This chapter is devoted to a discussion on further possible developments of some results, which we plan to study in our future work.

In section 5.1, we prove that if a ring $R$ is simple or prime or semiprime of char. $\neq 2$ with the associators in the right nucleus, then $R$ is associative. In section 5.2, we discuss some properties of the nonassociative rings satisfying weakly Novikov identity with the generalized commutators in the nuclei. Also we discuss some properties of weakly $M$-ring and we prove that if $R$ is a semiprime weakly $M$-ring satisfying $(R,R,R) \subseteq N_m$, then $R$ is associative.

Albert is an interactive computer system for building nonassociative algebras. Using Albert in section 5.3, we briefly discuss rings which satisfy a certain identity, assuming that these rings satisfy another given set of identities.
5.1: Rings with associators in the right nucleus:

Kleinfeld [18] proved that if $R$ is a semiprime ring of char. $\neq 2$ satisfying $(R,R,R) \subseteq N_l \cap N_m \cap N_r$, then $R$ is associative. Yen [39] generalized Kleinfeld's result under the weaker hypothesis $(R,R,R)$ contained in two of the three nuclei. In [26] E. Kleinfeld and M. Kleinfeld have shown that, assuming the $(R,R,R)$ in the left nucleus, if $R$ is a simple ring with identity 1 and char. $\neq 2$, then $R$ must be associative. Without using the identity 1, Yen [40] proved that a simple ring of char. $\neq 2$ with $(R,R,R)$ in the left nucleus is associative. In this section using the above results we prove that if a ring $R$ is simple or prime or semiprime of char. $\neq 2$, then $R$ is associative.

Throughout this section $R$ represents a nonassociative ring with the associator $(R,R,R)$ in the right nucleus $N_r$.

i.e., $(R,R,R) \subseteq N_r$.

We use the Teichmuller identity which is valid in any arbitrary ring.

$$(wx,y,z) - (w,xy, z) + (w,x,yz) = w(x,y,z) + (w,x,y)z,$$

for all $w, x, y, z \in R$.

As a consequence of 5.1.2, we know that $N_b, N_m$ and $N_r$ are associative subrings of $R$.

Let $n \in N_r$, then from 5.1.2, we obtain

$$(x,y,zn) = (x,y,z)n,$$

for all $x, y, z \in R$ and $n \in N_r$. 

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Let $I$ be the associator ideal of a ring $R$. Then from 5.1.2 $I$ can be characterized as all finite sums of associators and right (or left) multiples of associators. Hence, we obtain

$$I = (R,R,R) + (R,R,R)R,$$


...5.1.4

First we prove the following lemma.

**Lemma 5.1.1**: Let $T = \{ t \in R : tR = 0 \}$. Then $T$ is an ideal of $R$.

**Proof**: Let $t_1, t_2 \in T$. Then for every $x \in R$, we have $t_1x = 0$, $t_2x = 0$, $xt_1 = 0$ and $xt_2 = 0$.

Also $x(t_1 - t_2) = xt_1 - xt_2 = 0$,

$$(t_1 - t_2)x = t_1x - t_2x = 0.$$

Now for $r \in R$, $t \in T$, both $rT$ and $Tr$ are also in $T$.

Hence $(rt)x = 0$ and $x(tr) = 0$.

i.e., $rt \subseteq T$ and $tr \subseteq T$.

Hence $T$ is an ideal of $R$.

**Theorem 5.1.1**: Let $R$ be a simple ring and satisfies $(R,R,R) \subseteq N_R$. Then $R$ is associative.

**Proof**: We assume that $R$ is not associative. Then from 5.1.3 and 5.1.4, we obtain

$$R = R^2 = RI = R(R,R,R)$$

$$= R^2 = IR = (R,R,R)R.$$  

...5.1.5

Using 5.1.2 and 5.1.1, we obtain
\[ w(x,y,z) + (w,x,y)z \in N_r, \]  
\[ \text{for all } w, x, y, z \in R. \]

Then with \( y \in (R,R,R) \) in 5.1.6 and applying 5.1.1, we obtain

\[ w(x,y,z) \in N_r, \]

i.e., \( R(R,(R,R,R),R) \subseteq N_r. \)

Now using this, 5.1.1 and 5.1.3, we obtain

\[ 0 = (R,R,R(R,(R,R,R),R)) \]
\[ = (R,R,R) \cdot (R,(R,R,R),R). \]

Also \( R(R,R,R) \cdot (R,(R,R,R),R) = R \cdot (R,R,R) \cdot (R,(R,R,R),R) = 0 \) using 5.1.7.

