ABSTRACT

The study of nonassociative rings has yielded many interesting results in algebra. Many sufficient conditions are well known under which a given nonassociative ring becomes associative. The results on nonassociative rings in which one does assume associators in the nucleus have been scattered throughout the literature.


Throughout this work a ring $R$ is a synonym for a nonassociative ring, that is one in which the associative law of multiplication is not necessarily true and which does not have a unit element.

In this work we present some properties of nonassociative rings, in which associators are in the nucleus, namely, assosymmetric rings,
accessible rings, strongly $(-1,1)$ rings and Lie admissible rings. We prove that prime assosymmetric ring is third power associative and semiprime accessible ring is associative. We derive the nilpotency of the associator in a strongly $(-1,1)$ ring. Also it is shown that if $R$ is a nonassociative ring with associators in the left nucleus or middle nucleus or right nucleus, then $R$ is either associative or nucleus equals center.

The associator $(x, y, z)$ is defined by $(x, y, z) = (xy)z - x(yz)$, for all $x, y, z$ of a ring, plays a central role in the study of nonassociative rings. It is a measure of the nonassociativity of a ring. Obviously if it is zero for all $x, y, z$ in the ring, then the ring is associative. The commutator $(x, y)$ is defined as $(x, y) = xy - yx$ for all $x, y$ in a ring. We define the left nucleus $N_l = \{ n \in R / (n, R, R) = 0 \}$, right nucleus $N_r = \{ n \in R / (R, R, n) = 0 \}$ and the middle nucleus $N_m = \{ n \in R / (R, n, R) = 0 \}$. The nucleus $N$ of a ring $R$ is defined as $N = \{ n \in R / (n, R, R) = (R, n, R) = (R, R, n) = 0 \}$ i.e. $N = N_l \cap N_r \cap N_m$ and center $C$ of $R$ is defined as $C = \{ c \in N / (c, R) = 0 \}$. We define a ring $R$ to be of characteristic $\neq n$ if $nx = 0$ implies $x = 0$, for all $x$ in $R$.

An assosymmetric ring $R$ is one in which the associative law of multiplication has been weakened to the condition that $(P(x), P(y), P(z)) = (x, y, z)$ for every permutation $P$ of $x, y, z$. An accessible ring is a nonassociative ring in which
\((x, y, z) + (z, x, y) - (x, z, y) = 0\) and \(( (w, x), y, z) = 0\) for every \(w, x, y, z\) in \(R\). A ring \(R\) is said to be right alternative if \((x, y, y) = 0\) and left alternative if \((y, y, x) = 0\) for every \(x, y\) in \(R\). It is alternative if it is both right alternative and left alternative. A strongly \((-1, 1)\) ring is a nonassociative ring in which \((x, y, z) + (x, z, y) = 0\) and \(( (x, y), z) = 0\) for every \(x, y, z\) in \(R\).

The first chapter is devoted to present the necessary background. We give a brief survey of the work done by A.H.Boers, A.A.Albert, E.Kleinfeld, M.Kleinfeld, A.Thedy, C.T.Yen, I.R.Hentzel and D.P.Jacobs.

