and \( d^-_k, d^+_k \) are the under- and over- deviational variables, respectively of the \( k \)-th goal. Also the \( n \)-th priority factor \( (P_n) \) is assigned to the set of commensurable goals that are grouped together in the problem formulation and the priority factors have the relationship \( P_1 >>> P_2 >>> ... >>> P_n >>> ... >>> P_N \).

Where ‘>>>’ implies much greater than i.e., the membership goals at the highest priority level \( (P_1) \) are achieved to the extent possible before the set of membership goals at the next priority level \( (P_2) \) is considered, and so forth. \( P_n(d^-) \) is a linear function of the weighted under deviational variables at the \( n \)-th priority level, where \( P_n(d^-) \) is of the form:

\[
P_n(d^-) = \sum_{k=1}^{K} w^-_{nk} d^-_{nk}, w^-_{nk}, d^-_{nk} \geq 0, k = 1, 2, ..., K; N \leq K, \text{ where } d^-_{nk} \text{ is renamed for the actual deviational variable } d^-_k \text{ to represent it at the } n \text{-th priority level, } w^-_{nk} \text{ is the numerical weight associated with } d^-_{nk} \text{ and represents the weight of importance of achieving the aspired level of the } k \text{-th goal relative to the others which are grouped together at the } n \text{-th priority level. The values of } w^-_{nk} (k=1, 2, ..., K) \text{ are determined as:}
\]

\[
w^-_{nk} = \begin{cases} 
1 & \text{for the defined } \mu_k \text{ in (1.24)} \\
\frac{1}{(g_k^u - g_k)^n} & \text{for the defined } \mu_k \text{ in (1.25)} 
\end{cases}
\]
Where \((g_k^u - g_k)_n\), \((g_k - g_k^l)_n\) are used to represent \(g_k^u - g_k\), \(g_k - g_k^l\), respectively, at the \(n\)-th priority level.

1.3.3 Decision Making in Probabilistic Fuzzy environment

In real world decision making problems which closely describe and represent the actual decision making situation, the uncertainty inherent in real world complex system or human beings’ perception, such as the randomness of events related to the systems or the fuzziness of human judgments, should be reflected in the formulation of decision making problems. For dealing with the uncertainty in decision making, SP and fuzzy programming have been individually developed together with the introduction of various optimization models and the corresponding solution techniques.

However, recalling the vagueness or fuzziness inherent in human judgments for the decision making problems involving uncertainty, it is significant to realize that the uncertainty in real world decision making problems is often expressed by a fusion of fuzziness and randomness rather than either fuzziness or randomness.

Recent development in computational resources and scientific computing techniques, sophisticated optimization models [412] can now be solved efficiently. Yet many optimization applications are affected by uncertainty in input data or in model relationships. Zadeh’s incompatibility principle [100] also points to the importance of pondering uncertainty quantification in complex systems. Uncertainty can be described in several ways, depending on the information at hand. Among mathematical tools for coping with uncertainty, worst case scenario analysis, evidence theory, probability theory and fuzzy set theory, etc are played as important tools. Accessible accounts of these tools may be found in Sakawa [321], Shafer [334], Shiryaer [342], and in references therein. While research has progressed at a steady pace in the fields of stochastic optimization [171, 172, 331, 380, 385] and fuzzy mathematical programming [29, 209, 238, 415] the past decade, in particular, has witnessed a developing interest in situations where fuzziness and randomness are under one “roof” in an optimization framework [224, 239, 241, 263, 381, 382]. This interest has been motivated by the need for basing many human decisions on information which is both fuzzily imprecise and probabilistically uncertain [179, 423]. The multidisciplinary research field [240] that emerged as a result of this interest lies
at the boundary of stochastic optimization and fuzzy mathematical programming. Luhandjula [239, 240], describes the spectrum of research activities and the richness of ideas in development of theory, algorithms and applications in fuzzy stochastic optimization.