Using 5.1.5, we have \( R(R,(R,R,R),R) = 0. \)

From 5.1.8 and 5.1.5, we have

\[ 0 = R(R,(R,R,R),R) \]
\[ = (R,R,R)R \cdot (R,(R,R,R),R) \]
\[ = (R,I,R)R \cdot (R,(R,R,R),R), \text{ since } R \text{ is simple} \]
\[ = (R,(R,R,R),R)R \cdot (R,(R,R,R),R) \]
\[ = (R,(R,R,R),R)R \cdot (I,R) \]
\[ = (R,(R,R,R),R) \cdot R \]
\[ = (R,(R,R,R),R) \cdot R. \]

From 5.1.8 and 5.1.9, we have \( (R,(R,R,R),R) \subseteq T. \) Since \( T \) is an ideal of \( R \) and \( R \) is simple, we have either \( T = R \) or \( T = 0. \) But \( T = R \) implies \( RR = 0, \) a
contradiction. Thus $T = 0$. Hence $(R, (R, R, R), R) = 0$. Hence the associator $(R, R, R)$ is now in the middle and the right nucleus.

Now we show that the associator $(R, R, R)$ is also in the left nucleus.

For, we replace $n$ by $((a, b, c), d, e)$ in 5.1.3 where $a, b, c, d, e \in R$ and $((a, b, c), d, e) \in N$, from the hypothesis.

Thus, $(x, y, z((a, b, c), d, e)) = (x, y, z) ((a, b, c), d, e)$.

Now applying 5.1.2, we obtain

\[
(x, y, z((a, b, c), d, e)) + (x, y, z(a, b, c), d, e) = (x, y, z((a, b, c), d, e)) - (x, y, z(a, b, c), d, e) \quad \ldots 5.1.11
\]

i.e.,

\[
(x, y, z((a, b, c), d, e)) + (x, y, z(a, b, c), d, e) = 0
\]

or

\[
(x, y, z((a, b, c), d, e)) = -(x, y, z(a, b, c), d, e).
\]

Using the fact that associators are in the right and the middle nucleus, we obtain

\[
(x, y, z(a, b, c), d, e) = 0.
\]

Thus $(x, y, z((a, b, c), d, e)) = 0$. Hence we have,

\[
\]

Since $I = R$, we obtain $R((R, R, R), R, R) = 0$. \quad \ldots 5.1.13

From 5.1.13 and 5.1.5, we obtain

\[
0 = (R, R, R)R \cdot ((R, R, R), R, R)
= (I, R, R)R \cdot ((R, R, R), R, R) \quad \text{since } R \text{ is simple.}
= ((R, R, R), R, R)R \cdot (I, R, R)
\]

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Thus, from 5.1.13 and 5.1.14, \((R,R,R) \subseteq T\).

Since \(T\) is an ideal of \(R\) and \(R\) is simple we have either \(T = R\) or \(T = 0\). But \(T = R\) implies \(RR = 0\), which is again a contradiction. Thus \(T = 0\) and so \((R,R,R) = 0\). Hence we obtain the associators now in the left nucleus as well. Hence \((R,R) \subseteq N_1 \cap N_m \cap N_r\). Thus from [18] \(R\) must be associative. This completes the proof of the theorem. \(\square\)

**Corollary 5.1.1:** Let \(R\) be a prime ring and satisfying \((R,R,R) \subseteq N_r\). Then \(R\) is associative.

**Proof:** Let \(R\) be not associative. Then from 5.1.7, we have

\[0 = (R,R,R(R,(R,R,R),R))\]

\[= (R,R,R) (R, (R,R,R),R),\]

\[= I(R,(R,R,R),R).\]

From lemma 5.1.1 \((R,(R,R,R),R) \subseteq T\) and \(T\) is an ideal of \(R\). Hence we obtain \(IT = 0\). But since \(R\) is prime we have either \(I = 0\) or \(T = 0\). But \(I\) being an associator ideal is not equal to zero. Hence we have \(T = 0\) implying \((R, (R,R,R), R) = 0\). i.e., \((R,R,R)\) is in the middle and the right nucleus.