In chapter 2, we present some special properties of associators in the nonassociative rings. In section 2.1, we see that every commutator and associator are in the nucleus \(N\) of an assosymmetric ring \(R\) and commutators are in the center \(C\). Using this it is shown that a prime assosymmetric ring is third power associative. At the end of this section we present some examples of assosymmetric rings which are not power associative. In section 2.2, we prove that in a semiprime accessible ring \(R\) of characteristic \(\neq 2, 3\) the associator is in the nucleus. Using this it is shown that \(R\) is associative. In section 2.3, we show that the associator is in the nucleus of a strongly \((-1, 1)\) ring. Using this we prove the nilpotency of the associator in a strongly \((-1, 1)\) ring.
In chapter 3, we present some results related to nonassociative rings by assuming \((x,R,x)\) in the nucleus. In section 3.1, we consider a ring \(R\) with \((x,y,x)\) and commutators in the left nucleus. Using this it is shown that \(R\) must be a subdirect sum of a semiprime associative ring and a semiprime commutative ring. In section 3.2, we consider a ring with \((x,y,x)\) and \((N_+,R)\) in the right nucleus. By assuming \(N_i\) and \(N_r\) as the Lie ideals of \(R\), we show that if \(R\) is a prime ring with \((x,y,x)\in N_i\), then \(N_i = N_r\). Further we show that if \(R\) is a prime ring with \(N_i \neq 0\) then \(R\) is either associative or commutative. In section 3.3, we consider a ring \(R\) with \((x,y,x)\), commutator and the square of every element of the ring is in the nucleus. Using this it is shown that a prime ring \(R\) of characteristic \(\neq 2\) is either associative or a Lie ring.

In chapter 4, we present certain properties of nonassociative rings with associators in the middle nucleus and left nucleus. In section 4.1, we consider nonassociative ring \(R\) with associators in the middle nucleus. Using this it is shown that, if \(R\) is a semiprime right alternative ring of characteristic \(\neq 2\) satisfies \(R, R, R + (R, (R, R, R)) \subseteq N_+\), then \(R\) is associative. In section 4.2, we consider a simple ring \(R\) of characteristic \(\neq 2\) with associators in the left nucleus. Using this we prove that the associator is in the right nucleus and hence \(R\) is associative. In section 4.3, first we prove that if \(R\) is semiprime ring satisfies \((x,y,z)+(y,z,x)+(z,x,y)=0\),
then either $N = C$ or $R$ is associative. Also it is proved that if $R$ is a simple ring of characteristic $\neq 2,3$ which satisfies $(x,y,z)+(y,z,x)+(z,x,y)=0$ and $(y,x,x)=k(x,x,y)$, then either $N = C$ or $R$ is associative. Further it is shown that if $R$ is a prime ring which satisfies $(x,y,z)+(y,z,x)+(z,x,y)=0$ and $(x,y,z)=(z,y,x)$, then either $N = C$ or $R$ is associative.

Chapter 5 is devoted to discuss further possible developments of some results which we wish to study in future. In section 5.1, we prove that if $R$ is a simple ring with associators in the right nucleus, then $R$ is associative. We also extend this result to prime and semiprime rings as well. We wish to study some more properties of nonassociative rings with $(R,R,R)+(R,(R,R,R))$ in the right nucleus. In section 5.2, we discuss some results on the rings with the generalized commutators in the nuclei. Here we assume $R$ to satisfy weakly Novikov identity $(w,x,yz)=y(w,x,z)$, for all $w,x,y,z$ in $R$. We prove that if $R$ is a prime weakly Novikov ring such that commutator is contained in two of the three nuclei, then $R$ is associative or commutative. Also we prove that if $R$ is a semi prime weakly $M$-ring i.e., $(w,xy,z)=x(w,y,z)$, for all $w,x,y,z$ in $R$ with $(R,R,R)\subseteq N_a$, then $R$ is associative. We wish to try some more properties of the weakly $M$-ring in which the associator is in the left and right nucleus.
Albert is an interactive computer system for building nonassociative algebras. I.R.Hentzel, D.P.Jacobs and S.V.Muddana have proved that any semiprime ring $R$ satisfying $(a,b,c) = (a,c,b)$ and $(a,(b,c),d) = 0$ is associative, using Albert. Also I.R. Henzel and L.A.Peresi have used Albert to determine a basis of elements of degree 5 and degree 6 in the nucleus of the free alternative algebra. Using Albert we wish to study the properties of nonassociative rings with associators in the nucleus.

These are few ideas which arise in course of our study. We do hope several others will emerge as we proceed.

Some of the results of chapter 5 are communicated for publication in "European Journal of Pure and Applied Mathematics".