From the view point of the efficiency in capturing the nature of the different real life problems, the methodological development and upgradation of different optimization tools, probabilistic fuzzy decision making has become an emerging area of research. To describe the methodologies the concept of fuzzy random variable (FRV), fuzzy probability are described in the following sub-section.

1.3.3.1 Fuzzy random variable an overview

Fuzzy random variables (FRVs) were introduced to model and analyze ‘imprecisely valued’ measurable functions associated with the sample space of a random experiment, when the imprecision in values of these functions is formalized in terms of fuzzy sets. Different approaches to this concept have been developed in the literature; the most widely considered being that introduced by Kwakernaak [207, 208] and Kruse and Meyer [200], and the one by Puri and Ralescu [294] and Klement et al. [190]. In the literature combining Fuzzy Logic and Probability Theory we can find several notions and models, like fuzzy information systems [275], fuzzy probabilities [270, 298, 406] probabilistic sets [141, 142], and so on. FRVs represent a well-formalized concept underlying many recent probabilistic and statistical studies involving data obtained from a random experiment when these data are assumed to be fuzzily defined. FRVs have been considered in the setting of a random experiment to model, either a fuzzy perception or observation of a mechanism (the so-called ‘original random variable’) associating a real value with each experimental outcome, or an essentially fuzzy-valued mechanism, that is, a mechanism associating a fuzzy value with each experimental outcome.

For the first situation, Kwakernaak [207, 208] introduced a mathematical model which has been later formalized in a clearway by Kruse and Meyer [200]. In Kwakernaak or Kruse and Meyer’s approach, a FRV is viewed as a fuzzy perception of a classical real-valued random variable. Probabilistic and statistical studies for FRVs in Kwakernaak or Kruse and Meyer’s approach usually concern either ‘crisp’
parameters of the ‘original’ random variable or fuzzy-valued parameters defined on the basis of Zadeh’s extension principle.

For the last two decades many studies dealing with FRVs have been carried out. Most of them concern probabilistic aspects and results like, for instance, integration and differentiation of FRVs in probabilistic settings [119, 125, 199, 293, 308, 309, 310] fuzzy martingales, sub- and super-martingales [110, 217, 218, 295, 361, 373] etc. Several authors have examined Weak and Strong Laws of Large Numbers for FRVs to support the suitability of the fuzzy expected value concept by Puri and Ralescu as the limit of sample means of independent and identically distributed FRVs [161, 187, 189].

**Definition of FRV:**

**Borel Set:** we know that σ-algebra (σ-field) is considered as the set of all possible potentially interesting events in a probability space. This notion follows from the observation that, given any event, its complement is also an event, and given any two events, their union and intersect are also events. Formally, if $\mathcal{F}$ is a collection of subsets of the sample space $\Omega$, then $\mathcal{F}$ is said to be a $\sigma$-algebra if the following conditions hold: $\Omega \in \mathcal{F}$; if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$; and if $A = \bigcup_{i=1}^{\infty} A_i$ and $A_i \in \mathcal{F}$ for $i \in \mathbb{N}$, then $A \in \mathcal{F}$. The Borel $\sigma$-algebra, $\mathcal{B}$, is the smallest $\sigma$-algebra that contains the set of all open intervals in $\mathbb{R}$, the set of real numbers. Elements of $\mathcal{B}$ are called Borel sets, and $(\mathbb{R}; \mathcal{B})$ is called a Borel measurable ($\mathcal{B}$-measurable) space.

**FRV:**

A FRV on a probability space $(\Omega, \Phi, P)$ is a fuzzy valued function $X: \Omega \to \Phi(\mathbb{R})$, $\omega \to X_\omega$ such that for every Borel set $\mathcal{B}$ of $\mathbb{R}$ and for every $\alpha \in (0, 1)$, $(X[\alpha])^{-1}(\mathcal{B}) \in \Phi$. Here $\Phi(\mathbb{R})$ and $X[\alpha]$ denote respectively for the set of fuzzy numbers and the set valued function $X[\alpha]: \Omega \to 2^{\mathbb{R}}$, $\omega \to X_\omega[\alpha] = \{x \in \mathbb{R} | X_\omega(x) \geq \alpha\}$. By decomposition theorem of fuzzy numbers it is stated that if $\tilde{X}$ is a FRV then it can be represented as $\tilde{X} = \bigcup_{\alpha \in (0, 1]} \alpha X[\alpha]$