Now from 5.1.12 we have, \(0 = I((R,R,R),R,R)\). But \(((R,R,R),R,R) \subseteq T\) and hence \(IT = 0\). \(R\) being a prime ring we have either \(I = 0\) or \(T = 0\). But \(I\) is the associator ideal and hence is not equal to zero. Thus we have \(T = 0\) implying \(((R,R,R),R,R) = 0\). i.e., \((R,R,R)\) is in the left nucleus as well.
Therefore we have \((R, R, R) \subseteq N\). Now using [18] we obtain a contradiction. Hence \(R\) is associative.

\[\text{Theorem 5.1.2 : If } R \text{ satisfies the identity } (R, (R, R, R)) = 0 \text{ and if for all } a \in R, a^2 = 0 \text{ implies } a = 0, \text{ then } R \text{ must be associative.}\

\[\text{Proof : As in theorem 5.1.1 we obtain } (R, R, R) (R, (R, R, R), R) = 0. \text{ Thus } (v, (w, x, y), z)^2 = 0. \text{ Hence } (v, (w, x, y), z) = 0 \text{ shows that } (R, R, R) \text{ is in the middle nucleus of } R. \text{ This leads to } (R, R, R) \text{ ((R, R, R), R, R)} = 0. \text{ Then } ((x, y, z), v, w)^2 = 0. \text{ So } (R, R, R) \text{ is in } N, \text{ the nucleus of } R. \text{ Now we use [18] to see that } R \text{ is associative. This completes the proof of the theorem.}\

5.2. Weakly Novikov rings with associators in the nuclei:

Thedy [36] studied rings with commutators in the nuclei. Kleinfeld and Smith [23] studied a class of simple rings with commutators in the left nucleus. They in [24] also described flexible weakly Novikov rings. They proved that, the semiprime flexible weakly Novikov rings are associative. Yen [41] considered prime weakly Novikov ring and semiprime weakly M-ring. He proved that if \(T_k \subseteq N_l \cap N_r \text{ or } T_k \subseteq N_m \cap N_r \) where \(T_k = (((\ldots((R, R), R)\ldots), R), R)\) then the ring is associative or \(T_k = 0\). In this section we consider \(R\) to be a nonassociative ring satisfying weakly Novikov identity and \(T_k = (((\ldots((R, R), R)\ldots), R), R)\) where \(k\) is a positive
integer. We see that, if $R$ is a semiprime weakly Novikov ring satisfying $(R,R,R) \subseteq N_1$ or $(R,R,R) \subseteq N_2$, then $R$ is associative. Moreover, we prove that if $R$ is a semiprime weakly $M$-ring satisfying $(R,R,R) \subseteq N_m$, then $R$ is associative.

A ring $R$ is called simple if $R$ is the only nonzero ideal of $R$. Thus, $R^2 = R$. A ring $R$ is called semiprime if the only ideal of $R$ which squares to zero is the zero ideal. A ring $R$ is called prime if the product of any two nonzero ideals of $R$ is nonzero. If $S$ is a nonempty subset of a ring $R$, then the ideal of $R$ generated by $S$ is $<S>$.

Thought this section we consider rings with generalized commutators in the nuclei.

We know that a ring $R$ is weakly Novikov [24] if $R$ satisfy the following identity

$$(w,x,yz) = y(w,x,z), \quad \ldots 5.2.1$$

for all $w,x,y,z \in R$.

For any ring $R$, let $T_k = (((...(R,R),R),...,R),R),R)$, where $k$ is a positive integer.

From the above, we have $T_2 = (R,R)$ and $T_3 = ((R,R),R)$. Also $(R,T_k) = (T_k,R) \subseteq T_k$, where $k$ is a positive integer. Obviously we have the following identities.

$$T_k + T_k R = T_k + R T_k \quad \ldots 5.2.2$$
for all positive integers $k$.

In any arbitrary ring $R$, we have

$$S(x,y,z) = (x,y,z) + (y,z,x) + (z,x,y) = (xy,z) + (yz,x) + (xz,y)$$  \(\ldots5.2.3\)

for all $x,y,z$ in $R$.

We shall use the Teichmüller identity

$$(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z$$  \(\ldots5.2.4\)

for all $w,x,y,z$ in $R$.

As a consequence of 5.2.4, we have that $N_i$, $N_m$ and $N_r$ are associative subrings of $R$.

Suppose that $n \in N_i$. Then with $w = n$ in 5.2.4, we obtain

$$(nx,y,z) = n (x,y,z)$$  \(\ldots5.2.5\)

for all $x,y,z$ in $R$ and $n \in N_i$.

Suppose that $m \in N_r$. Then with $z = m$ in 5.2.4, we obtain

$$(w,x,ym) = (w,x,y)m$$  \(\ldots5.2.6\)

for all $w,x,y$ in $R$ and $m$ in $N_r$.

Suppose that $j \in N_i \cap N_m$. Then with $x = j$ in 5.2.4, we obtain

$$(wj,y,z) = (w, jy,z)$$  \(\ldots5.2.7\)

for all $w,y,z$ in $R$ and $j$ in $N_i \cap N_m$.

Let $I$ be the associator ideal of a ring $R$. As a consequence of 5.2.4 $I$ can be characterized as all finite sums of associators and left multiples of associators. In view of 5.2.1 it suffices to take all finite sums of associators if $R$ is a weakly Novikov ring. Hence, in this case $I = (R,R,R)$.

Now we prove the following lemma.
Lemma 5.2.1: If $R$ is a weakly Novikov ring, then $RN_r \subseteq N_r$ and $I \cdot N_r = (R,R,R) \cdot N_r = 0$.

Proof: Let $z \in N_r$ and $w,x,y \in R$.

Using 5.2.6 and 5.2.1, we have

$$(w,x,y)z = (w,x,yz)$$

$$= y(w,x,z)$$

$$= 0.$$ 

Thus, we get $I \cdot N_r = (R,R,R) \cdot N_r = 0$ and $RN_r \subseteq N_r$.

This completes the proof of the lemma.

For any ring $R$, let $V_k = T_k + RT_k$ for all positive integers $k$. In the sequel, for the convenience we denote $T_k$ and $V_k$ by $T$ and $V$ respectively.

Lemma 5.2.2: If $R$ is a ring such that $T$ is contained in two of the three nuclei, then $V$ is an ideal of $R$.

Proof: From 5.2.2, we have $V = T + TR = T + RT$.

Since $T$ is contained in two of the three nuclei, we have $TR \subseteq V$ and $RT \subseteq V$.

If $T$ is in the left nucleus, then $T + TR$ is a right ideal, i.e. $VR = TR + T \cdot R^2 \subseteq V$.

If $T$ is in the right nucleus, then $T + RT$ is a left ideal, i.e. $RV = RT + R^2 \cdot T \subseteq V$.

If $T$ is in the middle nucleus, then $V + RV = V + VR$ because of $RV = R(T + TR) = RT + RT \cdot R \subseteq V + VR$ and $VR = (T + RT)R = TR + R \cdot TR \subseteq V + RV$.

If $T$ is in both the left and right nuclei, then $V$ is a right and left ideal. So $V$ is an ideal. If $T$ is in the middle nucleus, then $V + VR = V + RV$. So if $V$ is
either a right ideal or a left ideal, then $V$ is an ideal. Since $T$ is either in the left or right nucleus, $V$ is a right or a left ideal, and in fact $V$ is an ideal. 

**Theorem 5.2.1:** If $R$ is a prime weakly Novikov ring such that $T \subseteq N_l \cap N_r$ or $T \subseteq N_m \cap N_r$, then $R$ associative or $T = 0$.

**Proof:** Using $T \subseteq N_l$ and lemma 5.2.1, we get

$$I\cdot V = I\cdot (T + RT) = 0.$$  \[\ldots 5.2.8\]

By lemma 5.2.2 and the primeness of $R$, 5.2.8 implies $I = 0$ or $V = 0$. Thus, $R$ is associative or $T = 0$. This completes the proof of the theorem.