### 1.3.3.2 Uncertain probabilities [242]

In probability distribution, if one or more parameters are not known with precision and are modeled by using fuzzy numbers are known as uncertain probability. Using fuzzy arithmetic, basic laws of uncertain probabilities can be developed [49, 405].
X be a continuous random variable with probability density function $f(x, \nu)$, where $\nu$ is a parameter describing the density function. If $\nu$ is considered as a fuzzy number $\tilde{\nu}$, then $X$ becomes a fuzzily described random variable with density $f(x, \tilde{\nu})$, and the event $P(c \leq X < d)$ becomes a fuzzy set whose $\alpha$-cut is defined as

$$[P(c \leq X < d)] [\alpha] = \left\{ \int_c^d f(x, \nu) dx | \nu \in \nu[\alpha]; \int_{-\infty}^{\infty} f(x, \nu) dx = 1 \right\}, \text{ for all } \alpha \in (0, 1].$$

The first two moments are also defined by their $\alpha$-cuts as for all $\alpha \in (0, 1]$

$$E(X)[\alpha] = \left\{ \int_{-\infty}^{\infty} x f(x, \nu) dx | \nu \in \nu[\alpha]; \int_{-\infty}^{\infty} f(x, \nu) dx = 1 \right\},$$

$$E[X - E(X)]^2[\alpha] = \left\{ \int_{-\infty}^{\infty} (x - E(X))^2 f(x, \nu) dx | \nu \in \nu[\alpha]; E(X) \in E(X)[\alpha]; \int_{-\infty}^{\infty} f(x, \nu) dx = 1 \right\}.$$

Also the FRVs $X_i, (i = 1, 2, \ldots, n)$ having joint density function $f_i(x_1, x_2, \ldots, x_n; \tilde{\nu})$ and marginal density function $f_i(x_i; \tilde{\nu})$ are said to be independent if $f_i(x_1, x_2, \ldots, x_n; \tilde{\nu}) = \prod_{i=1}^{n} f_i(x_i; \nu) \text{ for } \alpha \in (0, 1] \text{ and for all } \nu \in \nu[\alpha].$

### 1.3.3.3 Fuzzy Multiobjective Stochastic Programming (FMOSP)

In most practical situations, however, it is natural to consider that the uncertainty in real world decision making problems is often expressed by a fusion of fuzziness and randomness rather than either fuzziness or randomness. For handling not only DM’s vague judgments in multiobjective problems but also the randomness of the parameters involved in the objective or constraints, a fuzzy programming approach to MOSP problems was first taken by Hulsuekar et al. [151] by implicitly assuming that the fuzzy decision or the minimum operator in the proper representation of the DM’s fuzzy preferences. Realizing the drawbacks of the model of Hulsuekar et al. [151], Sakawa et al. [323, 324, 325] developed interactive fuzzy satisficing method for fuzzy multiobjective SP problems.

A brief survey of major fuzzy SP models including fuzzy random programming was found in the paper of Luhandjula [240]. Nevertheless, in some significant real world problems, one has to base decisions on information which is both fuzzily imprecise and probabilistically uncertain [87, 88, 170, 240, 350, 351]. In 2006 Luhandjula [240] surveyed the area of fuzzy multiobjective stochastic linear programming.
On the basis of possibility measure, possibilistic programming approaches to fuzzy random linear programming problems were introduced by Katagiri and Ishii [175], Katagiri and Sakawa [176, 178]. Also different multiobjective fuzzy random linear programming problems through possibility measure further developed by many researchers [177, 179, 180]. Recently, from a viewpoint of DM’s ambiguous understanding of means and variance of random variables, a concept of random fuzzy variables were proposed by Liu [225] in 2002. But the efficient methodology for modeling and solving FMOSP problems involving FRVs, where the characteristics of the random variables are fuzzily described is yet to appear in the literature.