**Lemma 5.2.3:** If $R$ is a weakly Novikov ring, such that $T \subseteq N_l \cap N_m$, then

$$(R, R, T) \subseteq 0.$$  \[\ldots 5.2.9\]

**Proof:** We have $(R, T) = (T, R) \subseteq T$. Using this, hypotheses, 5.2.4, 5.2.1, 5.2.7 and 5.2.5, for all $y \in T$, and $w, x, z \in R$ we have

$$(w, x, y)z = w(x, y, z) + (w, x, y)z$$

$$= (wx, y, z) - (wx, y, z) + (w, x, y)z$$

$$= - (w, (x, y), z) - (w, yx, z) + y(w, x, z)$$

$$= - (wy, x, z) + y(w, x, z)$$

$$= - ((w, y), x, z) - (yw, x, z) + y(w, x, z)$$

$$= 0.$$

Hence, we get $(R, R, T) \subseteq 0$.

This completes the proof of the lemma.
Theorem 5.2.2: Let $R$ be a prime weakly Novikov ring such that $T \subseteq N_1 \cap N_m$. If $S(x,y,z) \in N_m$ for all $x,y,z$ in $R$, or $(T,(R,R,R)) = 0$, then $R$ is associative or $T = 0$.

Proof: Assume that $S(x,y,z) \in N_m$ for all $x,y,z$ in $R$. Using this, 5.2.3 and the hypotheses, for all $x \in T$ and $y, z \in R$ we get

$$(y,z,x) = (x,y,z) + (y,z,x) + (z,x,y)$$

$$= S(x,y,z) \in N_m.$$  

Thus $(R,R,T) \subseteq N_m$. Applying this, 5.2.1 and 5.2.9, we have

$$(R,R,RT)R = R(R,R,T) \cdot R$$

$$= R \cdot (R,R,T) \cdot R$$

$$= 0.$$  

Combining the above equation with 5.2.9 we have

$$(R,R,V)R = 0. \quad \ldots \text{5.2.10}$$

Assume that $(T, (R,R,R)) = 0$. Using this, 5.2.1, 5.2.9 and 5.2.4, and noting that $(T,R) \subseteq T$, for all $w, x, y, t \in R$, and $z \in T$ we have

$$(w,x,y)z \cdot t = z(w,x,y) \cdot t$$

$$= (w,x,zy)t$$

$$= (w,x,(z,y))t + (w,x,yz)t$$

$$= w(x,y,z) \cdot t + (w,x,y)z \cdot t + (w,xy,z) t - (wx,y,z)t$$

$$= w(x,y,z) \cdot t + (w,x,y)z \cdot t$$

and $(x,y,wz)t = w(x,y,z)t = 0$. Combining this with 5.2.9, we also obtain 5.2.10.
Using 5.2.1 and 5.2.10, we see that \(<(R,R,T)> = (R,R,V)\). By the semiprimeness of \(R\), 5.2.10, implies \((R,R,V) = 0\). By Theorem 5.2.1, \(R\) is associative or \(T = 0\). This completes the proof of the theorem.

**Theorem 5.2.3**: If \(R\) is a prime weakly Novikov ring such that \((R,R)\) is contained in two of the three nuclei, then \(R\) is associative or commutative. In the latter case, \(N_r = 0\) or \(R\) is associative.

**Proof**: In view of Theorem 5.2.1, we may assume that \((R,R) \subseteq N_l \cap N_m\).

Let \(B = (B,R) + R (R,R)\). By lemma 5.2.2, \(B\) is an ideal of \(R\). Using lemma 5.2.3, we get \((R,R,(R,R))R = 0\).

Applying 5.2.3 and \((R,R) \subseteq N_l \cap N_m\), for all \(x,y,z\) in \(R\) we have \(S(x,y,z) = (x,y,z) + (y,z,x) + (z,x,y) \in N_l \cap N_m\). Let \(x \in (R,R)\). Then we get \((y,z,x) \in N_l \cap N_m\). Thus we obtain \((R,R,(R,R)) \subseteq N_l \cap N_m\). Using this and 5.2.11, we have \(R(R,R,(R,R))R = R(R,R,(R,R))R = 0\). Hence, applying this, 5.2.1 and 5.2.11, and noting that \(B\) is an ideal of \(R\), we obtain that \((R,R,B)R = 0\) and \(<(R,R,(R,R))> = (R,R,B)\). Thus, by the semiprimeness of \(R\) we get \((R,R,B) = 0\) and so \((R,R) \subseteq N_r\). By Theorem 5.2.1, \(R\) is associative or commutative.

Assume that \(R\) is commutative. Thus we have \(N_r = R = R N_r \subseteq N_r\) and \(I \cdot N_r = 0\) by lemma 5.2.1. Hence \(N_r\) is an ideal of \(R\). By the primeness of \(R\), \(I \cdot N_r = 0\) implies \(I = 0\) or \(N_r = 0\). i.e., \(N_r = 0\) or \(R\) is associative. This completes the proof of the theorem.
Theorem 5.2.4: If $R$ is a semiprime weakly Novikov ring such that $(R,R,R) \subseteq N_l$ or $(R,R,R) \subseteq N_r$, then $R$ is associative.

Proof: We know that the associator ideal $I$ of $R$ is all finite sums of associators. Assume that $(R,R,R) \subseteq N_l$. Then by this and 5.2.5, for all $w \in (R,R,R)$ and $x,y,z \in R$ we get

$$w(x,y,z) = (wx,y,z) \in (I,R,R) = 0.$$  

Thus, we have $(R,R,R) = 0$. So $I^2 = 0$. Assume that $(R,R,R) \subseteq N_r$. Then by lemma 5.2.1, we obtain


In either case, we have $I^2 = 0$. By the semiprimeness of $R$, this implies $I = 0$. Thus, $R$ is associative. This completes the proof of the theorem.

We know that a ring $R$ is weakly $M$-ring [41] if $R$ satisfies the following identity.

$$(w,xy,z) = z(w,y,z) \quad \ldots 5.2.12$$

for all $w,x,y,z$ in $R$.

If $R$ is a weakly $M$-ring then by 5.2.4 and 5.2.12, we have $I = (R,R,R)$.

Now we prove the following theorem.

Theorem 5.2.5: If $R$ is a prime weakly $M$-ring such that $T \subseteq N_l \cap N_m$ or $T \subseteq N_m \cap N_r$, then $R$ associative or $T = 0$. 

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Proof: We have \((T,R) \subseteq T\). Using this, \(T \subseteq N_m\) and 5.2.12, for all \(x \in T\) and \(w,y,z,t \in R\) we have
\[
x(w,y,z) = x(w,y,z) - y(w,x,z)
\]
\[
= (w,xy,z) - (w,yx,z)
\]
\[
= (w,(x,y),z)
\]
\[
= 0,
\]
and \(tx(w,y,z) = t x(w,y,z) = 0\).
The above two identities yield
\[
V I = 0. \quad \ldots 5.2.13
\]
Since \(V\) is an ideal of \(R\), by primeness of \(R\), 5.2.13 implies either \(I = 0\) or \(V = 0\). Hence, \(R\) is associative or \(T = 0\). This completes the proof of the theorem.

\[\square\]

**Theorem 5.2.6:** If \(R\) is a semiprime weakly \(M\)-ring such that \((R,R,R) \subseteq N_m\), then \(R\) is associative.

**Proof:** We know that the associator ideal \(I\) of \(R\) is all finite sums of associators.

Assume that \((R,R,R) \subseteq N_m\).

Since \(R\) is weakly \(M\)-ring, we have
\[
x(w,y,z) = (w,xy,z) \quad \text{for all } w, x, y, z \in R.
\]
Then by this, for all \(x \in (R,R,R)\) and \(w, y, z \in R\) we get
\[
\]
Thus we have \((R,R,R) (R,R,R) = 0\). So \(I^2 = 0\).
By the semiprimeness of \( R \), this implies \( I = 0 \). Thus, \( R \) is associative. This completes the proof of the theorem.

We wish to try for some properties of the weakly \( M \)-ring in which the associator is in the left and right nucleus.

5.3. Rings with certain identities: Using Albert.

Computers were used to study nonassociative algebras as early as the 1960's. Work on the system began in 1989. Albert is interactive computer system for building nonassociative algebras. I.R. Hentzel and D.P. Jacobs [6] suggested certain techniques for using Albert that allow us to test hypothesis effectively. This process provides a fast way to achieve new results and interacts with traditional methods. Using Albert in this section, we briefly discuss rings which satisfy a certain identity, assuming that these rings satisfy another given set of identities.

For example, we assume an algebra satisfying the identities

\[(xy)z = x(yz)\] ...

and \( x^2 = 0 \). ...

for all \( x, y, z \).