1.4 Some special issues of fuzzy stochastic Programming

During the last few years, the field of fuzzy SP has enriched a lot due to the pioneer contribution of the active researchers in the field. But the field is relatively young from the point of view of its extensive methodological study and applications to the real world problems. Hierarchical decision making is one of the current areas of study in the field of fuzzy stochastic optimization. In this context, fuzzy stochastic bilevel programming and fuzzy stochastic multilevel programming is emerging as a special area of study. But deep study for methodological development has yet to be widely circulated in the literature.

The different aspects of fuzzy SP formulation for different decision problems have been presented in detail in a subsequent chapter.

1.5 Outline of the work

Chapter 2 develops a FGP methodology for solving CCP problem involving fuzzy numbers and FRVs following normal distribution. In the model formulation process, the problem is converted into an equivalent FP problem by applying CCP technique in fuzzy environment. Then the problem is divided into equivalent sub problems by considering the tolerance limits of the fuzzy numbers relating to the system constraints having different forms of membership functions in the decision making context. Afterwards the objective of the problem and the system constraints relating to chance constraints are converted into fuzzy goals by assigning some
imprecise aspiration levels. In the decision process, a FGP methodology is introduced to find the most satisfactory solution in the decision making environment.

Chapter 3 presents two different FGP approach for modeling and solving multiobjective decision making problem having FRVs following normal distribution and fuzzy numbers associated with the system constraints. In the model formulation process, the problem is converted into an equivalent FP problem by using CCP technique in fuzzily described environment.

In first approach, the problem is decomposed on the basis of their tolerance ranges, considering fuzzy nature of the parameters. Afterwards the individual optimal solution of each objective is found to construct the membership goals of the objectives. A priority based FGP method is used for achievement of the highest membership degree to the extent possible under different priority structures. Finally the concept of Distance Function is used to identify the ideal-point dependent solution.

For the second approach, the problem is decomposed into two equivalent sub problems by considering the tolerance limits of the fuzzy numbers relating to the system constraints. The individual optimal solution of each objective of the decomposed problems is found to construct the membership goal of the objectives. Also the constraints associated with fuzzy numbers are converted into fuzzy goals by assigning some imprecise aspiration level. Then the best and worst values of each objective are calculated from the individual solution set and construct the membership goals of the objectives to find the compromise solution. Finally, the two-phase FGP solution technique is used to achieve the most satisfactory solution in the decision making environment.

Chapter 4 describes a methodology for solving fuzzy multiobjective CCP (FMOCCP) problem in which the right sided parameters associated with the system constraints are uniformly distributed FRV using FGP technique. In the proposed approach first the fuzzy CCP problem is converted into FP model by using the concept of \( \alpha \)-cuts and applying first decomposition theorem on fuzzy sets. After that, considering fuzzy nature of parameters associated with system constraints, the problem is decomposed on the basis of tolerance ranges of the parameters. Then the membership function of each of the individual objectives is defined in isolation to measure the degree of achievements of the goal levels of the objectives. Afterwards a
FGP model is developed to achieve the highest degree of each of the defined membership goals to the extent possible by minimizing the under deviational variables.

Chapter 5 is also represent a similar FGP technique like previous chapter 4 to solve multiobjective decision making problems in a probabilistic decision making environment having the right sided parameters associated with the system constraints are exponentially distributed FRVs.

Chapter 6 describes some mathematical techniques and modeling aspects for solving fuzzy multiobjective probabilistic decision making problems in which the constraints are jointly distributed and the right sided parameters are normally distributed FRVs. The probabilistic model is first converted into equivalent FP model by using incomplete gamma function described in a fuzzy decision making environment. Then independent optimal solution of each objective are determined under the decomposed set of system constraints which are obtained by considering fuzzy nature of parameters involved with them. The tolerance membership function for measuring the degree of satisfaction of the decision maker with the achievement of objective values is defined. The membership functions are then converted into fuzzy goals by assigning unity as aspiration level. Finally a weighted FGP technique is used to achieve the highest degree of each of the defined membership goal to the extent possible by minimizing under deviational variables and thereby obtaining most satisfactory solution in the decision making context which leads to an efficient as well as optimal compromise solution.