Identity 5.3.1 is the associative law. From identity 5.3.2 we have

\[0 = (x + y)^2 = x^2 + y^2 + xy + yx = xy + yx, \text{ i.e., } xy = -yx. \]

Together 5.3.1 and 5.3.2 imply
(xy)z = -(xy) = -(xz)y = (xz)y = x(zy) = -x(yz) = -(xy)z. Thus 2(xy)z = 0, and, assuming a field of char. ≠ 2, we have (xy)z = 0. The product of any three elements is zero.

The above example is a special case of a theorem by Nagata and Higman which states if \( R \) is an associative algebra satisfying \( x^n = 0 \) for some \( n \), and having no elements of additive order \( n \), then the product of any \( 2^n - 1 \) elements in \( R \) is zero ([43], p.126).

The identity

\[ (xy)x - x(yx) = 0 \]

is known as the flexible law.

The associator is available in Albert, and so flexibility can be expressed as:

\[ \text{identity } (x, y, x) \]

Clearly flexibility is a generalization of associativity. It is also implied by the commutative law as well, since commutativity would imply \( (xy)x = x(xy) = x(yx) \). In general a flexible algebra need not be commutative. While experimenting with Albert, we observed that, for char. ≠ 2, if the algebra was generated by a single element, then flexibility and commutativity seemed to be equivalent. This led to the following theorem, proven in [7]. Also given in [7] is a counter example showing that char. ≠ 2 is necessary.
Theorem 5.3.1: Let $R$ be an algebra over a field of char. $\neq 2$. Then $R$ is commutative if and only if it is flexible and generated by a set $S$ whose elements pairwise commute.

One of Albert's most interesting discoveries concerns the identity

$$(xy)z = y(zx). \quad \ldots 5.3.4$$

This identity seems to have been first studied by Chen in 1970 [4]. Obviously, any binary operation that is both commutative and associative must satisfy identity 5.3.4. However, the converse is very nearly true. For the moment let us consider groupoids. For example, 2–nice is equivalent to being commutative. Being both 2–nice and 3–nice is equivalent to being commutative and associative. Experiments with Albert suggested that in the presence of identity 5.3.4, an algebra's multiplication is 5–nice. This led to the following theorem, proven in [10] that

Theorem 5.3.2: A groupoid satisfying $(xy)z = y(zx)$ is $k$–nice for each $k \geq 5$.

The above Theorem 5.3.2 can then be used to prove the following theorem, which generalizes a result of Chen.

Theorem 5.3.3: A semiprime algebra satisfying $(xy)z = y(zx)$ is commutative and associative.

In [31], Paul studied algebras satisfying

$$(a,b,c) - (a,c,b) = 0 \quad \ldots 5.3.5$$

$$(a, (b,c), d) = 0 \quad \ldots 5.3.6$$
There he showed that a prime algebra satisfying these identities 5.3.5 and 5.3.6 is either associative or its nucleus and center coincide.

Assuming that $R$ is a semiprime algebra satisfying identities 5.3.5 and 5.3.6. Paul's insight [31] was to prove that in such rings

$$(a, b, c) ((d, e), f) = 0.$$ \hspace{1cm} \ldots 5.3.7

Let $I$ be the ideal generated by $D = (R, R, R)$ and let $C$ be the ideal generated by $((R, R), R)$. Using 5.3.7, we have

$$IC = 0.$$ \hspace{1cm} \ldots 5.3.8

By 5.3.8 the ideal $I \cap C$ is trivial. And since $R$ is semiprime, we must have $I \cap C = 0$.

**Lemma 5.3.1:** If $R$ satisfies identities 5.3.5, 5.3.6 and

$$(a, b, c) = 0,$$ \hspace{1cm} \ldots 5.3.9

then every commutator is in the center $C$ of $R$.

**Lemma 5.3.2:** The identities 5.3.5, 5.3.6, and 5.3.9 imply, for all $x$,

$$(x^2, x)^2 = 0.$$ \hspace{1cm} \ldots 5.3.10

**Lemma 5.3.3:** A semiprime algebra satisfying 5.3.5 and 5.3.6 must also satisfy

$$(x, x, x) = 0.$$ \hspace{1cm} \ldots 5.3.11

**Proof:** Let $u = (x, x, x) = (x^2, x)$, and let $\bar{u}$ be the image of $u$ in $R/C$. By lemma 5.3.1, $\bar{u}$ is in the center of $R/C$.