Chapter 7 contains a FGP method for modeling and solving multiobjective stochastic decision making problem involving FRVs associated with the parameters of the objectives as well as system constraints. In the proposed approach, an expectation model is generated on the basis of the mean of the FRVs involved with the objectives of the problem. Then the problem is converted into an equivalent FP model by considering the fuzzily defined chance constraints. Afterwards, the model is decomposed on the basis of the tolerance ranges of the fuzzy numbers associated with the fuzzy parameters of the problem. Now to construct the membership goals of the decomposed objectives under the extended feasible region defined by the decomposed system constraints, the individual optimal values of each objective is calculated in isolation. Then the membership functions are constructed to measure the degree of
satisfaction of each decomposed objectives in the decision making environment. Then a FGP model is developed to obtaining the optimal solution in the decision making environment.

Chapter 8 presents a FP method for modeling and solving bilevel stochastic decision making problems involving FRVs associated with the parameters of the objectives at different hierarchical decision making units as well as system constraints. In model formulation process an expectation model is generated as described in Chapter 7. The problem is then converted into a FP model as described earlier. After that the model is decomposed on the basis of tolerance ranges of fuzzy numbers associated with the parameters of the problem. To construct the fuzzy goals of the decomposed objectives of both decision making level under the extended feasible region defined by the decomposed system constraints, the individual optimal values of each objective at each level are calculated in isolation. Then the membership functions are formulated to measure the degree of satisfaction of each decomposed objectives in both the levels.

In the solution process, the membership functions are converted into membership goals by assigning unity as the aspiration level to each of them. Finally, a FGP model is developed to achieving the highest membership degree to the extent possible by minimizing the under deviational variables of the membership goals of the objectives of the decision makers in a hierarchical decision making environment.

Chapter 9 discussed a methodology for solving a multiobjective fuzzy random unbalanced transportation problem using weighted FGP technique. In the model formulation process the cost coefficients of the objectives are considered as fuzzy numbers and the supplies and demands associated with the system constraints are considered as FRVs with known fuzzy probability distribution. A FP model is first constructed by applying CCP methodology. Then, the problem is decomposed on the basis of the tolerance ranges of the fuzzy numbers associated with the model. The individual optimal solution of each decomposed objectives is found in isolation to construct the membership goals of the objectives. Finally, priority based FGP technique is used to achieve the highest degree of each of the defined membership goals to the extent possible by minimizing the under deviational variables and thereby obtaining optimal allocation of products by using distance function in a cost minimizing decision making situation.
In Chapter 10, a FMOCCP model is used for modeling and solving land allocation problems efficiently with the help of FGP for optimizing production of seasonal crops and related expenditures from the viewpoint of proper utilization of total cultivating land and different farming resources which are fuzzily described in agricultural systems. Some parameters of the model are also described using fuzzily defined random variables in capturing the possibilistic as well as probabilistic nature of the data. The resources parameters associated with the probabilistic constraints are taken as normally distributed FRVs. The fuzzy probabilistic model is first converted into an equivalent FP model using chance constrained methodology in fuzzy environment with the help of $\alpha$-cuts. Then independent optimal solution of each objective is determined under the decomposed set of system constraints which are obtained by considering the fuzzy nature of parameters involved with them. Afterwards, the membership goals of each of the objectives are defined for finding compromise solution of all the objectives. Finally, FGP technique is used to achieve the highest degree of each of the defined membership goals to the extent possible by minimizing the under deviational variables. The potential use of this methodology is illustrated by a case example of the Nadia District, west Bengal, India, in the context of land allocation problems.