By lemma 5.3.2, $\bar{u}^2 = 0$. Thus $\bar{u}$ generates a trivial ideal in $R/C$. If $M$ is the ideal generated by $u$ then $\bar{M}$ is the ideal generated by $\bar{u}$. So $\bar{M}^2 \subseteq R/C$. Since $u$ is an associator, $M$ and hence $M^2$ is contained in $I$. 

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Hence $M^2 \subseteq I \cap C = 0$. Since $R$ is semiprime, $M = 0$, and we have $u = 0$, i.e., $(x,x,x) = 0$. □

**Lemma 5.3.4:** identities 5.3.5, 5.3.6, and 5.3.11, together with

$$(ab,c,d) = (a,bc,d) = (a,b,cd) = 0,$$  ...5.3.12

imply $a(b,c,d) = 0$.  ...5.3.13

**Lemma 5.3.5:** The identities 5.3.5, 5.3.6, and 5.3.11 imply

$$(a,b,b) (c,d,d) = 0.$$  ...5.3.14

**Lemma 5.3.6:** In algebras satisfying 5.3.5, 5.3.6, and 5.3.11, the ideal $A = (R,R,R)$.

**Proof:** Since $A = D + RD$, it suffices to show $RD \subseteq D$. In the free algebra on $a, b, c, d$, lemma 5.3.4 tells us that $a(b, c, d)$ is in the $T$-ideal generated by the polynomials 5.3.5, 5.3.6, 5.3.11, 5.3.12. But the polynomials in 5.3.12 have the same degree as $a(b, c, d)$. Therefore, modulo the $T$-ideal generated by 5.3.5, 5.3.6, and 5.3.11, $a(b, c, d)$ must be a linear combination of the elements of the form $(RR,R,R)$, $(R,RR,R)$, and $(R,R,RR)$. This completes the proof. □

**Corollary 5.3.1:** Let $R$ be a semiprime algebra over $Z_{25}$ satisfying 5.3.5, 5.3.6. Then $R$ is associative.

**Proof:** By lemma 5.3.3, $R$ satisfies 5.3.11 as well. By lemma 5.3.5, $R$ must satisfy 5.3.14. Using 5.3.5 and the linearized form of 5.3.14 we get $(R,R,R) (R,R,R) = 0$. By lemma 5.3.6, we have $I^2 = 0$. Since $R$ is semiprime, we must have $I = 0$. This completes the proof. □
Theorem 5.3.4, shown below, was proven with the help of Albert. It generalizes Paul's result in two ways. First, it removes the ambiguity by eliminating the second condition in the conclusion of Paul's theorem. And second, it assumes only that \( R \) is semiprime, an assumption weaker than that of being prime.

**Theorem 5.3.4:** Any semiprime algebra, having char. \( \neq 2,3 \) and satisfying 5.3.5 and 5.3.6, is associative.

Theorem 5.3.4 was arrived at using a sequence of above lemmas 5.3.1, 5.3.2, 5.3.3, 5.3.4, 5.3.5 and 5.3.6. Initially the lemmas were established with Albert. Once Albert had confirmed their validity, Kleinfeld was able to prove the lemmas using traditional-style arguments. This experience suggests a top-down methodology whereby we first sketches out a high-level proof using a sequence of unproven lemmas. Next, using Albert, the lemmas are confirmed. Finally, being confident of their truth, we seeks conventional proofs. Description of this case study can be found in [11].

Algebras satisfying \((x, y, y) = 0\) are called right alternative. The right alternative identity is another generalization of associativity. The well known Cayley algebras [33] satisfy this identity. In [28], Miheev proved that any right alternative algebra also satisfies \((y, y, x)^4 = 0\), but, in general, \((y, y, x)^2 \neq 0\). In 1987 Hentzel and Jacobs used a combination of group representation theory and novel sparse
matrix methods to show that in free right alternative algebras \((y, y, x)^3 \neq 0\) [5]. Albert's efficiency is evidenced by the fact that it can now affirm this result in about an hour.

Albert is written in about 13,000 lines of C, and is free. It may be obtained by ftp'ing to ftp.cs.clemson.edu, typing anonymous at the login prompt, and then downloading the source modules in the albert directory.

Using Albert we wish to study the index of the nilpotency of the associator ideal in rings with associators in the nucleus.